Homework assignment 2*

Exercise 2.30. Let $v \in \mathbb{R}^3$ and $v \neq 0$ and consider the linear ODE on $\mathbb{R}^3$:

$$\dot{x} = v \times x,$$

where $\times$ denotes cross product.

Show that the solutions of this ODE are rigid rotations of the initial vector around the direction of the vector v.

Writing the ODE as:

$$\dot{x} = Sx,$$

show that $S = -S^T$ (that is, S is skew-symmetric). Show that the flow $\phi_t(x) = \mathrm{e}^{tS}\, x$ forms a group of orthogonal transformations.

Prove that every solution is periodic and determine the period in terms of v.

Solution. The main idea is to think geometrically about this problem, in particular about the geometric interpretation of the cross product of two vectors. Since $v \times v = 0$, it follows that $v/|v|$ is a unit eigenvector of the matrix S, corresponding to the eigenvalue 0. Choose two orthonormal vectors $v_1^\perp$ and $v_2^\perp$ in the orthogonal complement of the linear space spanned by v, and such that $v/|v|, v_1^\perp, v_2^\perp$ (in that order) form a right hand orthonormal basis of $\mathbb{R}^3$ (just like the standard basis e_1, e_2, e_3). Notice that $v \times v_1^\perp = |v| v_2^\perp$ and $v \times v_2^\perp = -|v| v_1^\perp$, and this implies that with respect to this particular basis, the system equations are very simple:

$$\dot{y} = S^* y, \;\; S^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -|v| \\ 0 & |v| & 0 \end{pmatrix}.$$

Of course the equation $\dot{x} = Sx$ is transformed to $\dot{y} = S^* y$ by means of the coordinate transformation:

$$x = Ty,$$

where T is a real orthogonal matrix (that is, $TT^T = T^T T = I$) such that $S^* = T^T S T$.

Let us first solve the transformed ODE by determining the principal fundamental matrix solution e^{tS^*}. Recalling the definition of e^{tS^*} and the Taylor series for $\cos(|v|t)$ and $\sin(|v|t)$, we find:

$$\mathrm{e}^{tS^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(|v|t) & -\sin(|v|t) \\ 0 & \sin(|v|t) & \cos(|v|t) \end{pmatrix}.$$

The assertion about the solutions being rigid rotations around the direction of v is now clear.

Then we can easily solve the original ODE by noting that:

$$\mathrm{e}^{tS} = \mathrm{e}^{tTS^*T^T} = T\, \mathrm{e}^{tS^*}\, T^T.$$

The simple -but not really elegant- way of proving that $S = -S^T$, is to start from $\dot{x} = v \times x$, and write the components of the vector field explicitly using the definition of the cross product. A nicer way is to first note that $S^* = -(S^*)^T$, and then observe that:

$$S = TS^*T^T = -T(S^*)^T T^T = -(TS^*T^T)^T = -S^T$$

Finally, note that:

$$\mathrm{e}^{tS}(\mathrm{e}^{tS})^T = T\,\mathrm{e}^{tS^*}\, T^T T (\mathrm{e}^{tS^*})^T T^T = I = T(\mathrm{e}^{tS^*})^T T^T T\, \mathrm{e}^{tS^*}\, T^T = (\mathrm{e}^{tS})^T\, \mathrm{e}^{tS},$$

from which it is immediate that the flow $\phi_t(x)$ forms a group of orthogonal transformations. It is also clear that every solution is periodic with period $2\pi/|v|$, since e^{tS^*} and hence e^{tS} is.

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