Homework assignment 1*

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The first problem assumes some knowledge of integration in Banach spaces, a topic we skipped in class (p. $100 - 105$ in the text).

1. Let X and Y be Banach spaces and U open in X. Let $\{f_n\}$ be a sequence of continuously differentiable functions $f_n : U \to Y$ and $\{Df_n\}$ the corresponding sequence of derivatives. Assume that both $\{f_n\}$ and $\{Df_n\}$ converge uniformly to f, and g respectively. ($\{f_n\}$ converges uniformly to f if for all $\epsilon > 0$, there is some N such that if $n > N$, then $||f_n(x) - f(x)|| < \epsilon$, for all $x \in U$).

   Prove that

   (a) The limits f and g are continuous.

   (b) f is differentiable and $Df = g$.

   Hint: Use the fact that for all n, there holds that

   $$\forall x \in U,\ \exists \delta > 0\ :\ h \in B_\delta(x) \Rightarrow f_n(x+h) - f_n(x) = \int_0^1 Df_n(x+th)h dt,$$

   where $B_\delta(x)$ denotes the open ball of radius δ centered at x. This is an application of Proposition 1.167. Show that this implies that the above equality remains valid if we replace f_n by f and Df_n by g. Conclude that $Df = g$. **Note**: Proposition 1.166 may be useful.

   (c) Consider the space of all continuously differentiable functions $f : U \to X$ and define $||f||_1 = ||f||_0 + ||Df||_0$, where $||.||_0$ denotes the sup norm. Obviously $||f||_1$ may be infinite (Example?). Show that the set of continuously differentiable functions $f : U \to X$ with *finite* $||.||_1$-norm is a Banach space.

   Note: Use the fact that the space of bounded continuous functions with the sup norm is a Banach space.

   Can you extend this idea to r-times differentiable functions $f : U \to X$ for r a positive integer larger than 1?

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The solution to the first part of the second problem will fill the gap in the proof of the theorem on uniform contraction mappings.

2. (a) Let X be a Banach space and $T \in L(X, X)$, that is, T is a bounded linear operator on X. Assume that $||T|| \leq \mu$ for some $\mu < 1$. Prove that $I - T$ is invertible (that is, $I - T$ is a bijection and $(I - T)^{-1} \in L(X, X)$) with inverse $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$ and $||(I - T)^{-1}|| \leq 1/(1 - \mu)$.

   (b) Let H be a Hilbert space and S a closed proper and nonempty subspace of H. It is well-known that every $h \in H$ can be uniquely decomposed as $h = h_1 + h_2$, where $h_1 \in S$ and $h_2 \in S^{\perp}$.

   Consider the mapping $P : H \to S$, defined by $Ph = h_1$ for all $h \in H$. Show that $P \in L(H, H)$ and calculate $||P||$.

   Consider now the equation $Px = x + b$, where $b \in H \setminus S^{\perp}$ is given. Note that this equation has no solution. Modify the equation to $Tx = x + b$ where $T := \lambda I \circ P$ where λ is a parameter taking values in the interval $(0, 1)$. Show that this equation does have a unique solution and determine that solution.

   Remark: The modified equation is relevant for the following reason. For every $\epsilon > 0$, there is a $\lambda \in (0, 1)$ so that $||T - P|| < \epsilon$. We can therefore approximate P by some T as closely as we want. Consequently, we can replace a nonsolvable equation by a solvable equation, which approximates the former as closely as we like.

The third problem offers an alternative -and more classical- proof for the property that solutions of ODE's depend continuously on initial conditions. This proof is based on Gronwall's lemma, a result which is of interest of its own, as it is a powerful tool in the theory of ODE's.

3. (a) **Gronwall's lemma**.

   Let $a < b$ be real numbers and suppose that $\phi : [a, b] \to [0, \infty)$ is continuous. Assume that c_1 and c_2 are nonnegative real numbers, and that

   $$\phi(t) \leq c_1 + c_2 \int_a^t \phi(s) ds, \quad t \in [a, b].$$

   Prove that

   $$\phi(t) \leq c_1 \, \mathrm{e}^{c_2(t-a)}, \quad t \in [a, b].$$

   Hint: Prove the lemma first for the case $c_1 > 0$.

   (b) Let $f : J \times U \to \mathbb{R}^n$, where J is open in $\mathbb{R}$ and U is open in $\mathbb{R}^n$, be continuously differentiable and Lipschitz with respect to its second variable. That is, there is some $K > 0$ such that for all pairs (t, x_1) and (t, x_2) in $J \times U$,

   $$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|.$$

   Suppose that the IVP's

   $$\dot{x} = f(t, x), \quad x(t_0) = x_0,$$

   and

   $$\dot{x} = f(t, x), \quad x(t_0) = y_0,$$

   have solutions $x(t)$ and $y(t)$, both defined on a common interval $J^* \subset J$, with $t_0 \in J^*$. Prove that

   $$|x(t) - y(t)| \leq |x_0 - y_0| \, \mathrm{e}^{K|t-t_0|}, \quad t \in J^*.$$