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Stabilization of positive linear systems ☆

Patrick De Leenheer *, Dirk Aeyels

Universiteit Gent, SYSTeMS, Technologiepark-Zwijnaarde 9, 9052 Gent, Belgium

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Abstract

We consider stabilization of equilibrium points of positive linear systems which are in the interior of the first orthant. The existence of an interior equilibrium point implies that the system matrix does not possess eigenvalues in the open right half plane. This allows to transform the problem to the stabilization problem of compartmental systems, which is known and for which a solution has been proposed already. We provide necessary and sufficient conditions to solve the stabilization problem by means of affine state feedback. A class of stabilizing feedbacks is given explicitly. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Positive linear systems are linear systems that leave the first orthant of $\mathbb{R}^n$ invariant for future times. These systems have been studied in different application fields ranging from biology and chemistry, over ecology to economy and sociology, see [4] or [3] for particular examples.

In this paper we deal with the stabilization problem of equilibrium points in the interior of the first orthant of $\mathbb{R}^n_+$. It turns out that the existence of an equilibrium point implies that the system matrix does not possess eigenvalues in the open right half plane. This implies that we have a nontrivial stabilization problem only if the largest eigenvalue of the system matrix lies on the imaginary axis. This eigenvalue is real and therefore equal to zero, by the Perron–Frobenius Theorem (see e.g. [2]) adapted to continuous-time systems (as outlined in e.g. [4]).

The next step is then to transform parts of the system to so-called linear compartmental systems. Linear compartmental systems constitute a subclass of positive linear systems, see [3] and [9]. Then results on stabilization of linear compartmental systems are invoked to solve the original stabilization problem. A class of stabilizing feedbacks is provided explicitly.

This paper is organized as follows.

Some definitions and known results are reviewed in Section 2. In Section 3 we show that existence of equilibrium points puts restrictions on the spectrum of the system matrix. We show in Section 4 that, to

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* Corresponding author. Current address: Mathematics Department, Arizona State University, Tempe, AZ 85287, USA.

E-mail addresses: leenheer@math.la.asu.edu (P. De Leenheer), dirk.aeyels@rug.ac.be (D. Aeyels).

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solve the stabilization problem, the constraint on the input should be relaxed. Finally necessary and sufficient conditions are given in Section 5 and a class of stabilizing feedbacks is provided. We also give examples illustrating the main results. Finally we link our results to results on positive controllability of linear systems in Section 6.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real and complex numbers, respectively, and $\mathbb{R}^n$ the set of n-tuples for which all components belong to $\mathbb{R}$. $\mathbb{R}^+ := [0, +\infty)$ ($\mathbb{R}_0^+ := (0, +\infty)$), while $\mathbb{R}_+^n$ ($\text{int}(\mathbb{R}_+^n)$) is the set of n-tuples for which all components belong to $\mathbb{R}^+$ ($\mathbb{R}_0^+$). With $\mathbb{R}^- := (-\infty, 0]$ and $\mathbb{R}_0^- := (-\infty, 0)$, $\mathbb{R}_-^n$ and $\text{int}(\mathbb{R}_-^n)$ have the obvious meaning.

The set of eigenvalues of a real $n \times n$-matrix A is denoted as $\sigma(A)$. A real $n \times n$-matrix A is singular if and only if $0 \in \sigma(A)$. For $z \in \mathbb{C}$, $\text{Re}(z)$ denotes the real part of z. The open right half plane of the complex plane is denoted as $\text{ORHP} := \{z \in \mathbb{C} | \text{Re}(z) > 0\}$ and the open left half plane is denoted as $\text{OLHP} := \{z \in \mathbb{C} | \text{Re}(z) < 0\}$. A real $n \times n$-matrix A is called *Hurwitz* if $\text{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$.

A real $n \times n$-matrix A is called *Metzler* (see e.g. [4], p. 204) if $a_{ij} \in \mathbb{R}^+$ for all $i \neq j$.

A real $n \times n$-matrix A is *compartmental* if the following two conditions are satisfied:

1. A is a Metzler matrix.
2. $\sum_{i=1}^n a_{ij} \leqslant 0$, for all $j = 1, \ldots, n$.

The second condition is often called the (*column*) *diagonal dominance condition*. Notice that it follows from Gerschgorin's Theorem (see e.g. [1]) that if A is a compartmental matrix, then $\sigma(A) \cap \text{ORHP} = \emptyset$.

A real $n \times n$ matrix $A = (a_{ij})$ is *reducible* if the index set $N := \{1, 2, \ldots, n\}$ can be split into two sets J and K with $J \cup K = N$ and $J \cap K = \emptyset$ such that $a_{jk} = 0$ for all $j \in J$ and $k \in K$.

Equivalently, A is reducible if there exists a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B and D are square matrices.

Reducibility can also be characterized as follows. An *m-dimensional coordinate subspace* of $\mathbb{R}^n$ with $m \in \{1, 2, \ldots, n-1\}$ is a subspace of $\mathbb{R}^n$ spanned by m distinct vectors belonging to the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$, where $e_i \in \mathbb{R}^n$ is such that the i-th entry equals 1 and the other entries are 0. A matrix A is reducible if its corresponding linear operator with respect to the standard basis of $\mathbb{R}^n$ maps an m-dimensional coordinate subspace into itself.

When A is not reducible, it is called *irreducible*.

Singular irreducible compartmental matrices are characterized as follows.

Proposition 1. *When A is a real $n \times n$ irreducible compartmental matrix, then A is singular if and only if $\sum_{j=1}^n a_{ij} = 0$ for all $i = 1, \ldots, n$.*

Proposition 1 is adapted from Theorem III in [8], see also [9].

We consider the class of linear systems with scalar inputs

$$\dot{x} = Ax + bu \tag{1}$$

where A is a real $n \times n$-matrix, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $u \in \mathbb{R}$. We assume that the input signals $u(t)$ are such that uniqueness of solutions of system (1) is guaranteed. The forward solution at time t of system (1) with initial condition x_0 and input $u(\tau)$, $\tau \in [0, t]$, is denoted as $x(t, x_0; \mathscr{U}_{[0,t]})$, where $\mathscr{U}_{[0,t]} := \{(\tau, u(\tau)) | \tau \in [0, t]\}$

Definition 1. System (1) is positive if and only if

$$\forall x_0 \in \mathbb{R}^n_+, \forall \mathcal{U}_{[0,t]} \subset [0,t] \times \mathbb{R}^+ \colon x(t, x_0; \mathcal{U}_{[0,t]}) \in \mathbb{R}^n_+, \forall t \in \mathbb{R}^+.$$

The following result can be found in the literature, see e.g. [4].

Proposition 2. *System* (2) *is positive if and only if* A *is a Metzler matrix and* $b \in \mathbb{R}^n_+$.

If A is a compartmental matrix (and therefore also a Metzler matrix) and if $b \in \mathbb{R}^n_+$, then the positive system (1) is called *compartmental*. Notice that compartmental systems without input ($u = 0$) possess a dissipation property since the total mass, given by the sum over all components of the state vector x, is not increasing along forward solutions of the system. Indeed, for $M := \sum_{i=1}^n x_i$ holds that $\dot{M} \leqslant 0$.

3. Equilibria and stability for positive linear systems

In this section it is shown that there is a link between the existence of an interior equilibrium point of a positive linear system and the stability properties of this equilibrium point.

The following hypotheses are introduced:

H1 A is Metzler, $b \in \mathbb{R}^n_+$.
H2 There exists an $\bar{x} \in \mathrm{int}(\mathbb{R}^n_+)$ and a $\bar{u} \in \mathbb{R}^+$ such that $A\bar{x} + b\bar{u} = 0$.

By Proposition 2 hypothesis H1 implies that system (1) is positive and hypothesis H2 means that system (1) admits an equilibrium point $\bar{x} \in \mathrm{int}(\mathbb{R}^n_+)$ for some fixed and constant input $\bar{u}$.

The following proposition is a slight modification of a theorem in [4].

Proposition 3. *If system* (1) *satisfies* H1 *and* H2, *then* $\sigma(A) \cap \mathrm{ORHP} = \emptyset$.

Proof. Suppose that A has an eigenvalue in ORHP. Since A is a Metzler matrix, it follows from the Perron–Frobenius Theorem adapted to continuous-time systems (see e.g. [4]) that A possesses a *real* eigenvalue λ_0 such that:

$$\forall \lambda \in \sigma(A) \setminus \{\lambda_0\} \colon \ \mathrm{Re}(\lambda) < \lambda_0. \tag{2}$$

Since there exists an eigenvalue of A in ORHP, we obtain that $\lambda_0 > 0$. Moreover, an (left) eigenvector f^{T} associated with λ_0, belongs to $\mathbb{R}^n_+ \setminus \{0\}$: $f^{\mathrm{T}} A = \lambda_0 f^{\mathrm{T}}$. Since $A\bar{x} + b\bar{u} = 0$ by H2, we obtain that $\lambda_0 f^{\mathrm{T}} \bar{x} + f^{\mathrm{T}} b\bar{u} = 0$. The first term in the left hand side of this equation is strictly positive, while the second term is nonnegative. The sum of these terms cannot be zero, so we have reached a contradiction. □

Proposition 3 shows that a positive system cannot posses equilibrium points in $\mathrm{int}(\mathbb{R}^n_+)$ if A has an eigenvalue in ORHP. If there exists an equilibrium point $\bar{x}$ in $\mathrm{int}(\mathbb{R}^n_+)$ for some $\bar{u} \in \mathbb{R}^+$, then A is Hurwitz or 0 is an eigenvalue of A and the other eigenvalues of A have a real part which is smaller than 0. If A is Hurwitz then it follows from linear systems theory that $\bar{x}$ is globally asymptotically stable (GAS) (even with respect to initial conditions in $\mathbb{R}^n$) with $u(t) = \bar{u}$, $\forall t \in \mathbb{R}^+$.

If 0 is an eigenvalue of A and the other eigenvalues of A have a real part which is smaller than 0, then $\bar{x}$ is not GAS with $u(t) = \bar{u}, \forall t \in \mathbb{R}^+$. Summarizing, we obtain

Corollary 1. *If system* (1) *satisfies* H1 *and* H2 *then* $\bar{x}$ *is an equilibrium point of system* (1) *with* $u(t) = \bar{u}$, $\forall t \in \mathbb{R}^+$. *If* $0 \notin \sigma(A)$, *then* $\mathrm{Re}(\lambda) < 0$, $\forall \lambda \in \sigma(A)$ *and* $\bar{x}$ *is GAS with respect to initial conditions in* $\mathbb{R}^n_+$. *If* $0 \in \sigma(A)$, *then* $\mathrm{Re}(\lambda) < 0$, $\forall \lambda \in \sigma(A) \setminus \{0\}$ *but* $\bar{x}$ *is not GAS with respect to initial conditions in* $\mathbb{R}^n_+$.

4. Stabilization by means of positive feedback is not possible

The discussion at the end of the previous section motivates the introduction of the following hypothesis:

H3 $0 \in \sigma(A)$ and for all $\lambda \in \sigma(A) \setminus \{0\}$ holds that $\mathrm{Re}(\lambda) < 0$.

Notice that if H3 holds it is possible that 0 is an eigenvalue with algebraic multiplicity higher than 1. We shall return to this matter below in Remark 5.

We assume that H1, H2 and H3 hold. An extra control vector field is added to the right-hand side of (1):

$$\dot{x} = Ax + b\bar{u} + gv \tag{3}$$

where $g \in \mathbb{R}^n$ and $v \in \mathbb{R}$. Notice that g is allowed to be equal to b. The role of the old control vector field b is to obtain a nontrivial equilibrium point (see H2). The role of the new control vector field g will become clear below when we state a stabilization problem. We stress that in the remainder of the paper $\bar{u}$ is a fixed scalar, while v is a scalar input.

The extended system (3) is positive (assuming that $v \in \mathbb{R}^+$) if and only if $g \in \mathbb{R}^n_+$. We assume in addition that $g \neq 0$:

H4 $g \in \mathbb{R}^n_+ \setminus \{0\}$.

The following stabilization problem naturally arises:

Stabilization problem 1. If system (3) satisfies H1, H2, H3 and H4, does there exist an appropriate continuous state feedback $v(x): \mathbb{R}^n_+ \to \mathbb{R}^+$ such that $\bar{x}$ is GAS for the resulting closed loop system when restricting initial conditions to $\mathbb{R}^n_+$?

If a continuous mapping $v(x): \mathbb{R}^n_+ \to \mathbb{R}^+$ solves the above stabilization problem, then $v(x)$ is called a *continuous stabilizing positive feedback*.

We have the following Proposition.

Proposition 4. *If system* (3) *satisfies* H1, H2, H3 *and* H4, *then there is no continuous stabilizing positive feedback*.

Proof. By Brockett's necessary condition for stabilization (see [7], p. 182 and [5]), the mapping $F: \mathbb{R}^n_+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $F(x, v) := Ax + b\bar{u} + gv$ should be onto a neighborhood of zero in some neighborhood of $(x, v) = (\bar{x}, 0)$.

Since 0 is an eigenvalue of A, $\dim(\mathrm{Im}\, A) < n$. Since $v \in \mathbb{R}^+$, this implies that the mapping F cannot be onto a neighborhood of zero in an arbitrary neighborhood of $(x, v) = (\bar{x}, 0)$. □

Proposition 4 motivates one to weaken the constraint on the sign of the feedback $v(x)$. Instead of a nonnegative feedback $v(x)$, one could look for a feedback that takes both positive and negative values on $\mathbb{R}^n_+$. On the other hand, the resulting closed loop system should remain a positive system. The search for such a feedback is the issue of the next section.

Remark 1. We formulate two more reasons to allow feedbacks for stabilization which take both positive and negative values:

1. The positivity constraint for the input is not natural in many applications. Indeed, examples of positive linear systems are found in e.g. biology, where the state components typically are concentrations of interacting species. A nonnegative input in this context means that all species can only be fed and can never be killed. It is clear that often this constraint is not realistic.
2. We want to stress that Definition 1 of a positive linear system is only meaningful when dealing with *open loop control*, i.e. when no information on the state is available when one applies an input to the system. This situation differs from a closed loop control context, where knowledge on the state is available.

It is clear that if the initial state of the system belongs to $\text{int}(\mathbb{R}^n_+)$, then by continuity of solutions the forward solution will remain inside $\text{int}(\mathbb{R}^n_+)$ for small t, *irrespective of the sign of the input* during this time interval. Only when the initial state belongs to the boundary of $\mathbb{R}^n_+$ the sign of the input can destroy positivity of the system. In a closed loop-scheme, one possesses information on the present state and therefore one can anticipate loss of positivity of the system. Inputs are therefore sometimes allowed to take negative values in this case. On the contrary, in an open loop scheme one does not possess information on the state. This lack of knowledge forces the input to be nonnegative to guarantee that the system is positive.

5. Stabilization by means of affine sign-indefinite feedback

Motivated by the discussion at the end of the preceding section we formulate a new stabilization problem:

Stabilization problem 2. If system (3) satisfies H1, H2, H3 and H4, does there exist an appropriate continuous state feedback $v(x): \mathbb{R}^n_+ \to \mathbb{R}$ such that:

1. The resulting closed loop system is a positive system.
2. $\bar{x}$ is GAS for the resulting closed loop system when restricting initial conditions to $\mathbb{R}^n_+$.

If a continuous mapping $v(x): \mathbb{R}^n_+ \to \mathbb{R}$ solves Stabilization problem 2, then $v(x)$ is called a *continuous stabilizing feedback*.

We will give necessary and sufficient conditions such that Stabilization problem 2 is solved by an appropriate *affine* feedback. Therefore we restrict the admissible class of feedback mappings to a class of affine mappings.

H5 $v(x) = k^{\mathrm{T}}(x - \bar{x})$

With H5 the resulting closed loop system is

$$\dot{x} = (A + gk^{\mathrm{T}})x + (b\bar{u} - gk^{\mathrm{T}}\bar{x}) \tag{4}$$

It is clear that with hypothesis H5, Stabilization problem 2 reduces to the following problem:

Affine stabilization problem. If system (3) satisfies H1, H2, H3 and H4 and if H5 holds, does there exist some $k \in \mathbb{R}^n$ such that:

1. $(A + gk^{\mathrm{T}})$ is Metzler and $(b\bar{u} - gk^{\mathrm{T}}\bar{x}) \in \mathbb{R}^n_+$.
2. $(A + gk^{\mathrm{T}})$ is Hurwitz.

Next we show that, in order to solve the Affine stabilization problem, it suffices to consider feedback gains k^{T} with nonpositive components. This result is based on Problem 6.9.1 in [4]. Below, $\lambda_0(A)$ is the dominating Perron–Frobenius eigenvalue of the Metzler-matrix A.

Lemma 1. *If A is a Metzler matrix and if C is a real $n \times n$-matrix with nonnegative entries such that $A - C$ is also a Metzler matrix, then $\lambda_0(A - C) \leqslant \lambda_0(A)$.*

Proposition 5. *If system* (3) *satisfies* H1, H2, H3, H4, *if* H5 *holds and if there exists an affine stabilizing feedback $k^{\mathrm{T}}(x - \bar{x})$ for some $k \in \mathbb{R}^n$, then there exists an affine stabilizing feedback $\tilde{k}^{\mathrm{T}}(x - \bar{x})$ for some $\tilde{k} \in \mathbb{R}^n_-$.*

Proof. Given $k \in \mathbb{R}^n$, the vector $\tilde{k} \in \mathbb{R}^n_-$ is defined as follows

$$\tilde{k}_i = \min(0, k_i) \quad \text{for } i \in N \tag{5}$$

and thus $\tilde{k}_i \leqslant k_i$ and $\tilde{k}_i \leqslant 0$ for $i \in N$. This implies that for all $i \neq j$

$$a_{ij} + g_i\tilde{k}_j = \begin{cases} a_{ij} + g_i k_j & \text{for } k_j \leqslant 0, \\ a_{ij} & \text{for } k_j > 0. \end{cases} \tag{6}$$

Since both A and $A + gk^{\mathrm{T}}$ are Metzler matrices, (5) implies that $A + g\tilde{k}^{\mathrm{T}}$ is also a Metzler matrix. Also, if $(b\bar{u} - gk^{\mathrm{T}}\bar{x}) \in \mathbb{R}_{+}^{n}$, then it is clear that $(b\bar{u} - g\tilde{k}^{\mathrm{T}}\bar{x}) \in \mathbb{R}_{+}^{n}$. On the other hand it follows from Lemma 1 that $\lambda_0(A + g\tilde{k}^{\mathrm{T}}) \leqslant \lambda_0(A + gk^{\mathrm{T}}) < 0$.

In conclusion, $A + g\tilde{k}^{\mathrm{T}}$ is Metzler and $(b\bar{u} - g\tilde{k}^{\mathrm{T}}\bar{x}) \in \mathbb{R}_{+}^{n}$, $A + g\tilde{k}^{\mathrm{T}}$ is Hurwitz and thus the feedback $\tilde{k}(x - \bar{x})$ solves the Affine stabilization problem. □

Proposition 5 implies that the affine stabilization problem is equivalent to the following problem.

Second affine stabilization problem. If system (3) satisfies H1, H2, H3 and H4 and if H5 holds, does there exist some $k^{\mathrm{T}} \in \mathbb{R}_{-}^{n}$ such that $(A + gk^{\mathrm{T}})$ is Metzler and Hurwitz.

Finally necessary and sufficient conditions on the pair (A, g) will be given such that the second affine stabilization problem can be solved. Two cases are distinguished: A is irreducible and A is reducible.

5.1. A is irreducible

In this subsection it is shown that if A is irreducible, then the affine stabilization problem can be transformed to the affine stabilization problem for *compartmental* systems, a problem which has already been dealt with in [9].

In the next proposition it is shown that every positive linear system with an irreducible system matrix can be transformed in a new positive linear system with an irreducible *compartmental* system matrix.

Proposition 6. *If system* (3) *satisfies* H1, *if* $\sigma(A) \cap \mathrm{ORHP} = \emptyset$ *and if* A *is irreducible then there exists a diagonal state transformation*

$$z = Tx := \mathrm{diag}(f)x \tag{7}$$

for some $f \in \mathrm{int}(\mathbb{R}_{+}^{n})$, *such that system* (3) *is transformed to*

$$\dot{z} = TAT^{-1}z + Tb\bar{u} + Tgv \tag{8}$$

where TAT^{-1} *is an irreducible compartmental matrix and* $Tb \in \mathbb{R}_{+}^{n}$. *If in addition* H4 *holds, then* $Tg \in \mathbb{R}_{+}^{n}$.

Proof. By the Perron–Frobenius Theorem adapted to continuous-time systems (as outlined in e.g. [4]), there exists a vector $f \in \mathrm{int}(\mathbb{R}_{+}^{n})$ and an associated dominating eigenvalue $\lambda_0 \in \mathbb{R}^{-}$ such that

$$f^{\mathrm{T}}A = \lambda_0 f^{\mathrm{T}} \tag{9}$$

where the right-hand side of Eq. (9) is a vector for which all components belong to $\mathbb{R}^{-}$. Eq. (9) can be rewritten as

$$(1, 1, \ldots, 1)\mathrm{diag}(f)A = \lambda_0 f^{\mathrm{T}} \tag{10}$$

Denoting $T = \mathrm{diag}(f)$ and multiplying (10) with T^{-1} on the right-hand side, we obtain

$$(1, 1, \ldots, 1)TAT^{-1} = \lambda_0(1, 1, \ldots, 1) \tag{11}$$

where the right-hand side of Eq. (11) is also a vector for which all components belong to $\mathbb{R}^{-}$. Clearly the state transformation $z = Tx$ meets the required specifications: TAT^{-1} is an irreducible compartmental matrix and $Tb \in \mathbb{R}_{+}^{n}$. If in addition H4 holds, then $Tg \in \mathbb{R}_{+}^{n}$. Notice that this implies in particular that system (8) is also a positive system. □

In [9] necessary and sufficient conditions are given for a pair $(\tilde{A}, \tilde{g})$ where $\tilde{A}$ is a singular, compartmental and irreducible matrix and $\tilde{g} \in \mathbb{R}_{+}^{n}$, such that there exists a vector $\tilde{k} \in \mathbb{R}_{-}^{n}$ such that $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is Metzler and Hurwitz. For completeness these conditions are included below. The main tool to prove the following result is Proposition 1.

Proposition 7. *Given a pair $(\tilde{A}, \tilde{g})$ where $\tilde{A}$ is a singular, compartmental and irreducible matrix and $\tilde{g} \in \mathbb{R}^n_+$, then there exists a $\tilde{k} \in \mathbb{R}^n_-$ such that $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is compartmental and Hurwitz if and only if*

$$\tilde{g} \neq 0 \text{ and there exists at least one } i \in N \text{ such that } \forall j \neq i \text{ with } \tilde{g}_j \neq 0, \text{ also } \tilde{a}_{ji} \neq 0. \tag{12}$$

This result can be rephrased as follows.

Corollary 2. *Suppose that A is an irreducible and compartmental matrix. The Second affine stabilization problem can be solved if and only if condition* (12) *holds for the pair (A, g).*

Remark 2. In linear systems theory a pair $(\tilde{A}, \tilde{g})$ is called *stabilizable* if and only if there exists a $\tilde{k}^{\mathrm{T}} \in \mathbb{R}^n$ such that $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is Hurwitz. It is known that $(\tilde{A}, \tilde{g})$ is stabilizable if and only if

$$\operatorname{rank}[\lambda I - \tilde{A}\ \tilde{g}] = n \text{ holds for all } \lambda \in \sigma(\tilde{A}) \cap \{z \in \mathbb{C} | \operatorname{Re}(z) \geqslant 0\}. \tag{13}$$

Suppose that $\tilde{A}$ is a singular, compartmental and irreducible matrix and that $\tilde{g} \in \mathbb{R}^n_+$. In this case, one could ask for the relation between condition (12) and condition (13). As we show next these conditions are equivalent, implying that classical stabilizability for a pair $(\tilde{A}, \tilde{g})$ where $\tilde{A}$ is singular, compartmental and irreducible, can be achieved by means of a feedback gain $\tilde{k}^{\mathrm{T}} \in \mathbb{R}^n_-$ and such that $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is compartmental.

Proposition 8. *Suppose that $\tilde{A}$ is a singular compartmental and irreducible matrix and that $\tilde{g} \in \mathbb{R}^n_+$. Then the pair $(\tilde{A}, \tilde{g})$ is stabilizable if and only if* (12) *holds.*

Proof. Since the pair $(\tilde{A}, \tilde{g})$ is stabilizable if and only if (13) holds, it suffices to prove that (13) and (12) are equivalent.

1. Condition (12) implies condition (13).
   This follows from linear systems theory. Indeed, suppose that condition (13) does not hold. Since $\tilde{A}$ is a singular, compartmental and irreducible matrix we know that the set $\sigma(\tilde{A}) \cap \{z \in \mathbb{C} | \operatorname{Re}(z) \geqslant 0\}$ equals singleton $\{0\}$ and thus this rank-condition is not satisfied for $\lambda = 0$. Then we obtain from linear systems theory that regardless of the choice of the vector $\tilde{k}$, 0 is eigenvalue of the matrix $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$. But then it is also impossible to find some $\tilde{k}^{\mathrm{T}} \in \mathbb{R}^n_-$ such that $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is Hurwitz, and thus by Proposition 7, condition (12) does not hold.
2. Condition (13) implies condition (12).
   Suppose that condition (12) does not hold. If $\tilde{g} = 0$, then condition (13) cannot hold for $\lambda = 0$ because $\tilde{A}$ is singular. Thus we assume that $\tilde{g} \neq 0$ and that

   $$\text{for all } i \in N \text{ there exists a } j \in N \text{ with } \tilde{g}_j \neq 0 \text{ and } \tilde{a}_{ji} = 0. \tag{14}$$

   On one hand it is obvious that since $\tilde{A}$ is singular, the dimension of $\operatorname{Im}(\tilde{A})$ is less than n. On the other hand it is obvious that the dimension of $\operatorname{Im}(\tilde{A})$ is at least equal to $(n-1)$ (Indeed, if we suppose that this dimension is less than $(n-1)$, then it is not possible to satisfy (13) for $\lambda = 0$). Therefore the dimension of $\operatorname{Im}(\tilde{A})$ equals $(n-1)$, which implies that we can choose $(n-1)$ linearly independent columns of $\tilde{A}$ that span $\operatorname{Im}(\tilde{A})$ (Recall that by definition $\operatorname{Im}(\tilde{A})$ is the linear space which is spanned by the columns of A). Then we obtain by (14) that $\operatorname{Im}(\tilde{A})$ is a subset of the set $\{x \in \mathbb{R}^n | x_i = 0 \text{ for at least one } i \in N\}$. But this contradicts with the fact that $\tilde{A}$ is irreducible. Indeed, the linear operator corresponding to an irreducible matrix and with respect to the standard basis of $\mathbb{R}^n$, does not map an m-dimensional coordinate subspace into itself for some $m \in \{1, 2, \ldots, n-1\}$. □

Finally we would like to point out that there exists an interpretation of condition (12) in terms of particular graphs associated with the pair $(\tilde{A}, \tilde{g})$, see [9] for details.

Remark 3. Suppose that the pair $(\tilde{A}, \tilde{g})$ satisfies the conditions of Proposition 7 and that we would like to find a vector $\tilde{k}^{\mathrm{T}} \in \mathbb{R}^n_-$ such that $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is compartmental and Hurwitz. Then we can construct such a $\tilde{k}^{\mathrm{T}}$ as follows:

We denote by $I^* \subset N$ the set of indices i for which (12) holds and define the sets Z_1, $Z_2 \subset I^*$ as follows:

1. $Z_1 := \{i \in I^* | \tilde{g}_j \neq 0$ for some $j \neq i\}$.
2. $Z_2 := \{i \in I^* | \tilde{g}_j = 0, \forall j \neq i$ and $\tilde{g}_i \neq 0\}$.

Notice that since (12) holds, $I^* \neq \emptyset$ and $I^* = Z_1 \cup Z_2$.

Then for every $\tilde{k}$, satisfying the following constraints, see [9]:

$$\max_{j \neq i, \tilde{g}_j > 0} \frac{-\tilde{a}_{ji}}{\tilde{g}_j} < \begin{cases} \tilde{k}_i \leqslant 0 & \text{for } i \in Z_1, \\ \tilde{k}_i \leqslant 0 & \text{for } i \in Z_2, \\ \tilde{k}_i = 0 & \text{for } i \in N \backslash (Z_1 \cup Z_2), \end{cases} \tag{15}$$

and such that at least one $\tilde{k}_i \neq 0$, the matrix $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is compartmental and Hurwitz.

Returning to the Second affine stabilization problem, we obtain

Theorem 1. *Suppose that* H1, H2, H3, H4 *and* H5 *hold. When A is an irreducible matrix, the Second affine stabilization problem can be solved if and only if*

$$g \neq 0 \text{ and there exists at least one } i \in N \text{ such that } \forall j \neq i \text{ with } g_j \neq 0, \text{ also } a_{ji} \neq 0. \tag{16}$$

Proof. *Necessity*. If condition (16) is not satisfied, then condition (12) is also not satisfied for the pair $(\tilde{A}, \tilde{g})$ where $\tilde{A} := TAT^{-1}$ and $\tilde{g} := Tg$ and T is the diagonal matrix with positive diagonal elements, defined in Proposition 6. Then it follows from Proposition 7 that there does not exist a $\tilde{k}^{\mathrm{T}} \in \mathbb{R}^n_-$ such that $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is compartmental and Hurwitz. This implies that the Second affine stabilization problem cannot be solved either.

Sufficiency. If condition (16) is satisfied, then condition (12) is also satisfied for the pair $(\tilde{A}, \tilde{g})$ where $\tilde{A} := TAT^{-1}$ and $\tilde{g} := Tg$ and T is the diagonal matrix with positive diagonal elements, defined in Proposition 6 (Notice that $\tilde{g} \neq 0$ because H4 holds and since T has positive diagonal elements). Then it follows from Proposition 7 that there exists a $\tilde{k}^{\mathrm{T}} \in \mathbb{R}^n_-$ such that $\tilde{A} + \tilde{g}\tilde{k}^{\mathrm{T}}$ is compartmental and Hurwitz ($\tilde{k}^{\mathrm{T}}$ can be chosen as in (15)). But then also $(\tilde{k}^{\mathrm{T}} T) \in \mathbb{R}^n_-$ and $A + g(\tilde{k}^{\mathrm{T}} T)$ is Metzler and Hurwitz, and thus the Second affine stabilization problem can be solved. □

Example. Consider the following pair (A, g):

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{17}$$

and suppose that we consider the equilibrium point $\bar{x} = (1\ 1)^{\mathrm{T}}$ with $\bar{u} = 0$. The pair (A, g) satisfies (16) and thus the Second affine stabilization problem can be solved. All $k \in \mathbb{R}^n_+$ such that $k_1 \in [-1, 0]$, $k_2 \in [-1, 0]$ and such that at least one of the components of k is not equal to zero, solves the second affine stabilization problem.

5.2. *A is reducible*

In this subsection we deal with the Second affine stabilization problem in case A is an reducible matrix. By definition a reducible matrix A can be brought in the following form by means of a suitable permutation:

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \tag{18}$$

where B and D are square matrices. Now if B is a reducible matrix, then it can also be brought in a form like (18) and then A takes the following form

$$\begin{pmatrix} B_1 & 0 & 0 \\ B_2 & B_3 & 0 \\ C_1 & C_2 & D \end{pmatrix}$$

where B_1 and B_3 are square matrices.

Continuing this process, the reducible matrix A can be brought in the following form by means of a linear transformation $y = P_1 x$ for some suitable permutation matrix P_1.

$$A_{\text{red}} := P_1 A P_1^{\mathrm{T}} = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{pmatrix}$$

for some $k \in \{2, \dots, n\}$ where

1. The blocks A_{ll}, $l \in \{1, \dots, k\}$, on the diagonal of A_{red} are square *irreducible* Metzler matrices. In addition, $\sigma(A_{ll}) \cap \text{ORHP} = \emptyset$ for all $l \in \{1, \dots, k\}$ by H3 and since $\sigma(A) = \bigcup_{l=1}^{n} \sigma(A_{ll})$.
2. The matrices A_{qr} with $q \neq r$ have nonnegative entries.

By Proposition 6 there exists a diagonal matrix

$$T = \begin{pmatrix} T_{11} & 0 & \dots & 0 \\ 0 & T_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T_{kk} \end{pmatrix}, \tag{19}$$

where for all $l \in \{1, \dots, k\}$, T_{ll} is a diagonal matrix with strictly positive diagonal elements such that the linear transformation $z = Ty$ puts A_{red} in the following form:

$$A' := T A_{\text{red}} T^{-1} = \begin{pmatrix} A'_{11} & 0 & \dots & 0 \\ A'_{21} & A'_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A'_{k1} & A'_{k2} & \dots & A'_{kk} \end{pmatrix}, \tag{20}$$

where

1. For all $l \in \{1, \dots, k\}$, $A'_{ll} := T_{ll} A_{ll} T_{ll}^{-1}$ is a square irreducible *compartmental* matrix.
2. A'_{qr} for $q \neq r$ are matrices with nonnegative entries.

Since $0 \in \sigma(A')$ by H3, there are a number of blocks A'_{jj} on the diagonal of A' such that $0 \in \sigma(A'_{jj})$. The sum of the entries of every column of these singular *irreducible* blocks equals zero by Proposition 1. Suppose that there are $m(\leqslant k)$ blocks with this property. The linear transformation $z = TP_1 x$ that transformed matrix A into the form (20) puts the vector g in the following form

$$g' := TP_1 g = \begin{pmatrix} g'_1 \\ \vdots \\ g'_k \end{pmatrix}, \tag{21}$$

where the entries of this vector are nonnegative by the nature of the linear transformation $TP_1 x$.

Summarizing, we obtain the following

Proposition 9. *If system* (3) *satisfies* H1, H3 *and* H4 *and if A is reducible and singular*, *then there exists a state transformation*

$$z = TP_1 x \tag{22}$$

for some suitable permutation matrix P_1 and some diagonal matrix T with strictly positive diagonal elements, *such that system* (3) *is transformed to the following positive system*:

$$\dot{z} = A'z + TP_1 b\bar{u} + g'v, \tag{23}$$

where A' and g' have form (20) *and* (21), *respectively*.

Next we provide two necessary conditions to solve the Second affine stabilization problem.

Proposition 10. *The Second affine stabilization problem can be solved only if $m = 1$*, *where m is the number of singular blocks on the diagonal of A'*.

Proof. Suppose that $m > 1$. Then there exist at least two blocks A'_{rr} en A'_{ss} on the diagonal of A' which are singular, compartmental and irreducible. Assume that $r < s$. From the existence of an affine stabilizing feedback and singularity of A'_{rr} and A'_{ss} follows that there exist nonpositive $k'_r \neq 0$ and $k'_s \neq 0$ such that $A'_{rr} + g'_r k'^{\mathrm{T}}_r$ and $A'_{ss} + g'_s k'^{\mathrm{T}}_s$ are Metzler and Hurwitz. This implies that the nonnegative vectors g'_r and g'_s are different from zero and thus that the matrix $g'_r k'^{\mathrm{T}}_s$ is nonpositive and different from zero (Here we have used the fact that the input v is scalar). But then $A' + g'k'^{\mathrm{T}}$ is not a Metzler-matrix since its (r,s)-block equals $g'_r k'^{\mathrm{T}}_s$. □

It follows from Proposition 10 that for the Second affine stabilization problem to be solvable, A' should contain only one *singular* block matrix on its diagonal. Thus we obtain that A' has the following form

$$A' = \begin{pmatrix} A'_{11} & \dots & 0 & \dots & 0 \\ A'_{21} & \ddots & \vdots & \dots & 0 \\ A'_{31} & \dots & 0 & \dots & 0 \\ \vdots & \dots & A'_{rr} & \dots & 0 \\ A'_{r+11} & \dots & A'_{r+1r} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & 0 \\ A'_{k1} & \dots & A'_{kr} & \dots & A'_{kk} \end{pmatrix} \tag{24}$$

where

1. A'_{rr} is a square, singular, compartmental and irreducible matrix.
2. For all $l \neq r$, A'_{ll} are square, nonsingular, compartmental and irreducible matrices.
3. A_{ij}, for $i \neq j$ are matrices with nonnegative entries.

Proposition 11. *If the Second affine stabilization problem is solvable*, *then there exists $i \in N$ such that the following conditions are satisfied*:

1. *A column of the matrix A'_{rr} is contained in the* ith *column of the matrix A'*.
2. $$g'_r \neq 0 \text{ and } \forall j \in N \text{ with } j \neq i\text{: if } [g']_j \neq 0 \text{ then also } [A']_{ji} \neq 0 \tag{25}$$

Remark 4. Notice that condition (25) is equivalent to the following three conditions:

1. The vectors $g'_1, g'_2, \dots, g'_{r-1}$ are zero.
2. The conditions of Proposition 7 hold for the pair (A'_{rr}, g'_r).
3. $\forall j \in N$ with $[g'_{r+l}]_j \neq 0$, holds that $[A'_{r+lr}]_{ji} \neq 0$, $l = 1, \dots, k-r$.

Proof. To prove the proposition it suffices to show that the Second affine stabilization problem is solvable only if the three conditions in Remark 4 are satisfied.

First we prove necessity of the first condition in Remark 4. Suppose that there exists a nonzero vector g'_s with $s \in \{1, \dots, r-1\}$. The Second affine stabilization problem is solvable only if there exists a nonzero and nonpositive vector $k_r'^{\mathrm{T}}$ such that $A'_{rr} + g'_r k_r'^{\mathrm{T}}$ is compartmental and Hurwitz. But then the matrix $A' + g' k'^{\mathrm{T}}$ is not Metzler for nonzero and nonpositive k'^{T} since its (s, r)-block is a nonzero nonpositive matrix.

The necessity of the second condition in Remark 4 follows immediately by application of Proposition 7.

Finally we prove necessity of the third condition in Remark 4. Suppose that this condition does not hold. Then for all $i \in N$ for which the ith column of A' contains a column of A'_{rr}, there exists a j^* and l^* such that

$$[g'_{r+l^*}]_{j^*} \neq 0 \quad \text{and} \quad [A'_{r+l^* r}]_{j^* i} = 0. \tag{26}$$

Furthermore, to solve the Second affine stabilization problem, there should exist a nonzero and nonpositive $k_r'^{\mathrm{T}}$ such that $A'_{rr} + g'_r k_r'^{\mathrm{T}}$ is compartmental and Hurwitz. But then $A' + g' k'^{\mathrm{T}}$ is not Metzler since by (26), $A' + g' k'^{\mathrm{T}}$ always possesses at least one negative off-diagonal entry. □

Now we are ready to state and prove the main result of this subsection.

Theorem 2. *The Second affine stabilization problem can be solved if and only if* $m = 1$, *where* m *is the number of singular blocks on the diagonal of* A', *and both conditions of Proposition* 11 *hold.*

Proof. *Necessity*. This follows immediately from Propositions 10 and 11.

Sufficiency. Suppose that both conditions of Proposition 11 hold. Define the set $I^* \subset N$ as the set of indices for which condition (25) holds. Next, define the sets Z_1, $Z_2 \subset I^*$ as follows:

1. $Z_1 := \{i^* \in I^* \mid [g']_j \neq 0 \text{ for some } j \neq i^*\}$.
2. $Z_2 := \{i^* \in I^* \mid [g']_j = 0, \forall j \neq i^* \text{ and } [g']_{i^*} \neq 0\}$.

Since condition (25) holds we obtain that $I^* \neq \emptyset$ and $I^* = Z_1 \cup Z_2$.

If k' is chosen as to satisfy the following constraints:

$$\max_{j \neq i^*, [g']_j > 0} \frac{-[A']_{ji^*}}{[g']_j} < \begin{cases} [k']_{i^*} \leqslant 0 & \text{for all } i^* \in Z_1, \\ [k']_{i^*} \leqslant 0 & \text{for all } i^* \in Z_2, \\ [k']_{i^*} = 0 & \text{for all } i^* \in N \backslash (Z_1 \cup Z_2) = N \backslash I^*, \end{cases} \tag{27}$$

and such that a least one component of k' is different from zero, then $A' + g' k'^{\mathrm{T}}$ is Metzler and Hurwitz. Indeed, denoting $k'^{\mathrm{T}} = (k_1'^{\mathrm{T}}, \dots, k_r'^{\mathrm{T}}, \dots, k_k'^{\mathrm{T}})$, we obtain by Proposition 7 that the matrix $A'_{rr} + g'_r k_r'^{\mathrm{T}}$ is compartmental and Hurwitz when k'^{T} satisfies (27). Also the off-diagonal elements of $A' + g' k'^{\mathrm{T}}$ are nonnegative when k'^{T} satisfies (27).

But then also $(k'^{\mathrm{T}} T P_1) \in \mathbb{R}^n_-$ by the nature of the transformation $z = T P_1 x$ defined in Proposition 9 and $A + g(k'^{\mathrm{T}} T P_1)$ is Metzler and Hurwitz. Therefore the Second affine stabilization problem can be solved. □

Remark 5. We point out that the class of positive linear systems for which the stabilization is nontrivial, are those with a system matrix that possesses zero as a *simple* eigenvalue and therefore only for a very small class of systems. Indeed, first we have shown in Proposition 3 that the dominating eigenvalue does not belong to the open right half plane. Since the system matrices are Metzler, this dominating eigenvalue should be zero to have a nontrivial problem. Notice that at this point the open loop system (3) with $v = 0$ could still be unstable. If for example zero has algebraic multiplicity 2, but geometric multiplicity 1, then the open loop system is unstable. However we have shown next that such a situation could not occur. Indeed, when the system matrix is irreducible, then zero is a simple eigenvalue by the Perron–Frobenius Theorem adapted to continuous-time systems, and when the system matrix is reducible we have shown in Proposition 10 that zero is also a simple eigenvalue.

Examples.

1. Consider the following pair (A, g):

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{28}$$

   and suppose that we are considering the Second affine stabilization problem for the equilibrium point $\bar{x} = (1\ 1)^{\mathrm{T}}$ when $b\bar{u} = (1\ 0)^{\mathrm{T}}$.

   Notice that A is already of the form (24) with $k = r = 2$ and that $m = 1$. But (25) is not satisfied. Indeed, $g_1 = 1 \neq 0$ and thus condition 1 in Remark 4 is not satisfied.

2. Consider the following pair (A, g):

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{29}$$

   and suppose that we are considering the Second affine stabilization problem for the equilibrium point $\bar{x} = (1\ 0\ \frac{1}{3})^{\mathrm{T}}$ when $b\bar{u} = (1\ 0\ 1)^{\mathrm{T}}$.

   Notice that A is already of the form (24) with $k = 3$, $r = 2$ and that $m = 1$. But (25) is not satisfied. Indeed, $g_3 = 1 \neq 0$ while $a_{32} = 0$ and thus condition 3 in Remark 4 does not hold. As a consequence, every nonzero $k^{\mathrm{T}} \in \mathbb{R}^n_-$ with $k_2 \neq 0$ (which is necessary to shift the eigenvalue in zero), is such that $[A + gk^{\mathrm{T}}]_{32} < 0$. This implies that $A + gk^{\mathrm{T}}$ cannot be Metzler.

6. Relationship with positive controllability

In [6], positive controllability of *arbitrary* linear systems (i.e. systems which are not necessarily positive) is investigated. In this section we review a result obtained in [6] and compare it to our results.

Consider the following linear system:

$$\dot{x} = Ax + bu, \tag{30}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^+$.

Definition 2. System (30) is completely positively controllable if for all x_0, $x_1 \in \mathbb{R}^n$, there exists an input $u(t)$, defined on some finite time interval $[0, T]$, such that the forward solution of system (30), starting in x_0 with input $u(t)$, exists for all $t \in [0, T]$ and such that $x(T, x_0, \mathscr{U}_{[0,T]}) = x_1$.

We single out the following result from [6].

Theorem 3 (Saperstone [6]). *System* (30) *is completely positively controllable if and only if the following conditions are satisfied*:

1. *The pair* (A, b) *is controllable in the classical sense* (*i.e.* $\mathrm{rank}[\lambda I - A\ b] = n$ *holds for all* $\lambda \in \sigma(A)$).
2. $\sigma(A) \cap \mathbb{R} = \emptyset$.

For our purposes we need a weaker concept, namely that of *asymptotic positive controllability*.

Definition 3. System (30) is asymptotically positively controllable if for all $x_0 \in \mathbb{R}$, there exists an input $u(t)$, defined on the infinite time interval $[0, +\infty)$, such that the forward solution of system (30), starting in x_0 with input $u(t)$, exists for all $t \in [0, +\infty)$ and such that $\lim_{t \to +\infty} x(t, x_0, \mathscr{U}_{[0,t]}) = 0$.

Using the results in [6] it is then possible to prove the following result.

Theorem 4. *System* (30) *is asymptotically positively controllable if and only if the following conditions are satisfied*:

1. *The pair* (A, b) *is stabilizable* (*see Remark* 2 *for a definition*) *in the classical sense*.
2. $\sigma(A) \cap \mathbb{R}^+ = \emptyset$.

Finally we introduce the concept of *positive stabilizability*.

Definition 4. System (30) is positively stabilizable if there exists a sufficiently smooth (say locally Lipschitz) feedback map $u(x): \mathbb{R}^n \to \mathbb{R}^+$ with $u(0) = 0$, such that the zero solution of the closed loop system (30) and $u(x)$ is globally asymptotically stable in $\mathbb{R}^n$.

It is straightforward that asymptotic positive controllability is a necessary condition for positive stabilizability.

Let us apply these results to the particular case where system (30) is a positive system. Thus according to Proposition 2, the matrix A is a Metzler matrix and $b \in \mathbb{R}^+$. Then by the Perron–Frobenius Theorem adapted to continuous-time linear systems, the matrix A possesses a *real* dominating eigenvalue. If we assume that system (30) is asymptotically positively controllable, then it follows from Theorem 4 that this dominating eigenvalue is negative and thus that A is Hurwitz. This implies that a necessary condition for positive stabilizability of a *positive* linear system is that the matrix A is Hurwitz. But then the trivial feedback $u(x) = 0$ is positively stabilizing. Summarizing, only trivial positive stabilizability problems occur for positive linear systems.

The origin of this triviality lies in the fact that the input is assumed to be *nonnegative*. We also encountered this in our stabilization problems. We were interested in stabilization of interior equilibria and were therefore led to consider systems for which the dominating eigenvalue is negative or zero (see Proposition 3 and Corollary 1). Thus the only nontrivial case was the one for which the dominating eigenvalue equals zero. If at this point we would have insisted on the use of *nonnegative* inputs for feedback, this problem would not have been solvable for the same reasons we encountered in the discussion of the previous paragraph. This is why we relaxed the input constraint and decided to allow negative inputs for feedback. On the other hand we required for obvious physical reasons, that the resulting closed loop system was a positive system.

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References

[1] R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York, 1970.
[2] F.R. Gantmacher, The Theory of Matrices, Chelsea, New York, 1959.
[3] J.A. Jacquez, C.P. Simon, Qualitative theory of compartmental systems, SIAM Review 35 (1) (1993) 43–79.
[4] D.G. Luenberger, Introduction to Dynamic Systems, Wiley, New York, 1979.
[5] E.P. Ryan, On brockett's condition for smooth stabilizability and its necessity in a context of smooth feedback, SIAM J. Control Optim. 32 (6) (1994) 1597–1604.
[6] S.H. Saperstone, Global controllability of linear systems with positive controls, SIAM J. Control 11 (3) (1973) 417–423.
[7] E.D. Sontag, Mathematical Control Theory, Springer, New York, 1990.
[8] O. Taussky, A recurring theorem on determinants, Amer. Math. Monthly 56 (4) (1949) 672–676.
[9] J.M. van den Hof, Positive linear observers for linear compartmental systems, SIAM J. Control Optim. 36 (2) (1998) 590–608.