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An Elementary Proof of a Matrix Tree Theorem for Directed Graphs*

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Abstract. We present an elementary proof of a generalization of Kirchhoff's matrix tree theorem to directed, weighted graphs. The proof is based on a specific factorization of the Laplacian matrices associated to the graphs, which involves only the two incidence matrices that capture the graph's topology. We also point out how this result can be used to calculate principal eigenvectors of the Laplacian matrices.

Key words. directed graphs, spanning trees, matrix tree theorem

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1. Introduction.

Kirchhoff's matrix tree theorem [3] is a result that allows one to count the number of spanning trees rooted at any vertex of an undirected graph by simply computing the determinant of an appropriate matrix—the so-called reduced Laplacian matrix—associated to the graph. A recent elementary proof of Kirchhoff's matrix tree theorem can be found in [5]. Kirchhoff's matrix theorem can be extended to directed graphs; see [4, 2, 7] for proofs. According to [2], an early proof of this extension is due to [1], although the result is often attributed to Tutte in [6] and is hence referred to as Tutte's Theorem. The goal of this paper is to provide an elementary proof of Tutte's Theorem. Most proofs of Tutte's Theorem appear to be based on applying the Leibniz formula for the determinant of a matrix expressed as a sum over permutations of the matrix elements to the reduced Laplacian matrix associated to the directed graph. Some proofs [4, 2] are based on counting schemes that apply the inclusion-exclusion principle to collections of particular subgraphs of the original directed graph. The strategy proposed here is to instead factor the Laplacian matrix as a product of two rectangular matrices and use the Binet–Cauchy determinant formula to break it up into simpler pieces that are easier to calculate. Factorization of the Laplacian matrix is not a novel idea and has been used before in the proof of Kirchhoff's matrix tree theorem for undirected graphs presented in [5]. However, the Laplacian matrix in the case of a directed graph is symmetric and negative semidefinite, and can therefore be factored as minus the product of a square matrix times its transpose. In the case of undirected graphs, the Laplacian matrix is no longer symmetric, hence a useful factorization is not immediately obvious. One of the key ideas presented here is that an interesting factorization still exists, although the two factors

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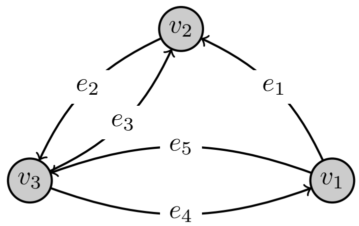


Fig. 1 *An example of a directed graph G with vertex set $V = \{v_1, v_2, v_3\}$ and directed edge set $E = \{e_1, e_2, e_3, e_4, e_5\}$.*

in the product are no longer square and instead are rectangular matrices. Nevertheless, both factors are closely related to the so-called incidence matrices associated to the directed graph, which are two algebraic objects that unambiguously capture the graph's topology and orientation.

Let $G = (V, E)$ be a *directed graph* with finite vertex set $V = \{v_1, \ldots, v_p\}$ and finite directed edge set $E = \{e_1, \ldots, e_q\}$. We assume that each directed edge points from some vertex v_i to another vertex $v_j \neq v_i$, and that there exists at most one directed edge from any vertex to any distinct vertex. For each vertex v_i of G, we define the *in-degree of* v_i as the number of distinct directed edges which point to v_i. Similarly, the *out-degree of* v_i is the number of distinct directed edges of G which point from v_i to some other vertex. A *directed cycle* of G is a collection of distinct vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$ and a collection of distinct directed edges $\{e_{k_1}, \ldots, e_{k_n}\}$ such that each e_{k_j} points from v_{i_j} to $v_{i_{j+1}}$, and where $v_{i_{n+1}} = v_{i_1}$.

Example. Consider the directed graph in Figure 1, which we shall use as a running example throughout this paper to illustrate the various concepts and notions. This directed graph has $p = 3$ vertices and $q = 5$ directed edges. The in-degrees of v_1, v_2, and v_3 are equal to $1, 2$, and 2, respectively. The out-degrees of v_1, v_2, and v_3 are equal to $2, 1$, and 2, respectively. There are several directed cycles, such as $\{v_2, v_3\}$ and $\{e_2, e_3\}$; $\{v_3, v_1\}$ and $\{e_4, e_5\}$; and $\{v_1, v_2, v_3\}$ and $\{e_1, e_2, e_4\}$.

Definitions. A *directed subgraph* of G is a directed graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$. Fix a vertex v_r in V. We say that a directed subgraph G' of G is an *outgoing (incoming) directed spanning tree rooted at* v_r if $V' = V$ and if the following three conditions hold:

1. Every vertex $v_i \neq v_r$ in V' has in-degree (out-degree) 1.
2. The root vertex v_r has in-degree (out-degree) 0.
3. G' has no directed cycles.

Note that any outgoing (incoming) directed spanning tree of G necessarily has $p-1$ distinct directed edges selected among the q directed edges of G. Indeed, an outgoing (incoming) directed spanning tree must have exactly p vertices. All its vertices except for the root v_r must have in-degree (out-degree) equal to 1, and the in-degree (out-degree) of the root v_r must be 0. Therefore, to identify outgoing (incoming) directed spanning trees, we should only consider directed subgraphs G' of G with the same number of vertices as G (namely, p) and with exactly $p - 1$ distinct directed edges chosen among the directed edges of G. There are a total of $\binom{q}{p-1}$ directed subgraphs G' of G with p vertices and $p-1$ directed edges, a possibly large number. Only some—and in some cases, none—of these directed subgraphs are outgoing (incoming) directed spanning trees, namely, those which do not contain directed cycles. In order to count

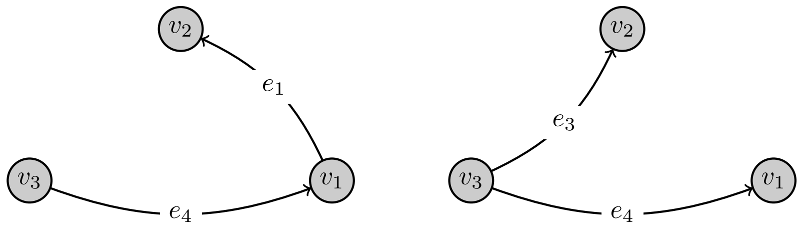


Fig. 2 *Outgoing directed spanning trees rooted at* $v_r = v_3$ *for the directed graph from Figure* 1.

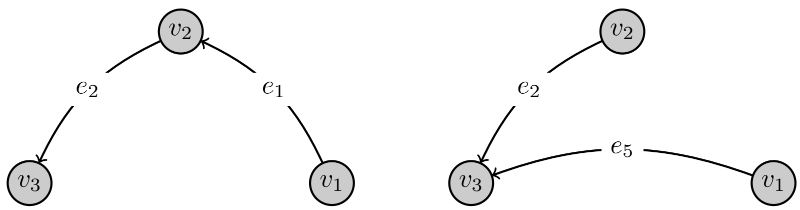


Fig. 3 *Incoming directed spanning trees rooted at* $v_r = v_3$ *for the directed graph from Figure* 1.

the number of outgoing (incoming) directed spanning trees at a given root, we need a counting scheme that recognizes the outgoing (incoming) directed spanning trees and ignores directed subgraphs which contain directed cycles. The proof of Tutte's Theorem presented below provides such a counting scheme.

Example. Consider the directed graph in Figure 1 and choose as root $v_r = v_3$. There are two outgoing (incoming) directed spanning trees rooted at v_3, which are depicted in Figure 2 (Figure 3).

To a directed graph G we associate two real $p \times p$ matrices, called the *Laplacians of* G, which are defined as follows:

$$L_1 = D_{in} - A_v \text{ and } L_2 = D_{out} - A_v^T, \tag{1.1}$$

where

- D_{in} is a diagonal matrix defined as $[D_{in}]_{ii}$ = in-degree of vertex v_i for all $i = 1, \dots, p$;
- A_v is the *vertex-adjacency matrix* of G, a real $p \times p$ matrix defined entrywise as follows:

$$[A_v]_{ij} = \begin{cases} 1 \text{ if there exists a directed edge from } v_i \text{ to } v_j, \\ 0 \text{ otherwise;} \end{cases}$$

- D_{out} is a diagonal matrix defined as $[D_{out}]_{ii}$ = out-degree of vertex v_i, for all $i = 1, \dots, p$.

Fix a vertex v_r in G and define the *reduced Laplacians* L_1^r *and* L_2^r by removing the rth row and rth columns from L_1 and L_2, respectively. Then Tutte's Theorem is given as follows.

Theorem 1 (Tutte's Theorem). *Let* $G = (V, E)$ *be a directed graph. The numbers of outgoing and incoming directed spanning trees rooted at* v_r *are equal to* $\det(L_1^r)$ *and* $\det(L_2^r)$, *respectively.*

Example. For the directed graph from Figure 1, and picking the root $v_r = v_3$, we have that

$$D_{in} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } A_v = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and thus

$$L_1 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \text{ and } L_1^r = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

Then $\det(L_1^r) = 2$ is indeed equal to the number of outgoing directed spanning trees rooted at $v_r = v_3$, confirming Tutte's Theorem for this example. Similarly,

$$D_{out} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and thus

$$L_2 = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix} \text{ and } L_2^r = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

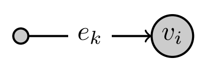
Then $\det(L_2^r) = 2$ is also indeed equal to the number of incoming directed spanning trees rooted at $v_r = v_3$, once again confirming Tutte's Theorem for this example.

2. Proof of Tutte's Theorem.

Although Tutte's Theorem is a remarkable result, it is expressed in terms of rather complicated matrices associated to a directed graph, namely, the reduced Laplacians. To a directed graph, one can associate two much more elementary matrices, known as *incidence matrices*, which arise quite naturally. For a given vertex, one can record the directed edges pointing *to* this vertex. This information will be captured by one of the incidence matrices, namely, by N_{in}. Similarly, one can record, for each vertex, the directed edges pointing *from* this vertex, and this will be captured by the second incidence matrix M_{out}.

Definitions. Let $G = (V, E)$ be a directed graph. The *incidence matrix* N_{in} is a real $q \times p$ matrix defined entrywise as follows:

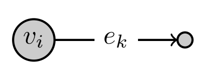
$$[N_{in}]_{ki} = \begin{cases} 1 \text{ if directed edge } e_k \text{ points } \textit{to} \text{ vertex } v_i, \\ 0 \text{ otherwise.} \end{cases}$$



One can identify the kth row of N_{in} with edge e_k. This row has exactly one nonzero entry, which equals 1 and is located in the ith column, where v_i is the vertex to which edge e_k points.

Similarly, the *incidence matrix* M_{out} is a real $p \times q$ matrix defined entrywise as

$$[M_{out}]_{ik} = \begin{cases} 1 \text{ if directed edge } e_k \text{ points } \textit{from} \text{ vertex } v_i, \\ 0 \text{ otherwise.} \end{cases}$$



One can identify the kth column of M_{out} with edge e_k. This column has exactly one nonzero entry, which equals 1 and is located in the ith row, where v_i is the vertex from which e_k points.

Example. For the directed graph from Figure 1,

$$N_{in} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } M_{out} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The two incidence matrices contain purely *local* information concerning a directed graph, namely, they record which directed edges point to, respectively, from each vertex. But on the other hand, both matrices also provide us with *global* information about the graph. Indeed, given these two matrices, we can unambiguously construct the graph. This suggests that perhaps the two Laplacians of a directed graph can be expressed in terms of just the two incidence matrices. The following factorization result shows that this is indeed the case.

LEMMA 1. *Let $G = (V, E)$ be a directed graph. Then*

$$D_{in} = N_{in}^T N_{in}, \ A_v = M_{out} N_{in}, \ \text{and } D_{out} = M_{out} M_{out}^T, \tag{2.1}$$

and thus

$$L_1 = (N_{in}^T - M_{out}) N_{in} \text{ and } L_2 = (M_{out} - N_{in}^T) M_{out}^T. \tag{2.2}$$

Fix a vertex v_r in V. Let N_{in}^r be the matrix obtained from N_{in} by removing the rth column in N_{in}, and let M_{out}^r be the matrix obtained from M_{out} be removing the rth row from M_{out}. Then

$$L_1^r = ((N_{in}^r)^T - M_{out}^r) N_{in}^r \text{ and } L_2^r = (M_{out}^r - (N_{in}^r)^T)(M_{out}^r)^T. \tag{2.3}$$

Proof. For all $i, j = 1, 2, \ldots, p$, consider

$$\begin{aligned} [N_{in}^T N_{in}]_{ij} &= \sum_{k=1}^q [N_{in}^T]_{ik} [N_{in}]_{kj} = \sum_{k=1}^q [N_{in}]_{ki} [N_{in}]_{kj} \\ &= \begin{cases} 0 \text{ if } i \neq j, \\ \text{in-degree of vertex } v_i \text{ if } i = j. \end{cases} \end{aligned}$$

Indeed, for each k, $[N_{in}]_{ki}[N_{in}]_{kj} = 0$ when $i \neq j$, since each edge e_k points to exactly one vertex, so at most one of the two factors in this product can be nonzero. On the other hand, when $i = j$, then $[N_{in}]_{ki}[N_{in}]_{kj} = ([N_{in}]_{ki})^2 = 1$ when e_k points to v_i, but equals 0 if e_k does not point to v_i. The sum over all q of these terms therefore yields the in-degree of vertex v_i. This establishes that $D_{in} = N_{in}^T N_{in}$. A similar proof shows that $D_{out} = M_{out} M_{out}^T$.

For all $i, j = 1, 2, \ldots, p$, consider

$$\begin{aligned} [M_{out} N_{in}]_{ij} &= \sum_{k=1}^q [M_{out}]_{ik} [N_{in}]_{kj} \\ &= \begin{cases} 0 \text{ if there is no edge pointing from } v_i \text{ to } v_j, \\ 1 \text{ if there is an edge pointing from } v_i \text{ to } v_j. \end{cases} \end{aligned}$$

Indeed, $[M_{out}]_{ik}[N_{in}]_{kj} = 1$ precisely when edge e_k points from vertex v_i to vertex v_j, and equals zero otherwise. The sum over all q of these terms cannot be larger than 1 because there is at most one distinct directed edge pointing from one vertex to another vertex. This establishes that $A_v = M_{out}N_{in}$.

The definitions of L_1 and L_2, together with (2.1), imply (2.2). Finally, it is immediate from the definition of L_1 (L_2) that $L_1^r = D_{in}^r - A_v^r$ ($L_2^r = D_{out}^r - (A_v^T)^r$), where D_{in}^r (D_{out}^r) and A_v^r ($(A_v^T)^r$) are obtained from, respectively, D_{in} (D_{out}) and A_v (A_v^T) by deleting the rth row and rth column from these matrices. Moreover, (2.1) implies that

$$D_{in}^r = (N_{in}^r)^T N_{in}^r \text{ and } A_v^r = M_{out}^r N_{in}^r, \text{ and}$$
$$D_{out}^r = (M_{out}^r)(M_{out}^r)^T \text{ and } (A_v^T)^r = (N_{in}^r)^T (M_{out}^r)^T,$$

which in turn yields (2.3). ∎

We are now ready for the proof.

Proof of Theorem 1. We shall only provide a proof for the reduced Laplacian L_1^r because the proof for the reduced Laplacian L_2^r is analogous. Lemma 1 implies that

$$\det(L_1^r) = \det\left(((N_{in}^r)^T - M_{out}^r)N_{in}^r\right).$$

For notational convenience we set

$$B = (N_{in}^r)^T - M_{out}^r \text{ and } C = N_{in}^r.$$

Then the Binet–Cauchy determinant formula implies that

$$\det(L_1^r) = \sum_{S\subseteq\{1,\ldots,q\},|S|=p-1} \det(B[S])\det(C[S]),$$

where the sum is over all subsets S of $\{1,\ldots,q\}$ containing $p-1$ elements. There are $\binom{q}{p-1}$ such subsets. Furthermore, $B[S]$ denotes the $(p-1)\times(p-1)$ submatrix obtained from the $(p-1)\times q$ matrix B by selecting precisely those columns of B in the set S. Similarly, $C[S]$ is the submatrix obtained from the $q\times(p-1)$ matrix C by selecting precisely those rows of C in the set S.

To complete the proof of Tutte's Theorem, we will show that:

1. When the $p-1$ elements in S correspond to the indices of the directed edges of an outgoing directed spanning tree rooted at v_r, then
$$\det(B[S])\det(C[S]) = 1.$$
2. When the $p-1$ elements in S correspond to the indices of the directed edges of a directed subgraph of G which is not an outgoing directed spanning tree rooted at v_r, then
$$\det(B[S])\det(C[S]) = 0.$$

1. Suppose that $S = \{k_1, k_2, \ldots, k_{p-1}\}$, with $k_1 < k_2 < \cdots < k_{p-1}$, is in a bijective correspondence to a set of $p-1$ indices of the directed edges $e_{k_1}, e_{k_2}, \ldots, e_{k_{p-1}}$ of an outgoing directed spanning tree rooted at v_r. Similarly, let $\tilde{S} = \{l_1, l_2, \ldots, l_{p-1}\}$, with $l_1 < l_2 < \cdots < l_{p-1}$, be in a bijective correspondence to the set of indices of the vertices in $V \setminus \{v_r\}$. Set $T = (V, E')$ to denote the directed subgraph of G corresponding to this tree, i.e., $E' = \{e_{k_1}, e_{k_2}, \ldots, e_{k_{p-1}}\}$.

Claim. The $(p-1)\times(p-1)$ matrix $C[S] = N^r_{in}[S]$ has precisely one nonzero entry in each row and in each column, and this nonzero entry equals 1. This follows from the fact that for an outgoing directed spanning tree rooted at v_r, each of the vertices distinct from v_r has in-degree equal to 1 (implying that each column of $C[S] = N^r_{in}[S]$ contains exactly one nonzero entry that equals 1), and each of the $p-1$ directed edges points to one of the $p-1$ nonroot vertices (implying that each row of $C[S] = N^r_{in}[S]$ contains exactly one nonzero entry that equals 1).

Consequently, the matrix $C[S] = N^r_{in}[S]$ is a permutation matrix, i.e., it is a matrix obtained from the identity matrix by finitely many column swaps, and thus $C[S](C[S])^T = I = (C[S])^T C[S]$, whence

$$\begin{aligned}\det(B[S])\det(C[S]) &= \det(B[S]C[S])\\ &= \det\left(I - M^r_{out}[S]N^r_{in}[S]\right)\\ &= \det(I-D),\end{aligned}\tag{2.4}$$

where

$$D = M^r_{out}[S]N^r_{in}[S].$$

Recall from Lemma 1 that $A_v = M_{out}N_{in}$. By reducing N_{in} to N^r_{in} and M_{out} to M^r_{out} as in Lemma 1, and then selecting the respective submatrices $N^r_{in}[S]$ and $M^r_{out}[S]$, we obtain that

$$D_{ij} = \begin{cases} 1 \text{ if there is a directed edge in } E' \text{ pointing from } v_{l_i} \text{ to } v_{l_j}, \text{ both in } V\setminus\{v_r\},\\ 0 \text{ otherwise.}\end{cases}$$

Claim. D is nilpotent, and hence there is an invertible $(p-1)\times(p-1)$ matrix S such that $S^{-1}DS = J$, where J, the Jordan canonical form of D, is strictly upper-triangular (all diagonal entries of J are zero), and then (2.4) implies that

$$\det(B[S])\det(C[S]) = \det\left(I-D\right) = \det\left(S(I-J)S^{-1}\right) = \det(I-J) = +1$$

To show that D is nilpotent, we will prove that $D^{p-1}=0$. Arguing by contradiction, assume that $[D^{p-1}]_{i_1 i_p} \neq 0$ for some i_1 and i_p in $\{1,\dots,p-1\}$. Then there exist $v_{l_{i_1}}, v_{l_{i_2}}, \dots, v_{l_{i_p}}$ in $V\setminus\{v_r\}$ such that for all $s = 1,\dots,p-1$ there is some directed edge in E' from $v_{l_{i_s}}$ to $v_{l_{i_{s+1}}}$. Since $V\setminus\{v_r\}$ contains $p-1$ distinct elements, it follows from the Pigeonhole Principle that the sequence $v_{l_{i_1}}, v_{l_{i_2}}, \dots, v_{l_{i_p}}$ must contain two identical terms. But then T contains a directed cycle, which is a contradiction.

2. Suppose that $S = \{k_1,\dots,k_{p-1}\}$ with $k_1 < \cdots < k_{p-1}$ corresponds to the index set of a subgraph $G' = (V,E')$ of G with $E' = \{e_{k_1},\dots,e_{k_{p-1}}\}$, which is not an outgoing directed spanning tree rooted at v_r. Our goal is to show that

$$\det(B[S])\det(C[S]) = 0.$$

There are three possible cases to consider.

Case 1: *There is a vertex v_i in G' with $v_i \neq v_r$, whose in-degree is not* 1. Then either the in-degree of v_i is 0, or it is at least 2. If the in-degree of v_i is 0, then the column of $N^r_{in}[S]$ that records all in-coming edges to v_i in G' is a zero column vector, and thus $\det(C[S]) = \det(N^r_{in}[S]) = 0$. If the in-degree of v_i is at least 2, then there are at least two identical rows in the matrix $N^r_{in}[S]$, and likewise $\det(C[S]) = \det(N^r_{in}[S]) = 0$.

Case 2: *The in-degree of the root* v_r *is not* 0. Then $N_{in}^r[S]$ has at least one zero row, and thus $\det(C[S]) = \det(N_{in}^r[S]) = 0$.

Case 3: G' *contains a directed cycle.* Let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$, be a collection of distinct vertices in $V' = V$, and let $\{e_{l_1}, \ldots, e_{l_n}\}$ be a collection of distinct directed edges in E' such that each e_{l_j} points from v_{i_j} to $v_{i_{j+1}}$, and where $v_{i_{n+1}} = v_{i_1}$. We claim that the sum of the l_1th, l_2th, $\ldots$, l_nth columns of $B[S] = (N_{in}^r)^T[S] - M_{out}^r[S]$ equals the zero vector, and thus $\det(B[S]) = \det\left((N_{in}^r)^T[S] - M_{out}^r[S]\right) = 0$. To see why, note that each of the aforementioned columns has exactly two nonzero entries, one being $+1$ and the other being -1. Moreover, the l_jth and the l_{j+1}th columns will have two nonzero entries of opposite sign in the same position. Therefore, by adding the columns, each $+1$ in some column is canceled by a -1 in another column, establishing the claim. ∎

3. Spanning Trees and Eigenvectors.

To a directed graph $G = (V, E)$, we have associated two Laplacians L_1 and L_2; see (1.1). The sum over all rows in both Laplacians is the zero vector; equivalently, all column sums in both Laplacians are equal to zero. This follows from (2.2) because

$$(1, 1, \ldots, 1)\left(N_{in}^T - M_{out}\right) = (1, 1, \ldots, 1)\left(M_{out} - N_{in}^T\right) = 0,$$

since each column of the matrix $N_{in}^T - M_{out}$ contains exactly two nonzero entries, one being a $+1$ and the other a -1. Thus, zero is an eigenvalue of both L_1 and L_2. We will find eigenvectors associated to the zero eigenvalues of L_1 and L_2 and relate them to the number of outgoing, respectively, incoming directed spanning trees rooted at each of the vertices of G.

THEOREM 2. *Let* $G = (V, E)$ *be a directed graph. Suppose that* x *and* y *are* p*-vectors, defined entrywise as follows:*

(3.1) $x_i =$ *number of outgoing directed spanning trees rooted at vertex* v_i, *and*

(3.2) $y_i =$ *number of incoming directed spanning trees rooted at vertex* v_i

for all $i = 1, \ldots, p$. *Then*

$$L_1 x = 0 = L_2 y. \tag{3.3}$$

Proof. We only give a proof for L_2 as it is similar for L_1. For notational convenience, we drop the subscript of L_2 and denote this matrix by L. Since zero is an eigenvalue of L, we have that $\det(L) = 0$. By expanding the determinant of L along each of the rows of L, we see that

$$\begin{aligned} L_{11}C_{11} + L_{12}C_{12} + \cdots + L_{1p}C_{1p} &= 0, \\ L_{21}C_{21} + L_{22}C_{22} + \cdots + L_{2p}C_{2p} &= 0, \\ &\vdots \\ L_{p1}C_{p1} + L_{p2}C_{p2} + \cdots + L_{pp}C_{pp} &= 0, \end{aligned}$$

where C_{ij} denotes the cofactor of L_{ij}.

We claim that for all i, j, k in $\{1, \ldots, p\}$,

$$C_{ij} = C_{kj}.$$

That is, the cofactors of elements of L in the same column are all equal. This is a standard exercise in linear algebra that relies on basic properties of determinants and exploits the fact that all the column sums of L equal zero, as remarked earlier. From Tutte's Theorem (Theorem 1), follows that for all $i = 1, \dots, p$,

$$C_{ii} = \det(L^i) = y_i = \text{number of incoming directed spanning trees rooted at } v_i.$$

This implies that $Ly = 0$. ∎

Remark. The vectors x and y in Theorem 2 are only eigenvectors of L_1 and L_2 when they are nonzero vectors. This requires that G should have at least one vertex such that there is an outgoing (or incoming) directed spanning tree rooted at that vertex. A sufficient condition for this to happen is that G is a *strongly connected* directed graph. This means that from every vertex of G there must exist a directed path to any other vertex of G. When G is strongly connected, there is a positive number of incoming and outgoing directed spanning trees rooted at every vertex of G. Hence the vectors x and y are entrywise positive vectors. This result also follows directly from the celebrated Perron–Frobenius Theorem applied to the irreducible Laplacian matrices $-L_1$ and $-L_2$. Note that these matrices have nonnegative off-diagonal entries, and we already know that both have an eigenvalue at zero. This eigenvalue is a *principal eigenvalue*, meaning that every other eigenvalue has negative real part. The Perron–Frobenius Theorem then implies that both matrices have unique (up to multiplication by nonzero scalars), entrywise positive eigenvectors associated to their zero eigenvalue. Theorem 2 above provides a way to compute these eigenvectors. Indeed, in principle they can be found by simply counting the number of outgoing and incoming directed spanning trees rooted at every vertex of G. In other words, we have established a purely graphical procedure to compute eigenvectors of the zero principal eigenvalues of the Laplacian matrices.

4. Extensions to Weighted Directed Graphs. In this section we generalize the preceding results to weighted directed graphs.

Let $G_w = (V, E, W)$ be a *weighted directed graph*, where $V = \{v_1, \dots, v_p\}$ is the vertex set, $E = \{e_1, \dots, e_q\}$ the directed edge set, and $W = \{w_1, \dots, w_q\}$ is the set of positive weights associated to each of the directed edges. The *weight of a weighted directed graph* is defined as the product of the weights of its edges:

$$\text{weight of } G_w = \Pi_{i=1}^q w_i.$$

A weighted directed subgraph of G_w is a weighted directed graph $G'_w = (V', E', W')$, where $V' \subseteq V$, $E' \subseteq E$, and W' is the subset of W that corresponds to the subset E' of E. Fix a vertex v_r in V. We say that a weighted directed subgraph G'_w is a *weighted outgoing (incoming) directed spanning tree rooted at* v_r if $G' = (V', E')$ is an outgoing (incoming) directed spanning tree rooted at v_r.

To a weighted directed graph G_w we can associate two real $p \times p$ matrices, also called the *Laplacians of* G_w, which are defined as follows:

$$L_{1,w} = D_{in,w} - A_{v,w} \text{ and } L_{2,w} = D_{out,w} - A_{v,w}^T, \tag{4.1}$$

where

- $D_{in,w}$ is a diagonal matrix such that for all $i = 1, \dots, p$, $[D_{in,w}]_{ii}$ is equal to the sum of the weights of all incoming edges to vertex v_i;

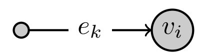
- $A_{v,w}$ is the *weighted vertex-adjacency matrix* of G_w, a real $p \times p$ matrix defined entrywise as follows:

$$[A_{v,w}]_{ij} = \begin{cases} w_k \text{ if } e_k \text{ is the weighted directed edge from } v_i \text{ to } v_j, \\ 0 \text{ if there is no weighted directed edge from } v_i \text{ to } v_j; \end{cases}$$

- $D_{out,w}$ is a diagonal matrix such that for all $i = 1, \dots, p$, $[D_{out,w}]_{ii}$ is equal to the sum of the weights of all outgoing edges of vertex v_i.

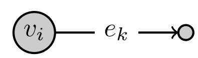
The *weighted incidence matrices* $N_{in,w}$ and $M_{out,w}$ can be associated to a weighted directed graph G_w as well. The matrix $N_{in,w}$ is a $q \times p$ matrix defined entrywise as follows:

$$[N_{in,w}]_{ki} = \begin{cases} w_k^{1/2} \text{ if directed edge } e_k \text{ points } \textit{to} \text{ vertex } v_i, \\ 0 \text{ otherwise,} \end{cases}$$



similarly, the matrix $M_{out,w}$ is a $p \times q$ matrix defined entrywise as follows:

$$[M_{out,w}]_{ik} = \begin{cases} w_k^{1/2} \text{ if directed edge } e_k \text{ points } \textit{from} \text{ vertex } v_i, \\ 0 \text{ otherwise.} \end{cases}$$



The key observation is that $L_{1,w}$ and $L_{2,w}$ can still be factored using the weighted incidence matrices, as in Lemma 2.2:

$$L_{1,w} = (N_{in,w}^T - M_{out,w})N_{in,w} \text{ and } L_{2,w} = (M_{out,w} - N_{in,w}^T)M_{out,w}^T. \tag{4.2}$$

Fix a vertex v_r in the weighted directed graph G_w. Denoting the reduced matrix $N_{in,w}^r$ ($M_{out,w}^r$) as the matrix obtained by deleting the rth column (rth row) from $N_{in,w}$ ($M_{out,w}^r$), and $L_{1,w}^r$ ($L_{2,w}^r$) by deleting the rth row and the rth column from $L_{1,w}$ ($L_{2,w}$), we also have that

$$L_1^r = ((N_{in}^r)^T - M_{out}^r)N_{in}^r \text{ and } L_2^r = (M_{out}^r - (N_{in}^r)^T)(M_{out}^r)^T. \tag{4.3}$$

These factorizations enable a proof of a generalization of Tutte's Theorem (Theorem 1) to weighted graphs, as well as a generalization of Theorem 2.

THEOREM 3. *Let $G_w = (V, E, W)$ be a weighted directed graph. Then the sum of the weights of the weighted outgoing (incoming) directed spanning trees rooted at v_r is equal to* $\det(L_{1,w}^r)$ *(*$\det(L_{2,w}^r)$*).*

Suppose that x and y are p-vectors, defined entrywise as

$x_i =$ *sum of weights of all weighted outgoing directed spanning trees rooted at v_i, and*
$y_i =$ *sum of weights of all weighted incoming directed spanning trees rooted at v_i*

for all $i = 1, \dots, p$. Then

$$L_{1,w}x = 0 = L_{2,w}y. \tag{4.4}$$

We shall skip the proof of Theorem 3 and highlight only two instances where the proof of Theorem 1 needs to be modified slightly. The proof of Theorem 2 does not require any modifications.

- As in the first part of the proof of Theorem 1, we set

$$B[S] = (N^r_{in,w})^T - M^r_{out} \text{ and } C[S] = N^r_{in,w}.$$

Here, $C[S] = N^r_{in,w}[S]$ is not necessarily a permutation matrix anymore, and (2.4) is modified to

$$\det(B[S])\det(C[S]) = \det(Q - D),$$

where

$$Q = (N^r_{in,w}[S])^T N^r_{in,w}[S] \text{ and } D = M^r_{out,w}[S]N^r_{in,w}[S].$$

It can be shown that Q is a diagonal matrix whose determinant equals the weight of the weighted spanning tree T. As before, D is still a nilpotent matrix (because $D^{p-1} = 0$), and thus

$$\det(B[S])\det(C[S]) = \det(Q - D) = \det(Q) = \text{weight of } T.$$

- Another modification must be made to the last paragraph of the proof of Theorem 1. We claim that the sum of the l_1th, l_2th, $\dots$, l_nth columns of $B[S]$ is still zero. In this case, each of the aforementioned columns has exactly two nonzero entries, one being the square root of the weight of some directed edge, and the other being minus the square root of the weight of a directed edge (instead of a $+1$ and a -1). But as before, the l_jth and the l_{j+1}th columns have two nonzero entries of opposite sign in the same position, and consequently these columns add up to the zero vector. Thus, we still conclude that $\det(B[S]) = 0$.

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