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Brief Paper

Stabilization of positive systems with first integrals☆

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Abstract

Positive systems possessing first integrals are considered. These systems frequently occur in applications. This paper is devoted to two stabilization problems. The first is concerned with the design of feedbacks to stabilize a given level set. Secondly, it is shown that the same feedback allows to globally stabilize an equilibrium point if it is asymptotically stable with respect to initial conditions in its level set. Two examples are provided and the results are compared with those in the literature. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Positive systems quite often exhibit first integrals (Sontag, 2001; Mierczyński, 1987). Consider for example a chemical reactor. This is a vessel containing a number of chemical components reacting with each other in a way dictated by chemistry. When no chemicals are added to or withdrawn from the reactor (such a reactor is *closed*), the total mass of the chemicals inside the reactor is conserved and is therefore a first integral for this system.

A natural question is whether it is possible to control the reactor by means of feedback such that the solutions of the closed-loop system converge to the set where the total mass equals a given constant. In other words, the goal is to stabilize the total mass of the system. Feedback laws achieving this goal will be provided.

If in addition the open-loop system possesses an equilibrium point, one may wonder whether it is possible to find a stabilizing feedback. It will be shown that if the equilibrium point is an asymptotically stable equilibrium point of the open-loop system with respect to all initial conditions belonging to the set of states of same total mass, the feedback solving the first stabilization problem also solves the second stabilization problem.

Our results are valid for a more general class of positive systems than the chemical reactor. The common property for systems of this more general class is that they possess first integrals.

We illustrate our results by two examples and compare them with results from Bastin (1999) and Sontag (2001).

2. Notation

With $\mathbb{R}^+ := [0, +\infty)$ ($\mathbb{R}_0^+ := (0, +\infty)$), define $\mathbb{R}_+^n$ ($\mathrm{int}(\mathbb{R}_+^n)$) as those n-tuples having all entries in $\mathbb{R}^+$ ($\mathbb{R}_0^+$). We call $x \in \mathbb{R}_+^n$ ($x \in \mathrm{int}(\mathbb{R}_+^n)$) a *nonnegative* (*positive*) vector and $\mathrm{bd}(\mathbb{R}_+^n) := \mathbb{R}_+^n \backslash \mathrm{int}(\mathbb{R}_+^n)$ is *the boundary of* $\mathbb{R}_+^n$. When $x, y \in \mathbb{R}^n$ we denote $x \leqslant y$ ($x < y, x \ll y$) if $y - x \in \mathbb{R}_+^n$ ($y - x \in \mathbb{R}_+^n \backslash \{0\}$, $y - x \in \mathrm{int}(\mathbb{R}_+^n)$). We denote the index set $\{1, 2, \ldots, n\}$ as N_n. If K is a nonempty subset of N_n, then $F_K := \{x \in \mathbb{R}_+^n \mid x_k = 0 \text{ for } k \in K\}$ is called a *face* of $\mathbb{R}_+^n$. The orthonormal *standard basis* of $\mathbb{R}^n$ is denoted by $\{e_i \mid i \in N_n\}$. Consider the following system:

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ and $f(x)$ is a continuous vector field. Suppose that solutions of system (1) are unique. The (*forward*, *backward*) *solution* of system (1) with initial condition $x_0 \in \mathbb{R}$ is denoted as $x(t, x_0)$, $t \in \mathscr{I}_{x_0}$ ($t \in \mathscr{I}_{x_0}^+$, $t \in \mathscr{I}_{x_0}^-$). The (*forward*, *backward*) *orbit* of $x(t, x_0)$ is the set

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$\{x(t,x_0)\mid t\in\mathscr{I}_{x_0}\}$ ($\{x(t,x_0)\mid t\in\mathscr{I}^+_{x_0}\}$, $\{x(t,x_0)\mid t\in\mathscr{I}^-_{x_0}\}$). We denote the ω limit set and α limit set of a solution as $\omega(x_0)$ and $\alpha(x_0)$.

Definition 1. A set $D\subset\mathbb{R}^n$ is a forward invariant set (invariant set) of system (1) if for all $x_0\in D$ holds that $x(t,x_0)\in D$ for all $t\in\mathscr{I}^+_{x_0}$ (for all $t\in\mathscr{I}_{x_0}$). System (1) is positive if $\mathbb{R}^n_+$ is a forward invariant set of system (1).

3. Positive systems with first integrals

Consider the following system:

$$\dot{x}=f(x), \tag{2}$$

where $x\in D$, D open and $\mathbb{R}^n_+\subset D$ and f is continuously differentiable in D. Introduce the hypothesis

(H1) $\forall x\in\mathbb{R}^n_+ : x_i=0\Rightarrow f_i(x)\geqslant 0$.

Hypothesis H1 implies—without proof—that the following holds.

Theorem 1. *If* H1 *is true*, *then system* (2) *is positive*.

Next we define the concept of a first integral.

Definition 2. A continuously differentiable function H: $\mathbb{R}^n_+\to\mathbb{R}$ is a first integral of system (2) if $(\partial H/\partial x)(x)^{\mathrm{T}}f(x)=0$ for all $x\in\mathbb{R}^n_+$.

We introduce the following hypothesis.

(H2) System (2) possesses k continuously differentiable first integrals $H_1,\dots,H_k$ in $\mathbb{R}^n_+$ of which at least one is radially unbounded. Moreover, $(\partial H_j/\partial x)(x)^{\mathrm{T}}>0$, for all $x\in\mathbb{R}^n_+$ and we assume (w.l.g) that $H_j(0)=0$ for all $j=1,\dots,k$.

Remark 1. We stress that, aside from these k known first integrals, the system may possess different unknown first integrals.

Notice that the first integrals H_j are positive semidefinite functions. Indeed, this follows from $H_j(x)-0=(\int_0^1(\partial H_j/\partial x)(tx)^{\mathrm{T}}\,\mathrm{d}t)x$ which implies that $H_j(x)\geqslant 0$ because $(\partial H_j/\partial x)(x)^{\mathrm{T}}>0$.

Hypothesis (H2) is true in some applications. Systems with constant mass for example possess the total mass $M(x):=\sum_{i=1}^n x_i$ as a first integral, for which (H2) is easily verified.

It will become clear in the subsequent analysis that our main results remain valid if the part of (H2) where at least one of the first integrals is required to be radially unbounded, is replaced by the seemingly weaker condition:

$$\sum_{i=1}^k |H_i(x)|\to+\infty\quad\text{if } |x|\to+\infty\quad\text{with } x\in\mathbb{R}^n_+. \tag{3}$$

However, we show next that if (3) is satisfied, we can always find a set of first integrals satisfying H2: Suppose that k first integrals $H_i(x)$ satisfying (3) are given. Since the first integrals are positive semidefinite, it follows that the absolute value signs in the sum of (3) can be left out. Define the first integral $H(x):=\sum_{i=1}^k H_i(x)$ and replace $H_1(x)$ in the original set of first integrals by $H(x)$. The new set of first integrals $H(x),H_2(x),\dots,H_k(x)$ *does* satisfy all conditions in (H2).

Definition 3. When H2 is true, k real constants $C_1,C_2,\dots,C_k$ define a corresponding level set H_{C} as follows:

$$H_{\mathrm{C}}=\{x\in\mathbb{R}^n_+\mid H_1(x)=C_1,\\ H_2(x)=C_2,\dots,H_k(x)=C_k\}. \tag{4}$$

An obvious but important property of a level set is that it is a forward invariant set for system (2).

Next we propose to control system (2) by means of a feedback in the following way:

$$\dot{x}=f(x)+Bu(x), \tag{5}$$

where $B\in\mathbb{R}^{n\times k}$ and $u:\mathbb{R}^n_+\to\mathbb{R}^k$ is a locally Lipschitz map, to be determined later. Notice that this choice implies that there are as many inputs available as there are known first integrals. A natural requirement is that the controlled system (5) is also positive and this leads to restrictions on B and u.

(H3) $b_j\in\mathbb{R}^n_+$, $\forall j=1,\dots,k$ (where b_j denote the columns of B) and for the map u holds that

$$\forall x\in\mathbb{R}^n_+ : x_i=0\Rightarrow[Bu(x)]_i\geqslant 0. \tag{6}$$

We have—without proof—that the following result holds.

Theorem 2. *If* (H1) *and* (H3) *are true*, *then system* (5) *is positive*.

Finally, we introduce a rather technical hypothesis.

(H4) $(\partial H_j/\partial x)(x)^{\mathrm{T}}b_j>0$ for all $x\in\mathbb{R}^n_+$ and $j=1,\dots,k$.

The geometric interpretation is that b_j and $\partial H_j/\partial x^{\mathrm{T}}$ enclose an acute angle. This is a mild hypothesis. Indeed both vectors are nonnegative in view of (H3) and (H2) implying that $(\partial H_j/\partial x)^{\mathrm{T}}b_j\geqslant 0$. So hypothesis (H4) excludes the situation where b_j and $\partial H_j/\partial x^{\mathrm{T}}$ are orthogonal.

4. Stabilization of a level set

Choose $C_1,\dots,C_k$ and define $H_{\mathrm{C}}:=\{x\in\mathbb{R}^n_+\mid H_j(x)=C_j,\ j=1,\dots,k\}$. We assume that $H_{\mathrm{C}}\neq\emptyset$.

First stabilization problem: If (H1)–(H4) are true, does there exist a feedback $u(x):\mathbb{R}^n_+\to\mathbb{R}^k$ such that $\lim_{t\to+\infty}H_j(x(t,x_0))=C_j$ for all $j=1,\dots,k$ and all $x_0\in\mathbb{R}^n_+$, where the solution of system (5) is denoted as $x(t,x_0)$.

Definition 4. A function $h:\mathbb{R}\to\mathbb{R}$ is a main sectors-function if it is locally Lipschitz and

(1) $h(0)=0$ and $xh(x)>0$ for all $x\neq 0$.
(2) There exist real numbers m and M with $m<M$ such that $h(x)\in[m,M]$ for all $x\in\mathbb{R}$.

We have chosen not to use the term *saturation function* because we do not require that the limits of the function as $x\to+\infty$ or $-\infty$ are the maximal or minimal saturation values.

Define the function $V:\mathbb{R}^n_+\to\mathbb{R}$ as follows:

$$V(x)=\frac{1}{2}\sum_{l=1}^{k}(C_l-H_l(x))^2. \tag{7}$$

Notice that V is continuously differentiable and positive semidefinite with $V(x)=0$ if and only if $x\in H_C$. Moreover, V is radially unbounded in $\mathbb{R}^n_+$ by hypothesis (H2).

Define Σ as follows: $\Sigma:=\{x\in\mathbb{R}^n_+\mid 0\leqslant H_j(x)<C_j,\ j=1,\dots,k\}$. Observe that Σ is bounded and $0\in\Sigma$.

Pick an arbitrary main sectors-function σ and define the feedback $u(x):\mathbb{R}^n_+\to\mathbb{R}^k$ component-wise:

$$u_i(x)=\begin{cases}-\sigma(\frac{\partial V}{\partial x}(x)^{\mathrm{T}}b_i) & \text{for } x\in\Sigma,\\ -\sigma(\frac{\partial V}{\partial x}(x)^{\mathrm{T}}b_i)\displaystyle\prod_{l:(b_i)_l\neq 0}\sigma(x_l) & \text{for } x\notin\Sigma.\end{cases} \tag{8}$$

Notice that u_i is locally Lipschitz and bounded for all $i=1,\dots,k$. Moreover, $u(x)$ satisfies (6) from H3 as we show next. From

$$\frac{\partial V^{\mathrm{T}}}{\partial x}b_i=-(C_i-H_i(x))\frac{\partial H_i}{\partial x}(x)^{\mathrm{T}}b_i-\sum_{l=1,l\neq i}^{k}(C_l-H_l(x))\frac{\partial H_l}{\partial x}(x)^{\mathrm{T}}b_i \tag{9}$$

and (H2) we obtain that $(\partial H_l/\partial x)(x)^{\mathrm{T}}b_i\geqslant 0$ if $l\neq i$, while from H4 follows that $(\partial H_i/\partial x)(x)^{\mathrm{T}}b_i>0$. Consequently, $(\partial V/\partial x)^{\mathrm{T}}b_i<0$ for all $x\in\Sigma$ implying that (6) is satisfied for $x\in\Sigma$. For $x\notin\Sigma$ the argument is simpler because of the factor $\prod_{l:(b_i)_l\neq 0}\sigma(x_l)$ in (8).

Next we calculate the derivative of V along solutions of system (5) with feedback (8):

$$\dot V=\begin{cases}-\displaystyle\sum_{j=1}^{k}(\tfrac{\partial V^{\mathrm{T}}}{\partial x}b_j)\sigma(\tfrac{\partial V^{\mathrm{T}}}{\partial x}b_j) & \text{for } x\in\Sigma,\\ -\displaystyle\sum_{j=1}^{k}(\tfrac{\partial V^{\mathrm{T}}}{\partial x}b_j)\sigma(\tfrac{\partial V^{\mathrm{T}}}{\partial x}b_j)\prod_{l:(b_j)_l\neq 0}\sigma(x_l) & \text{for } x\notin\Sigma\end{cases}$$

and thus $\dot V\leqslant 0$ for all $x\in\mathbb{R}^n_+$ because σ is a main sectors-function.

The forward solutions of system (5) with feedback (8) are bounded since $\dot V\leqslant 0$ and V is radially unbounded. From Lasalle's invariance principle follows that they converge to the largest forward invariant set E_0 in $E:=\{x\in\mathbb{R}^n_+\mid\dot V(x)=0\}$.[1] The task is thus to determine the set E_0.

Let us first determine the set E. For all $x\in\Sigma$ we have already proved that $\dot V<0$. This implies that $E\cap\Sigma=\emptyset$.

Next we determine which points of the set $\mathbb{R}^n_+\backslash\Sigma$ also belong to the set E. Notice first that the set $H_C=\{x\in\mathbb{R}^n_+\mid V(x)=0\}$ $(\subset\mathbb{R}^n_+\backslash\Sigma)$ is a subset of the set E. Indeed, since V is bounded below by zero and $V=0$ on H_C, the inequality $\dot V\leqslant 0$ implies that $H_C\subset E$. Now for $x\in\mathbb{R}^n_+\backslash\Sigma$,

$$\dot V=0\Leftrightarrow\sum_{j=1}^{k}\alpha_j(x)\beta_j(x)=0, \tag{10}$$

where for all $j=1,\dots,k$

$$\alpha_j(x):=\left(\frac{\partial V^{\mathrm{T}}}{\partial x}b_j\right)\sigma\left(\frac{\partial V^{\mathrm{T}}}{\partial x}b_j\right)\quad\text{and}$$

$$\beta_j(x):=\prod_{l:(b_j)_l\neq 0}\sigma(x_l). \tag{11}$$

This implies that for all $j=1,\dots,k$

$$\alpha_j(x)\begin{cases}\geqslant 0 & \text{for } x\in\mathbb{R}^n_+\\ =0 & \text{if and only if } \frac{\partial V^{\mathrm{T}}}{\partial x}b_j=0\end{cases}\quad\text{and}$$

$$\beta_j(x)\begin{cases}\geqslant 0 & \text{for } x\in\mathbb{R}^n_+\\ =0 & \text{if and only if } x\in F_{\mathrm{supp}(b_j)},\end{cases} \tag{12}$$

where $\mathrm{supp}(b_j):=\{l\in N_n\mid(b_j)_l\neq 0\}$ implying that $F_{\mathrm{supp}(b_j)}$ is a face of $\mathbb{R}^n_+$ and therefore a subset of $\mathrm{bd}(\mathbb{R}^n_+)$.

From (10) follows that the points of $\mathbb{R}^n_+\backslash\Sigma$ which also belong to E can be divided into two disjoint sets:

$$\Gamma_1:=\{x\in\mathbb{R}^n_+\backslash\Sigma\mid\alpha_j(x)=0 \text{ for all } j=1,\dots,k\} \tag{13}$$

and

$$\Gamma_2:=\{x\in\mathbb{R}^n_+\backslash\Sigma\mid\text{there exists } j^*\in N_n \text{ such that } \alpha_{j^*}(x)\neq 0 \text{ and } \beta_{j^*}(x)=0\}. \tag{14}$$

There holds that $E=\Gamma_1\cup\Gamma_2$ and $\Gamma_1\cap\Gamma_2=\emptyset$. Notice that $\Gamma_2\subset\mathrm{bd}(\mathbb{R}^n_+)$ because of (12).

(H5) $\Gamma_1=H_C$.

Remark 2. For the important case where all first integrals $H_1,\dots,H_k$ are linear and linearly independent, H5 is true when a generic condition holds which we shall determine next. Assume that $H_j(x):=h_j^{\mathrm{T}}x$ for all $j=1,\dots,k$ and that

[1] Actually, Lasalle's invariance principle states that the forward solutions converge to the largest invariant set in E, but for our purpose it is enough that solutions converge to the largest *forward* invariant set in E.

$\operatorname{rank}(H)=k$ where $H:=(h_1,\dots,h_k)$. There holds that

$$x\in\Gamma_1 \Leftrightarrow \alpha_j(x)=0 \quad \text{for all } j=1,\dots,k \Leftrightarrow (B^{\mathrm{T}}HH^{\mathrm{T}})\,x=C^*,$$

where $C_j^*:=\sum_{l=1}^k C_l h_l^{\mathrm{T}} b_j$, for all $j=1,\dots,k$. Hypothesis H5 is true if and only if

$$\operatorname{rank}(B^{\mathrm{T}}HH^{\mathrm{T}})=k \quad \text{or } \operatorname{rank}(B^{\mathrm{T}}H)=k. \tag{15}$$

Notice that in the case where $k=1$ it follows from (H4) that (15) holds and thus that (H5) is true. For example, for systems for which the total mass is the only known first integral, (H5) is true if (H4) is true. □

When (H5) holds, we obtain that $E=H_{\mathrm{C}}\cup\Gamma_2$. We already pointed out that $\Gamma_2\subset \operatorname{bd}(\mathbb{R}_+^n)$. The following hypothesis excludes the possibility that the set Γ_2 is a subset of E_0.

(H6) For all $x\in\Gamma_2$, holds that $x\notin E_0$.

We show next that $H_{\mathrm{C}}\subset E_0$. We already know that $H_{\mathrm{C}}\subset E$. Since $u_i(x)=0$ for all $x\in H_{\mathrm{C}}$ and $i=1,\dots,k$, H_{C} is a forward invariant set of system (5) with feedback (8), implying that $H_{\mathrm{C}}\subset E_0$.

If (H5) and (H6) also hold, then $E_0=H_{\mathrm{C}}$.

Remark 3. The following condition is sufficient to obtain that H6 is true:

$$\forall x\in\Gamma_2: x_i=0 \Rightarrow f_i(x)>0. \tag{16}$$

Indeed, if (16) holds, the forward solutions of (5) with feedback (8) starting in Γ_2, leave the set E instantaneously. Notice that (16) can be interpreted as a stricter version of H1 for states on $\operatorname{bd}(\mathbb{R}_+^n)$ which also belong to Γ_2. □

Summarizing we obtain the following result.

Theorem 3. *If* H1–H6 *hold then* $\lim_{t\to+\infty} H_j(x(t,x_0))=C_j$ *for all* $j=1,\dots,k$ *and all* $x_0\in\mathbb{R}_+^n$ *where* $x(t,x_0)$ *is the solution of system* (5) *with feedback* (8).

5. Stabilization of an equilibrium point

Again we choose a nonempty level set H_{C} and we introduce the following extra hypothesis.

(H)* There exists a $x^*\in H_{\mathrm{C}}$ such that $f(x^*)=0$ and such that x^* is asymptotically stable for system (2) with respect to initial conditions in the (forward invariant) set H_{C}.

Second stabilization problem: If (H1)–(H*) hold, does there exist a feedback $u(x):\mathbb{R}_+^n\to\mathbb{R}^k$ such that x^* is a globally asymptotically stable equilibrium point for system (5) with respect to initial conditions in $\mathbb{R}_+^n$.

Before stating and proving our main result we need to recall a stability result from Iggidr, Kalitine and Outbib (1996) and prove a lemma.

Theorem 4 (Iggidr, Kalitine and Outbib [2]). *Suppose that the system* $\dot x=g(x)$ *with* $x\in U$, U *open in* $\mathbb{R}^n$ *and* f *of class* C^1 *on* U, *satisfies* $g(0)=0$. *If there exists a function* $V:U\to\mathbb{R}$, *of class* C^1 *on* U *such that* $V(x)\geqslant 0$ *and* $\dot V:=(\partial V/\partial x)(x)^{\mathrm{T}}g(x)$ *for* $x\in U$, *and if* $x=0$ *is an asymptotically stable for* $\dot x=g(x)$ *with respect to initial conditions in the* (*forward invariant*) *set* $M_0:=\{x\in U\mid V(x)=0\}$, *then* $x=0$ *is stable with respect to all initial conditions in* U.

Lemma 1. *Suppose that the system* $\dot x=F(x)$ *with* $x\in G$ *for some subset* G *of* $\mathbb{R}^n$ *where* F *satisfies a local Lipschitz condition on* G, *satisfies* $F(x^e)=0$. *Assume that there exists a compact and forward invariant set* K *for system* $\dot x=F(x)$ *with* $x^e\in K$ *and that* $\omega(x_0)\subset K$ *for all* $x_0\in G$. *If* x^e *is asymptotically stable with respect to initial conditions in* K, *then every forward solution of system* $\dot x=F(x)$ *converges to* x^e.

Proof. The proof proceeds by contradiction. Suppose that there is a point $p\in K$, $p\neq x^e$, which belongs to $\omega(x_0)$. By invariance of ω limit sets, the orbit of the backward solution through p is also contained in $\omega(x_0)$. Notice that the backward solution through p remains in K, for if it would leave K this would contradict with $\omega(x_0)\subset K$. Therefore, it makes sense to introduce the α limit set $\alpha(p)\subset K$. Next we show that $x^e\in\alpha(p)$. Since $\alpha(p)\neq\emptyset$, there exists a $q\in K$ with $q\in\alpha(p)$. By forward invariance of α limit sets, the orbit of the forward solution through q is also contained in $\alpha(p)$. By assumption, this forward solution converges to x^e and since α limit sets are closed, $\alpha(p)$ must contain x^e as claimed. Summarizing, there exists $p\in K$, $p\neq x^e$, with $x^e\in\alpha(p)$. This contradicts that x^e is asymptotically stable with respect to initial conditions in K. Indeed, choose $\varepsilon>0$ such that $\varepsilon<d(x^e,p)$ ($d(x,y)$ is the Euclidean distance between x and y). Then the classical definition of stability implies that there exists a $\delta>0$ such that for every initial condition $y_0\in K$ satisfying $d(x^e,y_0)<\delta$, we have that $d(x^e,x(t,y_0))<\varepsilon$, $\forall t\geqslant 0$. Since $x^e\in\alpha(p)$, one can find a $z\in K$ satisfying $d(x^e,z)<\delta$ belonging to the orbit of the backward solution through p. But this implies that there exists a (possibly large) $T>0$ with $x(T,z)=p$, contradicting stability since $d(x^e,p)>\varepsilon$. □

Theorem 5. *If* (H1)–(H*) *hold, then* x^* *is a globally asymptotically stable equilibrium point of system* (5) *with feedback* (8) *with respect to initial conditions in* $\mathbb{R}_+^n$.

Proof. Again the behavior of V, defined in (7), along solutions of system (5) with feedback (8) is examined. All forward solutions are bounded because V is radially unbounded and $\dot V\leqslant 0$. By Theorem 3, the forward solutions converge to the *compact* level set H_{C}. Classical Lyapunov theorems are not sufficient to conclude that x^* is asymptotically stable or even that x^* is stable for the closed-loop system because V is only positive *semi-definite*. We claim that stability of x^* follows from Theorem 4. Indeed, V defined satisfies the two conditions of Theorem 4 and the set M_0 is the level set H_C. By H*, x^* is asymptotically stable for the closed-loop system with respect to initial conditions in $H_C = M_0$, proving the claim. Next we claim that convergence follows from Lemma 1. Indeed, we know from Theorem 3 that $\omega(x_0)$ is a subset of the compact set H_C for all $x_0 \in \mathbb{R}^n_+$. Moreover, by H*, x^* is asymptotically stable for the closed-loop system with respect to all initial conditions in H_C, proving the claim. □

6. Examples

Academic example: Consider the following system:

$$\dot{x}_1 = -x_1x_2 + x_2^2 + u,$$

$$\dot{x}_2 = -x_1x_2 + x_1^2 + u. \tag{17}$$

It is easily verified that the uncontrolled system ($u=0$) satisfies (H1) and possesses a first integral $H(x)=x_1^2+x_2^2$, which satisfies (H2). Since $(\partial H/\partial x)^{\mathrm{T}}(1,1)^{\mathrm{T}} > 0$, (H4) is readily verified. The set of equilibrium points of system (17) is $\{x \in \mathbb{R}^2_+ \mid x_1 = x_2\}$ and every level set $H_C := \{x \in \mathbb{R}^2_+ \mid H(x) = C\}$ for some $C > 0$ contains a unique equilibrium point which is shown to be asymptotically stable with respect to initial conditions in H_C (and thus that (H*) holds): Indeed, the dynamics of (17) restricted to a fixed level set H_C are $\dot{x}_1 = -x_1\sqrt{C - x_1^2} + C - x_1^2$ where $x_1 \in [0, \sqrt{(C)}]$. It is easily verified that the equilibrium point $x = \sqrt{C/2}$ is asymptotically stable with respect to initial conditions in $[0, \sqrt{C})$. Next we check (H5): the function $V = \frac{1}{2}(H(x) - C)^2$ satisfies $(\partial V/\partial x)(x)^{\mathrm{T}} b = 2(x_1 + x_2)(x_1^2 + x_2^2 - C)$ which equals 0 if $x \in H_C$ or if $x = 0$. But since $x = 0 \in \Sigma$ we obtain that (H5) holds. Finally, (H6) holds because (16) holds. Therefore the feedback law (8) satisfies (H3) and Theorem 5 holds.

Chemical engineering: Consider a reversible chemical reaction $X_1 \leftrightarrow X_2$ and denote the concentrations of the involved chemicals by x_1 and x_2. Assuming that the reactor is isothermal and well-stirred and postulating the mass action principle, we arrive at the following equations:

$$\dot{x}_1 = -k_1x_1^{\alpha} + k_2x_2^{\beta},$$

$$\dot{x}_2 = +k_1x_1^{\alpha} - k_2x_2^{\beta}, \tag{18}$$

where k_i, $i = 1,2$, is the rate constant of the ith reaction and $\alpha \geqslant 1$ and $\beta \geqslant 1$ are the *orders* of the respective reactions, not necessarily integers. For simplicity we assume henceforth that $k_1 = k_2 = 1$. Clearly (H1) is satisfied and $M(x) = x_1 + x_2$ is a first integral. Obviously (H2) is also true. Choosing $b = (1,0)^{\mathrm{T}}$, (H4) is immediately verified. Pick a level set $H_C := \{x \in \mathbb{R}^2_+ \mid M(x) = C\}$ for some $C > 0$. Then (15) and therefore also (H5) holds. Moreover (16) holds and thus (H6) is satisfied. Summarizing, all hypotheses of Theorem 3 are satisfied and H_C can be stabilized. Finally, consider the dynamics of system (18), restricted to $H_C : \dot{x}_1 = -x_1^{\alpha} + (C - x_1)^{\beta}$ where $x_1 \in [0, C]$. Since $C \neq 0$, this system has a unique equilibrium point $x^* \in (0, C)$ which is asymptotically stable. This implies that (H*) is satisfied and Theorem 5 holds.

7. Discussion of the results

7.1. Systems with constant mass

In Bastin (1999) the following positive systems are studied:

$$\dot{x} = f(x) - \mathrm{diag}(a)x + bu, \tag{19}$$

where $f(0) = 0$ and (H1) is true. Moreover, the total mass $M(x)$ satisfies $(\partial M^{\mathrm{T}}/\partial x)(x)^{\mathrm{T}} f(x) = 0$, for all $x \in \mathbb{R}^n_+$, implying that the total mass is a first integral for the system $\dot{x} = f(x)$. Finally $a, b \in \mathbb{R}^n_+ \backslash \{0\}$ and a detectability condition holds for the uncontrolled system $\dot{x} = f(x) - \mathrm{diag}(a)x$.

Notice that there is a fundamental difference between systems (5) and (19): The vector field of the latter contains a dissipative term—diag$(a)x$. Together with the detectability condition this implies global asymptotic stability of the zero solution of the uncontrolled system $\dot{x} = f(x) - \mathrm{diag}(a)x$ with respect to initial conditions in $\mathbb{R}^n_+$. Let us call this feature the *wash-out property* for future reference. An obvious problem for system (19) is to look for a feedback $u(x)$ such that for any given $M^* \neq 0$, $\lim_{t\to+\infty} M(x(t,x_0)) = M^*$ for all $x_0 \in \mathbb{R}^n_+$, where $x(t,x_0)$ is the solution of system (19) with feedback $u(x)$ starting in x_0. Of course a natural requirement is that the closed-loop system is positive. A sufficient condition is that $u(x) \in \mathbb{R}^+$ for all $x \in \mathbb{R}^n_+$ (although this condition is not necessary) and in Bastin (1999) feedbacks are restricted to this class of maps. A bounded feedback satisfying all these constraints is given in Bastin (1999).

Our stabilization problems are formulated for different classes of systems since no dissipative term—diag$(a)x$ is present in the vector field of the uncontrolled system (in our model $a = 0$) and the uncontrolled system does not exhibit the wash-out property. Despite this lack of wash-out, every level set associated to $M(x)$ contains an equilibrium point by Brouwer's fixed point theorem and one may wonder whether these can be globally stabilized. If it can be shown that an equilibrium point is asymptotically stable with respect to initial conditions in its associated level set, then our results imply that global stabilization is achievable. This might be possible for low-dimensional systems or for particular classes of systems as we will discuss in the next subsection. Finally, observe that stabilization of a level set but not of an equilibrium point was considered in Bastin (1999), while both problems are dealt with in this paper.

Although both our feedback and the feedback in Bastin (1999) are bounded, the latter is nonnegative in $\mathbb{R}^n_+$, while our feedback (8) takes both positive and negative values which would have been an undesirable feature in Bastin (1999). It is intuitively clear why level set stabilization is achievable in Bastin (1999) with a feedback taking only nonnegative values: when the total mass is large, control of the system is redundant because the total mass will decrease, thanks to the wash-out property. When the total mass is too small, a positive control signal should be given to increase the total mass. This is different from our situation. Instead of a wash-out property, a conservation law holds and the uncontrolled solutions evolve on level sets but do not converge to the zero solution. This necessitates the use of feedbacks taking both positive and negative values to achieve level set stabilization: For states with total mass higher than the desired value its mass should be removed from the system, while for states with lower total mass, its mass should be added. The level set stabilization problem for our systems would not be solvable with nonnegative control inputs.

7.2. Particular chemical networks

In Sontag (2001), the following class of polynomial systems [2] modeling particular chemical networks, is studied:

$$\dot{x} = g(x) := \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \left(\prod_{k=1}^{n} x_k^{c_{kj}} \right) (c_i - c_j), \tag{20}$$

where A is a nonnegative, irreducible $m \times m$ matrix with $m \leqslant n$ and C is a full rank, nonnegative $n \times m$ matrix having all entries equal to 0 or not less than 1 with nonzero rows. The most important results of Sontag (2001) are:

(1) System (20) is positive and possesses $n - m + 1$ linear first integrals. The solutions of system (20) belong to hyper planes H_C which are parallel to the linear space spanned by $\{c_1 - c_2, c_1 - c_3, \ldots, c_1 - c_m\}$.

(2)
- Every level set $H_C \subset \mathbb{R}^n_+$ contains at least one equilibrium point. Moreover, every level set H_C contains an equilibrium point that belongs to $\mathrm{int}(\mathbb{R}^n_+)$ and that is unique in $H_C \cap \mathrm{int}(\mathbb{R}^n_+)$. But a level set may also contain equilibria in $\mathrm{bd}(\mathbb{R}^n_+)$.
- If an equilibrium point belongs to $H_C \cap \mathrm{int}(\mathbb{R}^n_+)$, then it is asymptotically stable with respect to initial conditions in $H_C \cap \mathrm{int}(\mathbb{R}^n_+)$.
- An equilibrium point is asymptotically stable with respect to *all* initial conditions in its level set if and only if there are no equilibria in that level set belonging to $\mathrm{bd}(\mathbb{R}^n_+)$.

(3) Consider an equilibrium point $x^* \in H_C \cap \mathrm{int}(\mathbb{R}^n_+)$ of system (20). Then this equilibrium point is asymptotically stable with respect to initial conditions in $H_C \cap \mathrm{int}(\mathbb{R}^n_+)$. If in addition there are no equilibria in H_C belonging to $\mathrm{bd}(\mathbb{R}^n_+)$, then x^* is asymptotically stable with respect to initial conditions in H_C.

[2] In fact, a more general class of systems is studied, but here we restrict ourselves to the polynomial case. This restriction simplifies notation. Moreover, the results are the same in both cases.

In Sontag (2001) the following controlled version of system (20) is proposed:

$$\dot{x} = g(x) + \sum_{l=1}^{n-m+1} u_l e_{k_l}, \tag{21}$$

where e_{k_l} are pairwise distinct vectors of the standards basis of $\mathbb{R}^n$. An obvious question is whether there exists a feedback for system (21) such that the closed-loop system is positive and x^* is a globally asymptotically equilibrium point with respect to all initial conditions in $\mathbb{R}^n_+$? It is shown in Sontag (2001) that with an appropriate choice of standard basis vectors, the affine feedback $u_l(x) = \gamma_l(x^*_{k_l} - x_{k_l})$ for $l = 1, \ldots, n - m + 1$ where γ_l are arbitrary, strictly positive real numbers, solves this stabilization problem.

Let us discuss the most important differences and similarities between our systems and the systems of Sontag (2001): Although a common assumption for the uncontrolled systems (2) and (20) is that they exhibit first integrals, the properties of these first integrals are different. The first integrals of (20) are linear, while they can be nonlinear for (2). On the other hand, our hypothesis (H2) imposes certain restrictions for the first integrals. In particular the gradients of the first integrals are nonnegative vectors and the level sets are compact. Neither of these properties are required in Sontag (2001). In fact, examples are given of systems with first integrals having gradients which are not nonnegative and of systems with noncompact level sets.

In our second stabilization problem we assume that an equilibrium point is asymptotically stable with respect to initial conditions in this level set. One could wonder whether it is a feasible task to check this for a given system. For the class of systems given by Eq. (20) this is indeed the case by the availability of a simple criterion: The hypothesis holds if and only if no equilibrium point of the level set belongs to $\mathrm{bd}(\mathbb{R}^n_+)$.

An important difference between (5) and (21) is that the control vector fields in our system may be arbitrary [3] while they have to be standard basis vectors of $\mathbb{R}^n$ for (21).

Also, our feedback (8) is bounded in $\mathbb{R}^n_+$, while the feedback in Sontag (2001) is unbounded.

Another important difference is that our feedback for the second stabilization problem leaves the level set, associated with the equilibrium point that is stabilized, invariant for the closed-loop system and this is not the case for the closed-loop system in Sontag (2001).

Finally, we point out that the hypotheses (H5) and (H6) are technical and may sometimes be hard to check although we have provided some sufficient conditions for them to hold. In Sontag (2001) a similar role is played by the condition that the control vector fields are *non arbitrary* standard basis vectors of $\mathbb{R}^n$.

[3] Provided that they are nonnegative vectors to guarantee that the closed-loop system is positive of course.

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