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Stability Properties of Equilibria of Classes of Cooperative Systems

Patrick De Leenheer and Dirk Aeyels

***Abstract*—This note deals with the constant control problem for homogeneous cooperative and irreducible systems. These systems serve as models for positive systems. A necessary and sufficient condition for global asymptotic stability of the zero solution of this class of systems is known. Adding a constant control allows to shift the equilibrium point from zero to a point in the first orthant. We prove that for every nontrivial nonnegative control vector a unique nontrivial equilibrium point is achieved which is globally asymptotically stable if the zero solution of the uncontrolled system is globally asymptotically stable. In addition a converse result is provided. Finally a stability result for a particular class of Kolmogorov systems is established. We compare our main results to those in the literature.**

***Index Terms*—Cooperative systems, positive systems, stability.**

I. Introduction

A dynamical system is said to be positive if when initiated in the first orthant of $\mathbf{R^n}$, its state remains in this orthant for future times. Examples of these systems are found in a variety of applied areas such as biology, chemistry and sociology [1]–[3].

In [4] and [5] we considered homogeneous cooperative and irreducible systems and we have characterized the stability behavior of the zero solution. In particular we have shown that the zero solution is globally asymptotically stable (GAS) if and only if there exists a unique invariant ray in the interior of the first orthant such that the vector field on this ray points toward the origin.

In applications, it is often undesirable that the zero solution is GAS. Indeed, in a biological system for example, this implies that all species die out. On the other hand a (asymptotically) stable *nontrivial* equilibrium implies coexistence of several species.

The aim of this note is threefold.

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P. De Leenheer is with the Department of Mathematics, Arizona State University, Tempe, AZ 85287 USA (e-mail: leenheer@math.la.asu.edu).

D. Aeyels is with the SYSTeMS research group, Universiteit Gent, 9052 Gent, Belgium (e-mail: dirk.aeyels@rug.ac.be).

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1) To investigate the effect of a constant control on a particular class of positive systems with a GAS zero solution. The controlled system remains *positive* if and only if the control vector is nonnegative. We prove that the controlled system possesses a unique nontrivial GAS equilibrium point.
2) Providing a converse result by proving that if the controlled system possesses an equilibrium point in the first orthant of $\mathbf{R^n}$, then the uncontrolled system has a GAS zero solution. The first result and this converse result extend a well-known theorem for linear positive systems to a class of nonlinear positive systems [3]; see also [6] for a related result.
3) To establish a stability result for a particular class of so-called Kolmogorov systems.

II. Preliminaries

A. Notation

Let $\mathbf{R}$ be the set of real numbers and $\mathbf{R^n}$ the set of n-tuples for which all components belong to $\mathbf{R}$. $\mathbf{R}^+ = [\mathbf{0}, +\infty)$ and $\mathbf{R_0^+} = (\mathbf{0}, +\infty)$, while $\mathbf{R_+^n}$ ($\mathrm{int}(\mathbf{R_+^n})$) is the set of n-tuples for which all components belong to $\mathbf{R}^+$ ($\mathbf{R_0^+}$). Finally, the boundary of $\mathbf{R_+^n}$, $\mathbf{R_+^n} \setminus \mathrm{int}(\mathbf{R_+^n})$, is denoted as $\mathrm{bd}(\mathbf{R_+^n})$.

When $x, y \in \mathbf{R_+^n}$, then $x \le y$ means $x_i \le y_i, \forall\, i = 1, \ldots, n$. Furthermore, $x < y$ if $x \le y$ and $x \ne y$ and $x \ll y$ if $x_i < y_i$, $\forall\, i = 1, \ldots, n$.

Let I be a nonempty and proper subset of $\{1, 2, \ldots, n\}$. The set $F_I := \{x \in \mathbf{R_+^n} | \mathbf{x_i = 0} \text{ for } \mathbf{i \in I}\}$ is a *face* of $\mathbf{R_+^n}$. The *dimension* of F_I equals $\#I$, the cardinality of the set I.

When $x \in \mathbf{R_+^n}$ we define $[0, x] = \{z \in \mathbf{R_+^n} | \mathbf{0 \le z \le x}\}$ and $(0, x) = \{z \in \mathbf{R_+^n} | \mathbf{0 \ll z \ll x}\}$.

Given a vector $x \in \mathbf{R^n}$, $\mathrm{diag}(x)$ is a real $n \times n$ diagonal matrix where the ith diagonal entry equals x_i, the ith component of the vector x. A real $n \times n$ matrix $A = (a_{ij})$ is *Metzler* if and only if its off-diagonal entries $a_{ij}, \forall\, i \ne j$ belong to $\mathbf{R}^+$.

A is *irreducible* if and only if for every nonempty proper subset K of $N := \{1, \ldots, n\}$, there exists an $i \in K$ and a $j \in N \setminus K$ such that $a_{ij} \ne 0$. When A is not irreducible, it is called *reducible*.

Consider the system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbf{R^n}$ and $f(x)$ is a continuously differentiable vector field.

The *forward solution* of system (1) with initial condition $x_0 \in \mathbf{R^n}$ at $t = 0$ is denoted as $x(t, x_0)$ and is defined on the *maximal forward interval of existence* $\mathcal{I}_{x_0} := [0, T_{\max}(x_0))$. A set $D \subset \mathbf{R^n}$ is called *forward invariant* if and only if for all $x_0 \in D, x(t, x_0) \in D$ for all $t \in \mathcal{I}_{x_0}$. A system is called *positive* if and only if $\mathbf{R_+^n}$ is forward invariant. It is intuitively clear and shown in [5] that the following property is necessary and sufficient for positivity of system (1):

$$\mathbf{P}\ \forall\, x \in \mathrm{bd}(\mathbf{R_+^n}):\ \mathbf{x_i = 0 \Rightarrow f_i(x) \ge 0}.$$

The flow of system (1) is *monotone in* D if and only if for all x_0, $y_0 \in D$ with $x_0 \le (<, \ll) y_0$ holds that $x(t, x_0) \le (<, \ll) x(t, y_0)$ for all $t \in (\mathcal{I}_{x_0} \cap \mathcal{I}_{y_0}) \setminus \{0\}$.

The flow of system (1) is *strongly monotone in* D if and only if it is monotone in D and for all $x_0, y_0 \in D$ with $x_0 < y_0$ holds that $x(t, x_0) \ll x(t, y_0)$ for all $t \in (\mathcal{I}_{x_0} \cap \mathcal{I}_{y_0}) \setminus \{0\}$. We recall [7, Th. 3.1] for later reference.

Theorem 1: Suppose that system (1) is defined on some subset D of $\mathbf{R^n}$. If system (1) possesses a unique equilibrium point $\overline{x} \in D$ and if its flow is strongly monotone in D, then all solutions having compact forward orbit closure in D converge to $\overline{x}$.

B. Homogeneous, Cooperative and Irreducible Systems

Now we introduce the concept of a homogeneous vector field.

Definition 1: A vector field $f(x)$, $x \in \mathbf{R^n}$ is said to be *homogeneous* of order $\tau \in \mathbf{R}$ with respect to the dilation map $\delta_\lambda^r(x) := (\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n)^T (\forall i = 1, \ldots, n : r_i \in \mathbf{R}_0^+)$ if

$$\forall\, x \in \mathbf{R^n},\ \forall\, \lambda \in \mathbf{R}^+ : \mathbf{f}(\delta_\lambda^{\mathbf{r}}(x)) = \lambda^\tau \delta_\lambda^{\mathbf{r}}(\mathbf{f}(\mathbf{x})). \tag{2}$$

Assume

H1 $f(x)$ is a homogeneous vector field of order $\tau \in \mathbf{R}^+$ with respect to the dilation map $\delta_\lambda^r(x)$.

System (1) is called *homogeneous* if **H1** holds.

Suppose **H1** holds. If there exists a point $\overline{x} \in \mathbf{R^n}$, $\overline{x} \neq 0$ such that $f(\overline{x}) = \gamma_{\overline{x}}\mathrm{diag}(r)\overline{x}$, $\gamma_{\overline{x}} \in \mathbf{R}$, then the vector field at each point of the *ray* $R_{\overline{x}} = \{\delta_\lambda^r(\overline{x}) | \lambda \in \mathbf{R}_0^+\}$ *through* $\overline{x}$ is tangent to $R_{\overline{x}}$. Indeed, $d/d\lambda(\delta_\lambda^r(\overline{x}))|_{\lambda=1} = \mathrm{diag}(r)\overline{x}$. This means that the vector field is tangent to $R_{\overline{x}}$ in the point $\overline{x}$. In addition, $f(\delta_\lambda^r(\overline{x})) = (\gamma_{\overline{x}}\lambda^\tau)\mathrm{diag}(r)\delta_\lambda^r(\overline{x})$ and thus the vector field is also tangent to $R_{\overline{x}}$ in *every* point of $R_{\overline{x}}$. As a consequence the forward (and backward) solution of system (1) starting in a point of the ray through $\overline{x}$ will remain on this ray for all future (and past) times for which this solution is defined. We call such a ray an *invariant ray* for system (1).

Next, we call on the concept of a cooperative vector field, which has been widely studied, see for example [7].

Definition 2: A vector field $f(x)$, $x \in \mathbf{R}^n$ is said to be *cooperative in* $W \subset \mathbf{R^n}$ if the Jacobian matrix $\partial f/\partial x$ is Metzler for all $x \in W$.

Assume

H2 $f(x)$ is cooperative in $\mathbf{R}_+^\mathbf{n}$.

System (1) is called cooperative if **H2** holds. The meaning of the term cooperative is best explained in a biological context. Suppose that the state of system (1) consists of n interacting species. If the system is cooperative this implies that the presence of species i induces the growth of species j, for all $j \neq i$.

Finally, we introduce the concept of an irreducible vector field.

H3 For $x \in \mathrm{int}(\mathbf{R}_+^\mathbf{n})$, the Jacobian matrix $\partial f/\partial x$ is irreducible.

For $x \in \mathrm{bd}(\mathbf{R}_+^\mathbf{n}) \setminus \{0\}$ holds that $\exists i \in N$ with $x_i = 0$ such that $f_i(x) > 0$.

System (1) is called irreducible if **H3** holds. The following result was proved in [4] and [5].

Theorem 2: If **H1**, **H2** and **H3** hold, then system (1) is positive. In addition the zero solution of system (1) is GAS with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$ if and only if there exists a unique invariant ray $R_{\overline{x}}$ in $\mathbf{R}_+^\mathbf{n}$ for system (1) such that $R_{\overline{x}} \subset \mathrm{int}(\mathbf{R}_+^\mathbf{n})$ and $f(\overline{x}) = \gamma_{\overline{x}}\mathrm{diag}(r)\overline{x}$ with $\gamma_{\overline{x}} < 0$.

A very brief and informal outline of the proof is given here to provide some insight in this result. First it follows from **H2** and $f(0) = 0$ (which is implied by **H1** and continuity of f) that the system is positive. Relying on homogeneity and positivity of the system, a particular $(n-1)$-dimensional system (a projection of the original system) can be defined that evolves on the intersection of $\mathbf{R}_+^\mathbf{n}$ and the unit sphere in $\mathbf{R^n}$. Application of Brouwer's Fixed Point Theorem leads to existence of an equilibrium point for the flow of the projected system which in turn implies existence of an invariant ray in $\mathbf{R}_+^\mathbf{n}$ for the original system. Finally, it follows from **H2** and **H3** that the flow of the system is strongly monotone in $\mathbf{R}_+^\mathbf{n}$. This allows to prove uniqueness of the invariant ray which can be shown to belong to $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ because of **H3**. Relying on the strong monotonicity property of the flow, it is possible to show that the zero solution is GAS.

Theorem 2 gives a criterion for GAS of the zero solution of system (1). From a practical point of view a GAS zero solution is not very interesting. Indeed in a biological system for example this implies that all the species die out. To counteract this undesired property we examine the effect of applying a nonnegative constant control to system (1). In the sequel we will frequently consider systems satisfying the conditions of Theorem 2. To avoid a cumbersome notation we introduce the following hypothesis:

H Hypotheses **H1**, **H2** and **H3** hold and the zero solution of system (1) is GAS with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$. The unique invariant ray in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ of system (1) is denoted as $R_{\overline{x}}$ and has the property that $f(\overline{x}) = \gamma_{\overline{x}}\mathrm{diag}(r)\overline{x}$ with $\gamma_{\overline{x}} < 0$.

III. Main Results

A. Constant Control

Consider the following controlled system:

$$\dot{x} = f(x) + b \tag{3}$$

where $x \in \mathbf{R^n}$, $b \in \mathbf{R^n}$ and $f(x)$ satisfies **H1** and **H2**. We have the following (for a proof we refer to [5]).

Proposition 1: If **H1** and **H2** hold, then system (3) is positive if and only if $b \in \mathbf{R}_+^\mathbf{n}$.

The proof of sufficiency is obtained by observing that **P** holds. Conversely, if $b \notin \mathbf{R}_+^\mathbf{n}$ the forward solution starting in 0 leaves $\mathbf{R}_+^\mathbf{n}$.

When **H** holds and $b \in \mathbf{R}_+^\mathbf{n}$, we show next that (3) possesses at least one equilibrium point.

Proposition 2: If **H** holds and if $b \in \mathbf{R}_+^\mathbf{n}$, then there exists $y_b \in R_{\overline{x}}$ such that for all $y \in R_{\overline{x}}$ with $y \geq y_b$, the set $[0,\, y]$ is forward invariant for system (3). For all $y \in R_{\overline{x}}$ with $y \geq y_b$, the set $[0,\, y]$ contains at least one equilibrium point of system (3).

Proof: First, we show that for $b \in \mathbf{R}_+^\mathbf{n}$ there exists $y_b \in R_{\overline{x}}$ such that

$$f(y_b) + b \ll 0. \tag{4}$$

Indeed, for all $y \in R_{\overline{x}}$ we have that $f(y) = (\gamma_{\overline{x}}\lambda^\tau)\mathrm{diag}(r)y$ where λ is such that $y = \delta_\lambda^r(\overline{x})$. Since $\gamma_{\overline{x}} < 0$ by **H**, $\tau \geq 0$ by **H1** and both r and $y \in \mathrm{int}(\mathbf{R}_+^\mathbf{n})$, it is clear that when $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$, there exists $y_b \in R_{\overline{x}}$ (sufficiently far away from the origin) satisfying (4).

Next, we show that the set $[0,\, y_b]$ is forward invariant for system (3).

For all $z \in [0,\, y_b]$ with $z_i = (y_b)_i$ for some i, we have that

$$(f(y_b) + b) - (f(z) + b) = \left(\int_0^1 \frac{\partial f}{\partial x}(ty_b + (1-t)z)\, dt\right)(y_b - z) \tag{5}$$

which, by **H2** and (4), implies that $f_i(z) + b_i \leq f_i(y_b) + b_i < 0$. Together with Proposition 1 this implies that $[0,\, y_b]$ is forward invariant for (3).

Notice that for all $y \in R_{\overline{x}}$ with $y \geq y_b$ also holds that $f(y) + b \ll 0$ since $f(y) \ll f(y_b) \ll 0$. This implies that $[0,\, y]$ is forward invariant for system (3).

Since every set $[0,\, y]$ is homeomorphic to the standard unit disk, the last assertion of the Theorem follows from an application of Brouwer's Fixed Point Theorem. ■

It can be shown that equilibrium points of system (3) never belong to $\mathrm{bd}(\mathbf{R}_+^\mathbf{n})$.

Proposition 3: If **H** holds and if $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$, then every equilibrium point of system (3) in $\mathbf{R}_+^\mathbf{n}$, belongs to $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$.

Proof: Notice that by Proposition 2 the existence of at least one equilibrium point of system (3) in $\mathbf{R}_+^\mathbf{n}$ is guaranteed. Consider an arbitrary equilibrium point $z \in \mathbf{R}_+^\mathbf{n}$ and assume that $z \in \mathrm{bd}(\mathbf{R}_+^\mathbf{n})$. Since $b \neq 0$ and $f(0) = 0$ it is impossible that $z = 0$ and therefore

$z \in \text{bd}(\mathbf{R}_+^\mathbf{n}) \setminus \{0\}$. Then, it follows from **H3** and since $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$ that $f(z) + b$ is not equal to zero, contradicting that z is an equilibrium point of system (3). ■

Next we investigate the behavior of system (3) in the vicinity of an arbitrary equilibrium point.

Proposition 4: If **H** holds and if $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$, then the Jacobian matrix of $f(x) + b$, evaluated at an equilibrium point of system (3) in $\mathbf{R}_+^\mathbf{n}$, is a Hurwitz matrix.

Proof: Pick any equilibrium point $z \in \mathbf{R}_+^\mathbf{n}$. From Proposition (3) we obtain that $z \in \text{int}(\mathbf{R}_+^\mathbf{n})$.

Assume that $(\partial f/\partial x)(z)$ is not a Hurwitz matrix.

Since $(\partial f/\partial x)(z)$ is an irreducible Metzler matrix it follows from the Perron–Frobenius Theorem that there exists a *real* dominating eigenvalue λ_0 (dominating, in the sense that the real part of any other eigenvalue of the matrix is strictly less than λ_0) which is simple. Associated to λ_0 is a (left) eigenvector $p^T \in \text{int}(\mathbf{R}_+^\mathbf{n})$. The assumption that $(\partial f/\partial x)(z)$ is not a Hurwitz matrix implies that $\lambda_0 \geq 0$. Thus, we have that

$$p^T \frac{\partial f}{\partial x}(z) = \lambda_0 p^T. \tag{6}$$

On the other hand Euler's formula gives:

$$\frac{\partial f}{\partial x}(z)\text{diag}(r)z = \text{diag}(r + \tau^*)f(z) \tag{7}$$

where $\tau^* := (\tau, \tau, \ldots, \tau)$.

Multiplying (7) with p^T on the left and invoking (6) we find that

$$\lambda_0 p^T \text{diag}(r)z = p^T \text{diag}(r + \tau^*)f(z). \tag{8}$$

We know that $f(z) = -b \leq 0$ and that there exists at least one j such that $f_j(z) < 0$ since $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$. This implies that the right hand side of (8) is strictly negative, while the left hand side is nonnegative. We have therefore reached a contradiction and conclude that $(\partial f/\partial x)(z)$ is a Hurwitz matrix. ■

Now, we are ready to state the main result of this section.

Theorem 3: If **H** holds and if $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$, then there exists a unique equilibrium point z^* in $\mathbf{R}_+^\mathbf{n}$ for system (3). This equilibrium point belongs to $\text{int}(\mathbf{R}_+^\mathbf{n})$ and is GAS for system (3) with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$.

Proof: Let us first prove that there exists a unique equilibrium point for system (3).

Invoking Proposition 2, we shall establish that for all $y \in R_{\overline{x}}$ with $y \geq y_b$ the sets $[0, y]$ contain *exactly one* equilibrium point of system (3). This implies uniqueness of the equilibrium point of system (3) in $\mathbf{R}_+^\mathbf{n}$ [which, as we know from Proposition 3, must belong to $\text{int}(\mathbf{R}_+^\mathbf{n})$].

To prove that all the mentioned sets $[0, y]$ contain exactly one equilibrium point we introduce the concept of degree of $f(x) + b$ relative to $(0, y)$

$$\deg(f(x) + b, (0, y)) = \sum_{f(z)+b=0} \text{sign det}\left(\frac{\partial f}{\partial x}(z)\right) \tag{9}$$

where sign denotes the sign of a real number (0, $+1$ or -1) and $\det((\partial f/\partial x)(z))$ stands for the determinant of the Jacobian matrix $((\partial f/\partial x)(z))$.

To define the concept of degree, $f(x) + b$ and $(0, y)$ should satisfy the following conditions:

1) $f(x) + b$ is C^1 on the open set $(0, y)$ and C^0 on the closure of $(0, y)$ (This holds because of **H2**);
2) $f(x) + b$ is has no zeros on the boundary of $(0, y)$ (This is the case as can be seen from Proposition 3);
3) The Jacobian matrices of all zeros of $f(x) + b$ in $(0, y)$ are nonsingular (this is the case as can be seen from Proposition 4).

The degree has the property that it is a homotopy invariant. We will show that $f(x) + b$ is homotopic to the vector field $g(x) = -(x - z)$ where z is any equilibrium point of system (3) in the set $[0, y]$. Define the function $h(x, t) = -t(x - z) + (1 - t)(f(x) + b)$. Then it is clear that

$$h(x, 0) = f(x) + b$$
$$h(x, 1) = -(x - z)$$

and that $h(x, t)$ is C^0 in $[0, y] \times [0, 1]$. Also it can be checked that $h(x, t)$ does not vanish on the boundary of $[0, y] \times [0, 1]$, implying that $h(x, t)$ is a homotopy as claimed.

This leads to

$$\deg(f(x) + b, (0, y)) = \deg(-(x - z), (0, y)) = (-1)^n. \tag{10}$$

Proposition 4 implies that $\text{sign det}((\partial f/\partial x)(z)) = (-1)^n$ when z is an equilibrium point of (3) in $[0, y]$. Then it follows from (9) and (10) that there exists exactly one equilibrium point of (3) in $[0, y]$.

So far we have shown that system (3) contains a unique equilibrium point in $\mathbf{R}_+^\mathbf{n}$. Let us denote this equilibrium point as z^*. It remains to be shown that z^* is GAS for system (3) with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$. (Local asymptotic) stability of z^* is clear from Proposition 4, while convergence of all trajectories in $\mathbf{R}_+^\mathbf{n}$ to z^* follows from an application of Theorem 1 with $D = \mathbf{R}_+^\mathbf{n}$. Indeed, there holds that

1) the flow of system (3) is strongly monotone in $\mathbf{R}_+^\mathbf{n}$. This follows from Kamke's Theorem which can be found in e.g., [7];
2) all the solutions of system (3) starting in $\mathbf{R}_+^\mathbf{n}$ have compact forward orbit closure in $\mathbf{R}_+^\mathbf{n}$. This follows from the fact that all the compact sets $[0, y]$ with $y \in R_{\overline{x}}$ and $y \geq y_b$ are known to be forward invariant sets (see Proposition 2);
3) the equilibrium point z^* of system (3) is unique in $\mathbf{R}_+^\mathbf{n}$.

■

B. A Converse Result

Next, we will prove a converse result. Together with Theorem 3 we extend in this way a Theorem for controlled linear positive systems to a class of controlled nonlinear positive systems.

Theorem 4: Suppose that **H1**, **H2** and **H3** hold and that $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$. If system (3) possesses an equilibrium point $x^* \in \mathbf{R}_+^\mathbf{n}$, then the zero solution is a GAS equilibrium point of system (1) with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$.

Proof: Suppose that the zero solution of system (1) is not GAS with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$. It follows from the Main Theorem in [4] or from [5] that system (1) possesses an invariant ray in $\mathbf{R}_+^\mathbf{n}$ such that the vector field on this ray does not point toward the origin. In addition this invariant ray belongs to $\text{int}(\mathbf{R}_+^\mathbf{n})$ but is not necessarily unique. Denoting this invariant ray as $R_{\overline{x}}$, we obtain that the following holds

$$f(\overline{x}) = \gamma_{\overline{x}} \text{diag}(r)\overline{x} \text{ for some } \gamma_{\overline{x}} \geq 0. \tag{11}$$

By assumption system (3) possesses an equilibrium point $x^* \in \mathbf{R}_+^\mathbf{n}$. It is easily checked that $x^* \in \text{int}(\mathbf{R}_+^\mathbf{n})$ because **H1**, **H2** and **H3** hold and since $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$. Indeed, this follows immediately from the proof of Proposition 3.

Then a $y \in R_{\overline{x}}$ can be found such that

$$y \leq x^* \tag{12}$$

and such that at least one component of y and x^* are equal, i.e., there exists a $k \in N$ such that $y_k = x_k^*$.

Notice that it is not possible that $y \in R_{\overline{x}}$ or in other words that $y = x^*$. This is impossible since from (11) and because $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$ would imply that $f(x^*) + b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$. But then x^* would not be an equilibrium point of system (3). Hence there exists $l \in N$ such that $y_l < x_l^*$. Now we can define two nonempty sets K and L as follows

$$K := \{k \in N | y_k = x_k^*\} \quad \text{and} \quad L := \{l \in N | y_l < x_l^*\}. \qquad (13)$$

It follows from (12) that $K \cup L = N$.

Consider the following equalities:

$$(f(x^*) + b) - (f(y) + b) = 0 - (f(y) + b) = \left[\int_0^1 \frac{\partial f}{\partial x}|_{(tx^* + (1-t)y)}\, dt\right] (x^* - y). \qquad (14)$$

From **H3** follows that there exists $k^* \in K$ such that the k^*th component of the vector of the right hand side of (14) is a strictly positive number. On the other hand it follows from (11) and since $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$ that the k^*th component of the vector in the left hand side, $-(f_{k^*}(y) + b_{k^*})$, is not positive. Thus, we have reached a contradiction. ■

Theorem 3 and Theorem 4 together give rise to the following.

Corollary 1: Suppose that **H1**, **H2** and **H3** hold and that $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$. System (3) possesses an equilibrium point $x^* \in \mathbf{R}_+^\mathbf{n}$ if and only if the zero solution of system (1) is GAS with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$. If x^* exists, it is unique in $\mathbf{R}_+^\mathbf{n}$ and belongs to $\text{int}(\mathbf{R}_+^\mathbf{n})$. Moreover, x^* is a GAS equilibrium point of system (3) with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$.

Corollary 1 implies that the existence of an equilibrium point in $\mathbf{R}_+^\mathbf{n}$ for the controlled system is equivalent with a GAS zero solution of the uncontrolled system.

IV. Applications

A. Control of Dissipative Cyclic Chemical Reactions

In this section, we consider a particular class of chemical reactions taking place inside a chemical reactor. For more on modeling of chemical reactions we refer to [2]. The class of reactions under considerations is that of *cyclic* chemical reactions:

$$X_1 \to X_2 \to \cdots \to X_n \to X_1. \qquad (15)$$

If we assume that the reactions proceed according to the so-called *mass action principle* [2], we obtain that the concentrations x_i, $i \in N$, of the chemicals X_i obey the following differential equation:

$$\dot{x} = Cr(x) \qquad (16)$$

where

$$C = \begin{pmatrix} -1 & 0 & \dots & 1 \\ 1 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -1 \end{pmatrix} \quad \text{and} \quad r(x) = \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \\ \vdots \\ x_n^\alpha \end{pmatrix} \qquad (17)$$

for some $\alpha \geq 1$. We have assumed that the reaction rate constants of all reactions are equal to 1 (to simplify the notation and the calculations). It is clear that system (16) is homogeneous of order $\tau := \alpha - 1$, cooperative and irreducible. An important feature of this system is that the exponent of every x_i in $r(x)$ is the same. This means that the exponent that dictates how fast a reaction proceeds is the same for all reactions, a nontrivial assumption. Another property of this reaction scheme is that we are dealing with a *closed* chemical reactor: No chemicals are exchanged with the outside world.

We have proved in [4] and [5] that the zero solution of system (16) is stable, but not asymptotically stable.

Suppose that we introduce a dissipative term that models the extraction of one of the chemicals from the reactor. Without loss of generality we assume that the first chemical X_1 is extracted. Moreover we are looking for a dissipation term that results in a homogeneous, cooperative and irreducible system. The following choice satisfies these constraints:

$$\dot{x} = Cr(x) - d(x) \qquad (18)$$

where $d(x) = (x_1^\alpha \ 0 \ \cdots \ 0)^T$. The dissipative term is $-x_1^\alpha$. Notice that the exponent α in this term should be the same as the exponents appearing in $r(x)$. Our strategy implies that α is known. In practice however, it is often hard to determine α.

We show next that the zero solution of system (18) is GAS with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$. First, we will show that it follows from [4, Th. 3] or from [5] that system (18) possesses an invariant ray in $\text{int}(\mathbf{R}_+^\mathbf{n})$. It turns out that this ray is asymptotically stable and an application of Theorem 2 then concludes the proof.

We are looking for a $\overline{x} \in \text{int}(\mathbf{R}_+^\mathbf{n})$ such that

$$Cr(\overline{x}) - d(\overline{x}) = \gamma_{\overline{x}} \overline{x} \qquad (19)$$

for some $\gamma_{\overline{x}} \in \mathbf{R}$. We assume without loss of generality (by homogeneity) that $\overline{x}_1 = 1$. Adding all components of both vectors in (19) results in $-1 = \gamma_{\overline{x}}(1 + \overline{x}_2 + \cdots + \overline{x}_n)$. But since $\overline{x} \in \text{int}(\mathbf{R}_+^\mathbf{n})$ we obtain that $\gamma_{\overline{x}} < 0$. Therefore, the zero solution of system (18) is GAS with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$.

To conclude we would like to control system (18) with a constant control vector. We obtain the following system:

$$\dot{x} = Cr(x) - d(x) + b \qquad (20)$$

where $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$. Since hypothesis **H** has shown to be true for system (18), Theorem 3 can be invoked: For all $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$ system (20) possesses a unique equilibrium point in $\text{int}(\mathbf{R}_+^\mathbf{n})$ that is GAS with respect to initial conditions in $\mathbf{R}_+^\mathbf{n}$.

We point out that the chemical reactor associated to system (20) is *open*: Chemical X_1 is withdrawn from the reactor and all chemicals for which the corresponding component of the vector b is different from zero are fed to the reactor at a constant rate.

In conclusion, we have designed a very simply control methodology for the class of cyclic chemical reactors. Both a dissipative term and a constant control vector define this methodology. The restrictions in our example are that the exponent α that determines the speed of a reaction is the *same* for all reactions. and that this exponent has to be *known* to introduce an appropriate dissipative term.

B. Kolmogorov Systems

Next we consider a particular class of Kolmogorov systems. Kolmogorov systems are described by the following differential equation:

$$\dot{x} = \text{diag}(x) F(x) \qquad (21)$$

where $F(x)$ is C^1 on $\mathbf{R}^\mathbf{n}$. They are often encountered in mathematical biology, see, e.g., [3].

Notice that the well-known Volterra–Lotka systems are examples of Kolmogorov systems where $F(x)$ is an affine map. The (biological) interpretation for the map $F(x)$ in a Kolmogorov system is the following: a component $F_i(x)$ of the map $F(x)$ is the per-capita growth-rate of species i.

We shall restrict ourself to the study of the following particular class of Kolmogorov systems

$$\dot{x} = \mathrm{diag}(x)(f(x) + b) \tag{22}$$

where it will be assumed that **H** holds and that $b \in \mathrm{int}(\mathbf{R}_+^\mathbf{n})$ (notice the slightly stronger restriction on b compared to the one in the previous Section). Hypothesis **H** implies in particular that system (22) is cooperative in $\mathbf{R}_+^\mathbf{n}$ and irreducible in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$.

It can be established (for a proof we refer to [5]) that

Proposition 5: The sets $\mathbf{R}_+^\mathbf{n}$, $\mathrm{bd}(\mathbf{R}_+^\mathbf{n})$, $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ and all the faces of $\mathbf{R}_+^\mathbf{n}$ are invariant sets for system (21). In particular system (21) is a positive system.

System (21) is positive because property **P** holds. Invariance of the faces follows from the fact that if $x_i = 0$, then also $\dot{x}_i = x_i f_i = 0$. Notice that the dynamics of system (21) on the invariant faces are also of the Kolmogorov type.

It is clear that every equilibrium point of system (3) is also an equilibrium point of system (22). In addition, the equilibrium points of system (3) and (22) in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ are the same. However, notice that system (22) may have equilibrium points on $\mathrm{bd}(\mathbf{R}_+^\mathbf{n})$ which are not equilibrium points of system (3).

When we assume that **H** holds for system (3) and that $b \in \mathbf{R}_+^\mathbf{n}\backslash\{0\}$, it follows from Theorem 3 that system (3) possesses an equilibrium point z^* in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ which is unique in $\mathbf{R}_+^\mathbf{n}$. The previous discussion implies that z^* is also an equilibrium point of system (22) and that it is the unique equilibrium point in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ of system (22). From Proposition 5 we have that $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ and $\mathrm{bd}(\mathbf{R}_+^\mathbf{n})$ are forward invariant sets for system (22). This implies that z^* can *at best* be a GAS equilibrium point of system (22) *with respect to initial conditions in* $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$. We will show that this is indeed the case, but before doing so we need an auxiliary result. Its proof is omitted since it is similar to the proof of Proposition 2.

Proposition 6: If **H** holds and if $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$, then there exists $y_b \in R_{\overline{x}}$ such that for all $y \in R_{\overline{x}}$ with $y \geq y_b$, the set $[0,\, y]$ is forward invariant for system (22).

Theorem 5: If **H** holds and if $b \in \mathrm{int}(\mathbf{R}_+^\mathbf{n})$, then there exists an equilibrium point z^* of system (22) which is unique in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ and GAS with respect to initial conditions in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$.

Proof: Existence of the equilibrium point z^* and its uniqueness in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ has been shown already. We are left with proving that z^* is GAS for system (22) with respect to initial conditions in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$.

(Local asymptotic) Stability of z^* follows from the fact that the Jacobian of $\mathrm{diag}(x)(f(x) + b)$ evaluated at z^* equals $\mathrm{diag}(z^*)(\partial f/\partial x)(z^*)$. We know from Proposition 4 that $(\partial f/\partial x)(z^*)$ is a Hurwitz matrix, implying that $\mathrm{diag}(z^*)(\partial f/\partial x)(z^*)$ is also a Hurwitz matrix (This follows from the fact—which we do not prove here—that if M is a Hurwitz Metzler matrix then DM is also a Hurwitz Metzler matrix for all diagonal matrices D having strictly positive diagonal elements).

Convergence of all trajectories in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ to z^* follows from an application of Theorem 1 with $D = \mathrm{int}(\mathbf{R}_+^\mathbf{n})$. Recall that this Theorem was also applied in the proof of Theorem 3. Notice however the slight difference: previously D was equal to $\mathbf{R}_+^\mathbf{n}$, while now it equals $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$. This difference complicates the proof as will become clear in the second item below.

1) The flow of system (22) is strongly monotone in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$. This follows from Kamke's Theorem which can be found in, e.g., [7].
2) All the solutions of system (22) in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ have compact forward orbit closure in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$. [We stress that they must have a compact closure in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ and not in $\mathbf{R}_+^\mathbf{n}$. Here lies the difference with the proof of Theorem 3.]

   Indeed, the compact sets $[0,\, y]$ with $y \in R_{\overline{x}}$ and $y \geq y_b$ are known to be forward invariant for system (22) (see Proposition 6), implying that all the forward solutions of system (22) starting in $\mathbf{R}_+^\mathbf{n}$ are bounded. We still need to establish that the closure of every forward orbit is *compact in* $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$. For this purpose we use the same reasoning as in [8].

   For all $y \in \mathrm{int}(\mathbf{R}_+^\mathbf{n})$, sufficiently close to the origin, holds that $\mathrm{diag}(y)(f(y) + b) \gg 0$ by continuity of f and since $f(0) = 0$. This implies that for all these y, the sets $\{x \in \mathrm{int}(\mathbf{R}_+^\mathbf{n}) | \mathbf{x} \geq \mathbf{y}\}$ are forward invariant sets for system (22) as this system is cooperative on $\mathbf{R}_+^\mathbf{n}$. Then it follows that the closure of the forward orbit of solutions of system (22) starting in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$, belongs to $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$.
3) The equilibrium point z^* of system (22) is unique in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$.

V. Discussion of the Main Results

In this section, we discuss the main results of this note (Corollary 1 and Theorem 5) and compare them to some known results.

Let us first discuss Corollary 1. Consider the following affine system:

$$\dot{x} = Ax + b \tag{23}$$

where A is an irreducible Metzler matrix and $b \in \mathbf{R}_+^\mathbf{n} \setminus \{0\}$. It can be shown that system (23) is a positive system. A classical result as proved, for example, in [3] states that A is a Hurwitz matrix if and only if system (23) possesses a unique equilibrium point in $\mathbf{R}_+^\mathbf{n}$. If this equilibrium point exists then it belong to $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ and it is GAS. Corollary 1 can be interpreted as a generalization of this result to a particular class of nonlinear systems. For a related result we refer to [6].

To conclude we compare Theorem 5 with a result from [8]. In that paper a particular class of Kolmogorov systems is studied and the following Theorem is proved.

Theorem 6: Consider system (21) and suppose that the following holds:

C1 $F(x)$ is cooperative in $\mathbf{R}_+^\mathbf{n}$;
C2 $F(0) \gg 0$;
C3 $(\partial F/\partial x)(y) \geq (\partial F/\partial x)(z)$ (where the inequality is to be interpreted entry-wise) when $z \geq y \geq 0$.

If system (21) possesses an equilibrium point in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$, then this equilibrium point is unique in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$ and it is GAS with respect to initial conditions in $\mathrm{int}(\mathbf{R}_+^\mathbf{n})$.

This result and Theorem 5 are dealing with the same problem: determining the stability properties of an interior equilibrium point for particular Kolmogorov systems. Both results have in common that a cooperativity condition holds and the fact that the vector fields $f(x)+b$ and $F(x)$ point toward the interior of the first orthant. More importantly we point out the differences between both Theorems: In our result there is no concavity condition **C3** (and typically this condition is not satisfied for systems for which our result applies). On the other hand, our systems are subject to a homogeneity condition and an irreducibility condition, both absent in the result of [8].

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On the Consistency Between an LFT Described Model Set and Frequency Domain Data

Tong Zhou

***Abstract*—The main objective of this technical note is to derive a simple necessary and sufficient condition for a linear fractional transformation (LFT) perturbed model set being consistent with frequency domain plant input–output data. Only discrete-time models and unstructured modeling errors are dealt with. Compared with the available results in which the eigenvalues of a matrix are involved, this condition is related only to the Euclidean norms of two vectors. Moreover, these vectors linearly depend on measurement errors. Some of its applications to model set validation have been briefly discussed. Based on this condition, an almost analytic solution has been established for model set validation under a deterministic framework when the measurement errors are energy bounded. Numerical simulations show that this consistency condition can lead to a significant computation cost reduction.**

***Index Terms*—Convex optimization, linear fractional transformation, model set validation, robust control.**

I. INTRODUCTION

An LFT described model set (MS) is widely considered to be the most general one adopted in robust controller design. This MS can easily capture all the other representations, such as additive perturbed MS, coprime factor perturbed MS, etc., as its special case. To validate a MS through experimental data, the consistency condition plays an essential role in both deterministic and stochastic frameworks [2], [13], [14], [12], [6], [15], [16]. Compared with time domain experimental data (TDFD), the utilization of frequency domain experimental data (FDED) is more computationally attractive in MS validation. It is because that owing to the boundary tagential Nevanlinna–Pick interpolation theory, the consistency of a MS with *all* the FDED can be *independently* verified at each *individual* frequency point. This property significantly reduces the computational complexity in falsifying a MS [6], [15].

The problem of validating an LFT perturbed MS through FDED was originally attacked by Smith and Doyle under a quite general framework [14]. In their problem formulation, modeling errors are permitted to be structured and disturbances may affect a plant at arbitrary locations. In their research, the relations have been made clear between MS validation and structured singular values. Moreover, a necessary condition has been derived for the reproduction of FDED. Furthermore, some convexity properties have been established under a condition concerning the dimension of a subspace when the number of modeling error blocks is not greater than 2. And so on. The implementation of the developed validation test, however, seems not very easy. To establish a computable test, this problem has been reinvestigated by Chen in [6]. In his study, through an introduction of auxiliary signals, it is proved that an LFT described MS is consistent with FDED at a frequency point if and only if a matrix is negative semi-definite. As the involved matrix depends linearly on the measurement errors of the plant output, the model set validation problem (MSVP) is finally reduced to a convex optimization problem which is computationally tractable. All these results are derived from the assumptions that the modeling errors are unstructured and the measurements of the plant input are perfect. Based on that consistency condition, an algorithm is proposed for validating an LFT described MS with structured uncertainties. The convergence of the algorithm, however, remains unclear.

When the eigenvalues of a matrix are involved in convex optimization, however, eigenvalue decomposition (ED) or singular value decomposition (SVD) is usually necessary in order to obtain a subgradient of the cost function. And these decompositions are generally time consuming [3], [4]. Moreover, when discussing a MSVP under a stochastic framework, it seems very hard to derive from the results of [6] an elegant and physically meaningful expression for the unfalsified probability of a MS. This is due to that there are uncertain elements in the involved matrix and the eigenvalues of the matrix are related to the unfalsified probability [10], [15], [16].

To overcome these difficulties, in this technical note, a simpler condition is derived for the consistency of an LFT described MS with FDED. Only discrete time LFT models will be discussed. Under the condition that modeling errors are unstructured, it is shown that a MS with LFT uncertainties is consistent with FDED if and only if the square of the Euclidean norm of a vector is not greater than that of another vector. An appealing characteristic of this condition is that both of these vectors are linear functions of measurement errors. Based on this condition, a MSVP is investigated under a deterministic framework in which the measurement errors of the plant output are energy bounded. An analytic solution is established for the existence of a measurement error and a model error such that the experimental data can be reproduced, except that the zeros of a simply structured function must be numerically computed. It is also proved that this function is monotonically increasing in the interested domain. All the derivations are based on routine linear algebra.

Throughout this technical note, the following standard notation is adopted. $\|H(z)\|_\infty$ represents the $\mathcal{H}_\infty$-norm of $H(z)$, while $\|v\|_2$ the Euclidean norm of a vector v. $\overline{\sigma}(X)$ denotes the maximal singular value of matrix X. $\mathcal{R}^{m\times n}$ and $\mathcal{C}^{m\times n}$ stand respectively for the sets of real and complex $m\times n$ dimensional matrices. When $n=1$, $m\times n$ is always abbreviated to m. $\mathcal{R}e\{x\}$, $\mathcal{I}m\{x\}$ are, respectively, the real and imaginary parts of x. I_m and $0_{m\times n}$ represent respectively the $m\times m$ dimensional identity matrix and the $m\times n$ dimensional matrix with all the elements being zero. When the dimension is not very important, the subscript is often omitted. X^H and X^T are respectively the conjugate transpose and transpose of matrix X. Finally, $\mathrm{diag}\{\lambda_i|_{i=1}^n\}$ is the $n\times n$ diagonal matrix with its ith row ith column element equal to λ_i.

This technical note is organized as follows. The next section states the main results and briefly compares them with the available results when applying to MS validation, while Section III investigates an appli-

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The author is with the Department of Automation, Tsinghua University, Beijing 100084, China (e-mail: tzhou@mail.tsinghua.edu.cn).

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