Linear Algebra by Example

These notes provide a group of examples reviewing the basic mathematical operations of linear algebra. For more details, refer to the readings listed in the syllabus. Most sections are followed by a problem for you to attempt, with solutions given on the last page. A great reference for many of these ideas is **Boas** §3.3 and §3.6.

1 Matrix Addition

For matrix addition to be defined, both matrices must be of the same dimension, that is, both matrices must have the same number of rows and columns. Addition then proceeds by adding corresponding components, as in

$$C_{ij} = A_{ij} + Bij.$$

For example, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 18 \end{pmatrix}.$$

However,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{pmatrix}$$

is undefined.

2 Scalar Multiplication

A matrix can be multiplied by a scalar, in which case each element of the matrix is multiplied by the scalar. In components,

$$C_{ij} = \lambda A_{ij}$$

where λ is a scalar, that is, a complex number. For example, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$3A = 3 \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix}.$$

A) Try it for yourself by computing:

$$i \cdot \begin{pmatrix} 1 & i \\ -2i & 3 \end{pmatrix}$$

(Answers are on the last page.)

3 Matrix Multiplication

Matrices can also be multiplied together, but this operation is somewhat complicated. Watch the progression in the examples below; basically, the elements of the row of the first matrix are multiplied by the corresponding elements of the column of the second matrix. Matrix multiplication can be written in terms of components as

$$C_{ij} = \sum_k A_{ik} B_{kj}.$$

The simplest example is

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix} = ae + bg$$

which should remind you of the dot product. Somewhat more complicated examples are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} ae + bg \\ ce + dg \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \end{pmatrix}.$$

A more general example, combining these ideas, is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

and a numerical example is

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}.$$

Note however that

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix}$$

is undefined. For matrix multiplication to be defined, the number of columns of the matrix on the left must equal the number of rows of the matrix on the right.

B) Try it for yourself by computing

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(Answers are on the last page.)

4 Transpose

The transpose of a matrix is obtained by interchanging rows and columns. In terms of components,

$$(A_{ij})^T = A_{ji}.$$

For example,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

and

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Longrightarrow B^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}.$$

A square matrix is called **symmetric** if it is equal to its transpose, that is, if $A = A^T$. Non-square matrices also have transposes, for example

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longrightarrow v^T = \begin{pmatrix} x & y & z \end{pmatrix}.$$

5 Hermitian Adjoint

The Hermitian adjoint of a matrix is the same as its transpose except that along with switching row and column elements you also complex conjugate all the elements. If all the elements of a matrix are real, its Hermitian adjoint and transpose are the same. In terms of components,

$$(A_{ij})^\dagger = A^*_{ji}.$$

For example, if

$$A = \begin{pmatrix} 1 \\ i \\ -2i \end{pmatrix}$$

then

$$A^\dagger = \begin{pmatrix} 1 & -i & 2i \end{pmatrix}.$$

A matrix is called **Hermitian** if it is equal to its adjoint, $A = A^\dagger$.

C) Try it for yourself. What is $B^\dagger$ if

$$B = \begin{pmatrix} 1 & i \\ -5i & i \end{pmatrix}$$

(Answers are on the last page.)

6 Trace

The trace of a matrix is just the sum of all of its diagonal elements. In terms of components,

$$\mathrm{tr}(A) = \sum_i A_{ii}.$$

For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

then

$$\mathrm{tr}(A) = 1 + 5 + 9 = 15.$$

D) Try it for yourself by computing

$$\mathrm{tr}\begin{pmatrix} 1 & 34 & 5 \\ 23 & 5 & 98 \\ 132 & 7 & 9 \end{pmatrix}.$$

(Answers are on the last page.)

7 Determinants

The determinant of a matrix is somewhat complicated in general, so you may want to check one of the reference books. The 2×2 and 3×3 cases can be memorized using the examples below.

The determinant of a 2×2 matrix is given by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

The determinant of a 3×3 matrix is computed as follows:

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &= a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a \cdot (ei - hf) - b \cdot (di - gf) + c \cdot (dh - ge) \\ &= aei - ahf - bdi + bgf + cdh - cge. \end{aligned}$$

The smaller 2×2 determinants are called the **cofactors** of the elements a, b, and c, respectively. The minus sign in front of b is part of the cofactor. Cofactors are formed by keeping only what is left after eliminating everything from the row and column where the element desired resides. So, for a, the row elements, b and c, and the column elements, d and g, are eliminated, leaving the 2×2 matrix shown above.

Computing 4×4 matrices is a straightforward extension of the above procedure, but it is easier to just go to a computer!!!

E) Try it for yourself by determining

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

(Answers are on the last page.)

8 Inverses

The matrix inverse of a matrix A, denoted A^{-1}, is the matrix such that when multiplied by the matrix A the result is the **identity matrix**. (The identity matrix is the matrix with ones down the diagonal and zeroes everywhere else.)

For 2×2 matrices, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For 3×3 matrices B, B^{-1} is the transpose of the matrix made of all cofactors of B, divided by the determinant of B. This is easier said in symbols, so if

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

then

$$B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

In both cases, the inverse only exists if the determinant is nonzero.

F) Try it for yourself for the matrix

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

What is B^{-1}? Verify that BB^{-1} is the identity matrix.

(Answers are on the last page.)

9 Bra-Ket Notation

Column matrices play a special role in physics, where they are interpreted as **vectors** or **states**. To remind us of this uniqueness they have their own special notation; introduced by Dirac, called "bra-ket" notation. In bra-ket notation, a column matrix can be written

$$|v\rangle := \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The adjoint of this vector is denoted

$$\langle v| := (|1\rangle)^\dagger = \begin{pmatrix} a^* & b^* & c^* \end{pmatrix}.$$

As one quick application, if we take $|v\rangle$ to be a 3-vector with components a, b, and c as above, then the magnitude of the vector is

$$\langle v|v\rangle = \begin{pmatrix} a^* & b^* & c^* \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = |a|^2 + |b|^2 + |c|^2$$

which is exactly the result expected. This property is one example of the overlap between linear algebra and vector analysis.

10 Solutions

A)

$$i \cdot \begin{pmatrix} 1 & i \\ -2i & 3 \end{pmatrix} = \begin{pmatrix} i & -1 \\ 2 & 3i \end{pmatrix}$$

B)

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-i \\ -i-1 \end{pmatrix}$$

C)

$$B = \begin{pmatrix} 1 & i \\ -5i & i \end{pmatrix} \Longrightarrow B^{\dagger} = \begin{pmatrix} 1 & 5i \\ -i & -i \end{pmatrix}$$

D)

$$\mathrm{tr} \begin{pmatrix} 1 & 34 & 5 \\ 23 & 5 & 98 \\ 132 & 7 & 9 \end{pmatrix} = 1 + 5 + 9 = 15$$

E)

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(45-48) - 2(36-42) + 3(32-35) \\ &= -3 + 12 - 9 = 0 \end{aligned}$$

F)

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Longrightarrow B^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$BB^{-1} = -\frac{1}{2} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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