Laplacian in Orthogonal Curvilinear Coordinates

The metric in orthogonal curvilinear coordinates is:

$$\begin{aligned} ds^2 &= h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2 \\ &= dr^2 + r^2 d\theta^2 + dz^2 && \text{cylindrical} \\ &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 && \text{spherical} \end{aligned}$$

The Laplacian is:

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial x_3}\right)\right]$$

Example: Cylindrical Coordinates

$$\begin{aligned} x &= r\cos\phi & dx &= dr\cos\phi - r\sin\theta\sin\phi\, d\phi \\ y &= r\sin\phi & dy &= dr\sin\phi - r\cos\phi\, d\phi \\ z &= z & dz &= dz \end{aligned}$$

Then:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= (dr\cos\phi - r\sin\phi\, d\phi)^2 + (dr\sin\phi - r\cos\phi\, d\phi)^2 + (dz)^2 \\ &= dr^2 + r^2 d\phi^2 + dz^2 \end{aligned}$$

Therefore $h_1 = 1$, $h_2 = r$, $h_3 = 1$.

$$\begin{aligned} \nabla^2 &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)\right] \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

Notice that the transformation of coordinates from rectangular to spherical does not change the signs of the eigenvalues.

$$A_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In Maple: laplacian($\psi(r,\phi,z)$, $[r,\phi,z]$, coords=cylindrical);