Central Forces Homework 4

Due 5/19/17, 4 pm

For every problem, before you start the problem, make a brief statement of the form that a correct solution should have, clearly indicating what quantities you need to solve for. This statement will be graded.

PRACTICE:

1. (McIntyre 7.20)

   In each of the following sums, shift the index $n \to n + 2$. Don't forget to shift the limits of the sum as well. Then write out all of the terms in the sum (if the sum has a finite number of terms) or the first five terms in the sum (if the sum has an infinite number of terms) and convince yourself that the two different expressions for each sum are the same:

   (a)
   $$\sum_{n=0}^{3} n$$

   (b)
   $$\sum_{n=1}^{5} e^{in\phi}$$

   (c)
   $$\sum_{n=0}^{\infty} a_n n(n-1) z^{n-2}$$

REQUIRED:

2. **Legendre Polynomials**

   (a) Use *Mathematica* or *Maple* to find the first 5 Legendre polynomials.

   (b) Use Rodrigues' formula to calculate the first 5 Legendre polynomials. (You are encouraged to use *Mathematica* or *Maple* to help with the derivatives.

   (c) Look up two recurrence relations for Legendre polynomials and use them to find $P_3(z)$ and $P_3'(z)$ assuming that all you know is that $P_0(z) = 1$ and $P_1(z) = 2z$. Do this part of the problem by hand.

3. (McIntyre 7.28)

   Use your favorite tool (*e.g.* Maple, Mathematica, Matlab, pencil) to generate the Legendre polynomial expansion to the function $f(z) = \sin(\pi z)$. How many terms do you need to include in a partial sum to get a "good" approximation to $f(z)$ for $-1 < z < 1$? What do you mean by a "good" approximation? How about the interval

$-2 < z < 2$? How good is your approximation? Discuss your answers. Answer the same set of questions for the function $g(z) = \sin(3\pi z)$

The following problems were postponed from the last homework set. They are now due with this homework set.

4. NASA has launched a satellite into a **circular** orbit around the earth and wants to increase the radius slightly while maintaining a circular orbit. NASA scientists propose to fire the engines briefly, applying a small impulse to the satellite.

   One scientist says that it doesn't matter if the impulse is applied in a direction tangential to the satellite motion or perpendicular to the motion, arguing that both approaches will simply fine tune the total energy of the system.

   A second scientist disagrees and argues that one of the options would work but the other would definitely not work.

   A third scientist says that neither option would work. Which scientist would you side with, and why?

5. Consider the frictionless motion of a hockey puck of mass m on a perfectly circular bowl-shaped ice rink with radius a. The central region of the bowl ($r < 0.8a$) is perfectly flat and the sides of the ice bowl smoothly rise to a height h at $r = a$.

   (a) Draw a sketch of the potential energy for this system. Set the zero of potential energy at the top of the sides of the bowl.

   (b) Situation 1: the puck is initially moving radially outward from the exact center of the rink. What minimum velocity does the puck need to escape the rink?

   (c) Situation 2: a stationary puck, at a distance $\frac{a}{2}$ from the center of the rink, is hit in such a way that it's initial velocity $\vec{v}_0$ is perpendicular to its position vector as measured from the center of the rink. What is the total energy of the puck immediately after it is struck?

   (d) In situation 2, what is the angular momentum of the puck immediately after it is struck?

   (e) Draw a sketch of the effective potential for situation 2.

   (f) In situation 2, for what minimum value of $\vec{v}_0$ does the puck just escape the rink?

6. In a solid, a free electron doesn't "see" a bare nuclear charge since the nucleus is surrounded by a cloud of other electrons. The nucleus will look like the Coulomb potential close-up, but be "screened" from far away. A common model for such problems is described by the Yukawa or screened potential:

$$U(r) = -\frac{k}{r}e^{-\frac{r}{\alpha}}$$

(a) Graph the potential, with and without the exponential term. Describe how the Yukawa potential approximates the "real" situation. In particular, describe the role of the parameter α.

(b) Draw the effective potential for the two choices $\alpha = 10$ and $\alpha = 0.1$ with $k = 1$ and $\ell = 1$. For which value(s) of α is there the possibility of stable circular orbits?