16 Change of Variables

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P}{\partial\theta}\right) - A\sin^2\theta P - m^2 P = 0 \tag{1}$$

Change variables to $z = \cos\theta$.

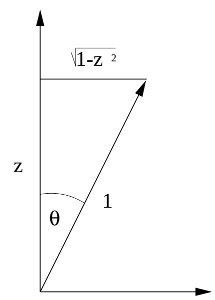

Figure 1: Relationship between z and θ.

$$\sqrt{1-z^2} = \sin\theta \tag{2}$$

$$\frac{\partial}{\partial\theta} = \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z} = -\sin\theta\frac{\partial}{\partial z} = -\sqrt{1-z^2}\frac{\partial}{\partial z} \tag{3}$$

$$\sin\theta\frac{\partial}{\partial\theta} = -\left(1-z^2\right)\frac{\partial}{\partial z} \tag{4}$$

The second derivative involves a product rule:

$$\begin{aligned}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) &= \left(1-z^2\right)\frac{\partial}{\partial z}\left(\left(1-z^2\right)\frac{\partial}{\partial z}\right) \\ &= \left(1-z^2\right)^2\frac{\partial^2}{\partial z^2} - 2z\left(1-z^2\right)\frac{\partial}{\partial z}\end{aligned} \tag{5}$$

Divide by $(1-z^2)^2$:

$$\frac{\partial^2 P}{\partial z^2} - \frac{2z}{1-z^2}\frac{\partial P}{\partial z} - \frac{A}{1-z^2}P - \frac{m^2}{(1-z^2)^2}P = 0 \tag{6}$$

17 SERIES SOLUTIONS OF ODE'S

Assume $m = 0$, the simplest case. Multiply by $1 - z^2$ to clear the denominators.

$$(1 - z^2)\frac{\partial^2 P}{\partial z^2} - 2z\,\frac{\partial P}{\partial z} - \frac{A}{P} = 0 \tag{7}$$

Assume:

$$\begin{aligned} P(z) &= \sum_{n=0}^{\infty} a_n\, z^n && (8)\\ \frac{dP}{dz} &= \sum_{n=0}^{\infty} a_n\, n\, z^{n-1} && (9)\\ \frac{d^2P}{dz^2} &= \sum_{n=0}^{\infty} a_n\, n(n-1)\, z^{n-2} && (10) \end{aligned}$$

$$0 = \sum_{n=2}^{\infty} a_n\, n(n-1)\, z^{n-2} - z^2 \sum_{n=0}^{\infty} a_n\, n(n-1)\, z^{n-2} - 2z \sum_{n=0}^{\infty} a_n\, n\, z^{n-1} - A \sum_{n=0}^{\infty} a_n\, z^n \tag{11}$$

Let $n \to n + 2$:

$$\begin{aligned} &\sum_{n=0}^{\infty} a_{n+2}\,(n+2)(n+1)\, z^n - \sum_{n=0}^{\infty} a_n\, n(n-1)\, z^n - 2\sum_{n=0}^{\infty} a_n\, n\, z^n - A\sum_{n=0}^{\infty} a_n\, z^n && (12)\\ &= \sum_{n=0}^{\infty} \left[a_{n+2}\,(n+2)(n+1) - a_n\, n(n-1) - 2\,a_n\, n - A\,a_n\right] z^n = 0 && (13) \end{aligned}$$

Now comes the MAGIC part:

$$a_{n+2}\,(n+2)(n+1) - a_n\, n(n-1) - 2\,a_n\, n - A\,a_n = 0 \tag{14}$$

Solve for (the recurrence relation) a_{n+2} in terms of a_n

$$a_{n+2} = \frac{n(n+1) + A}{(n+2)(n+1)}\, a_n \tag{15}$$

Choose $A = -\ell(\ell + 1)$ for the series to terminate.