Central Forces

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Abstract

The two body problem is treated classically. The reduced mass is used to reduce the two body problem to an equivalent one body problem. Conservation of angular momentum is derived and exploited to simplify the problem. Spherical coordinates are chosen to respect this symmetry. The equations of motion are obtained in two different ways: using Newton's second law, and using energy conservation. Kepler's laws are derived. The concept of an effective potential is introduced. The equations of motion are solved for the orbits in the case that the force obeys an inverse square law. The equations of motion are also solved, up to quadrature (i.e. in terms of definite integrals) and numerical integration is used to explore the solutions.

2 INTRODUCTION

In the Central Forces paradigm, we will examine a mathematically tractable and physically useful problem - that of two bodies interacting with each other through a force that has two characteristics: (a) it depends only on the separation between the two bodies, and (b) it points along the line connecting the two bodies. Such a force is called a central force. Perhaps the most common examples of this type of force are those that follow the $\frac{1}{r^2}$ behavior, specifically the Newtonian gravitational force between two point masses or spherically symmetric bodies and the Coulomb force between two point or spherically symmetric electric charges. Clearly both of these examples are idealizations - neither ideal point masses or charges nor perfectly spherically symmetric mass or charge distributions exist in nature, except perhaps for elementary particles such as electrons. However, deviations from ideal behavior are often small and can be neglected to within a reasonable approximation.

These notes discuss two solutions to the central force problem—classical behavior exemplified by the gravitational interaction and quantum behavior exemplified by the Coulomb interaction. In this way, we will be able to explore the strong similarities and the important differences between classical and quantum physics. Notice the difference in length scale: the archetypal gravitational example is planetary motion—at astronomical length scales, the archetypal Coulomb example is the hydrogen atom—at atomic length scales. We will also consider forces that depend on r in other ways and the kinds of motion they produce.

One of the unifying themes of this topic is the importance of *angular momentum*. You should have covered angular momentum in your introductory physics course. Before starting these notes, you might find it helpful to review the definition of angular momentum, how it enters into dynamical equations (Newton's laws and kinetic energy, for example), and the law of conservation of angular momentum.

You should read these notes in conjunction with the assigned readings in your textbooks. You should note that the development of the classical central force problem in other textbooks may use a formulation based on Lagrangians, which you will not cover until the Classical Mechanics Capstone. We will use a different approach in these notes. You are not responsible for learning the Lagrangian formalism for this course, but your reading in other books will be clearer if you know that the Lagrangian is defined simply as the difference between kinetic energy and potential energy: $\mathcal{L} = T - U$. And be sure you don't confuse the various symbols. Some books use L to represent the Lagrangian instead of $\mathcal{L}$, $\boldsymbol{L}$ to represent the angular momentum vector, and l to represent the magnitude of the angular momentum. We will also use L_u $(u = x, y, z)$ to represent the components of the angular momentum vector. Some authors use K to represent kinetic energy or V to represent potential energy.

We will obtain the equations of motion in two equivalent ways, 1) using Newton's second law and 2) using energy conservation. The second approach is slightly more sophisticated in that it exploits more of the symmetries from the beginning.

3 Systems of Particles

Consider a system of n different masses m_i, interacting with each other and being acted on by external forces. We can write Newton's second law for the positions $\boldsymbol{r}_i$ of each of these masses with respect to a fixed origin $\boldsymbol{O}$, thereby obtaining a system of equations governing the motion of the masses.

$$\begin{aligned}
m_1 \frac{d^2\boldsymbol{r}_1}{dt^2} &= \boldsymbol{F}_1 + \ 0 \ + \boldsymbol{f}_{12} + \boldsymbol{f}_{13} + \ \dots \ + \boldsymbol{f}_{1n} \\
m_2 \frac{d^2\boldsymbol{r}_2}{dt^2} &= \boldsymbol{F}_2 + \boldsymbol{f}_{21} + \ 0 \ + \boldsymbol{f}_{23} + \ \dots \ + \boldsymbol{f}_{2n} \\
&\vdots \\
m_n \frac{d^2\boldsymbol{r}_n}{dt^2} &= \boldsymbol{F}_n + \boldsymbol{f}_{n1} + \boldsymbol{f}_{n2} + \dots + \boldsymbol{f}_{n(n-1)} + 0
\end{aligned} \tag{1}$$

Here, we have chosen the notation $\boldsymbol{F}_i$ for the net external forces acting on mass m_i and $\boldsymbol{f}_{ij}$ for the internal force of mass m_j acting on m_i.

In general, each internal force $\boldsymbol{f}_{ij}$ will depend on the positions of the particles $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ in some complicated way, making (1) a set of **coupled** differential equations. To solve (1), we first need to **decouple** the differential equations, i.e. find an equivalent set of differential equations in which each equation contains only one variable.

The weak form of Newton's third law states that the force $\boldsymbol{f}_{12}$ of m_2 on m_1 is equal and opposite to the force $\boldsymbol{f}_{21}$ of m_1 on m_2. We see that each internal force appears twice in the system of equations (1), once with a positive sign and once with a negative sign. Therefore, if we add all of the equations in (1) together, the internal forces will all cancel, leaving:

$$\sum_{i=1}^{n} m_i \frac{d^2\boldsymbol{r}_i}{dt^2} = \sum_{i=1}^{n} \boldsymbol{F}_i \tag{2}$$

Notice what a surprising equation (2) is. The right-hand side directs us to add up all of the external forces, each of which acts on a different mass; something you were taught never to do in introductory physics.

The left-hand side of (2) directs us to add up (the second derivatives of) n "weighted" position vectors pointing from the origin to different masses. We can simplify the left-hand side of (2) if we multiply and divide by the total mass $M = m_1 + m_2 + \ldots + m_n$ and use the linearity of differentiation to "factor out" the derivative operator:

$$\begin{aligned}
\sum_{i=1}^{n} m_i \frac{d^2\boldsymbol{r}_i}{dt^2} &= M \sum_{i=1}^{n} \frac{m_i}{M} \frac{d^2\boldsymbol{r}_i}{dt^2} && (3)\\
&= M \frac{d^2}{dt^2}\left(\sum_{i=1}^{n} \frac{m_i}{M}\, \boldsymbol{r}_i\right) && (4)\\
&= M \frac{d^2\boldsymbol{R}}{dt^2} && (5)
\end{aligned}$$

We recognize (or define) the quantity in the parentheses on the right-hand side of (4) as the position vector $\boldsymbol{R}$ from the origin to the "center of mass" of the system of particles.

$$\boldsymbol{R} = \sum_{i=1}^{n} \frac{m_i}{M}\, \boldsymbol{r}_i \tag{6}$$

With these simplifications, equation (2) becomes:

$$M \frac{d^2\boldsymbol{R}}{dt^2} = \sum_{i=1}^{n} \boldsymbol{F}_i \tag{7}$$

which has the form of Newton's 2nd Law for a fictitious particle with mass M sitting at the center of mass of the system of particles and acted on by all of the external forces from the original system.

We can define the momentum of the center of mass as the total mass times the time derivative of the position of the center of mass:

$$\boldsymbol{P} = M\frac{d\boldsymbol{R}}{dt} \tag{8}$$

If there are no external forces acting, then the acceleration of the center of mass is zero and the momentum of the center of mass is constant in time (conserved).

$$M\frac{d^2\boldsymbol{R}}{dt^2} = \frac{d\boldsymbol{P}}{dt} = 0 \tag{9}$$

Notice that the entire discussion above applies even if all of the internal forces are zero $\boldsymbol{f}_{ij} = 0$, i.e. none of the particles have any way of knowing that the others are even present. Such particles are called non-interacting. The position of the center of mass of the system will still move according to equation (7).

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1 Problems

1. (TM 9.6) Consider two particles of equal mass m. The forces on the particles are $\boldsymbol{F}_1 = 0$ and $\boldsymbol{F}_2 = F_0\hat{\imath}$. If the particles are initially at rest at the origin, find the position, velocity, and acceleration of the center of mass as functions of time. Solve this problem in two ways, with or without theorems about the center of mass motion and write a short description comparing the two solutions.)

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4 REDUCED MASS

So far, we have found one decoupled equation to replace (2.1). What about the other $n-1$ equations? It turns out that, in general, there is no way to decouple and solve the other equations. Physicists often say, "The n-body problem can not be solved in general." Whenever you are stuck trying to solve a general problem, it often pays to start with simpler examples to build up your intuition. We will make several assumptions to simplify this problem and keep track of them in a list.

1. Assume that there are no external forces acting.

2. Assume that there are only two masses.

The system of equations (2.1) reduces to:

$$\begin{aligned} m_1 \frac{d^2 \boldsymbol{r}_1}{dt^2} &= -\boldsymbol{f}_{21} \\ m_2 \frac{d^2 \boldsymbol{r}_2}{dt^2} &= \boldsymbol{f}_{21} \end{aligned} \tag{10}$$

Because we added the two equations of motion to find the equation of motion for the center-of-mass, we are led now to consider subtracting the equations so as to get $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$. Figure 1 shows the basic geometry of our problem. $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the position vectors of the two masses measured with respect to an arbitrary coordinate origin **O**. We call the displacement

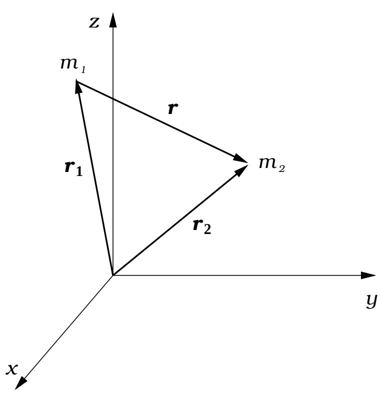

Figure 1: The position vectors for m_1 and m_2 and the displacement vector between them.

between the two masses $\boldsymbol{r}$. The magnitude of this displacement is r and the direction is $\hat{\boldsymbol{r}}$. These quantities can be found from $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ by:

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \tag{11}$$

$$r = |\boldsymbol{r}| = |\boldsymbol{r}_2 - \boldsymbol{r}_1| \tag{12}$$

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} \tag{13}$$

We see that before we subtract, we should multiply the first equation in (11) by m_2 and the second equation by m_1 so that the factors in front of the second derivative are the same. Subtracting the first equation from the second and regrouping, we obtain:

$$m_1 m_2 \frac{d^2}{dt^2}(\boldsymbol{r}_2 - \boldsymbol{r}_1) = m_1 m_2 \frac{d^2}{dt^2}(\boldsymbol{r}) = (m_1 + m_2)\,\boldsymbol{f}_{21} \tag{14}$$

or rearranging:

$$\frac{m_1 m_2}{m_1 + m_2}\frac{d^2\boldsymbol{r}}{dt^2} = \mu \frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{f}_{21} \tag{15}$$

The combination of masses

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{16}$$

is called the reduced mass. This equation is in the same form as Newton's law for a single fictitious mass μ, with position vector $\boldsymbol{r}$, moving subject to the force $\boldsymbol{f}_{21}$. For the rest of these notes, we will talk about "the mass", meaning this fictitious particle. Note that to solve the original two mass problem we started with, we will need to transform the solutions for $\boldsymbol{r}$ back to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. See Problem 1.1.

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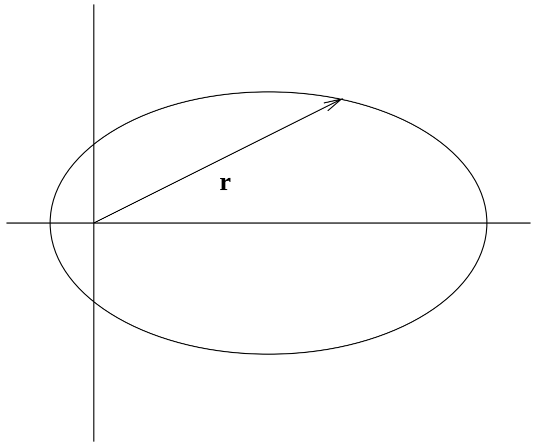
1 Problems

1. The figure below shows the orbit of a "fictitious" reduced mass, μ, traveling around the center-of-mass at the origin. The position vector $\mathbf{r}$ locates the particle at a particular instant t. Assume that $m_2 = m_1$ and draw on the figure the position vectors for m_1 and m_2 corresponding to $\mathbf{r}$. Also sketch the orbits for m_1 and m_2. Give an example of a physical situation that might produce this type of motion. (NOTE: Do this problem "by hand." Do not use MAPLE or a graphing calculator.)



Repeat this problem for $m_2 > m_1$ and $m_2 >> m_1$.

2. Find $\boldsymbol{r}_{\text{sun}} - \boldsymbol{r}_{\text{cm}}$ and μ for the Sun–Earth system. Compare $\boldsymbol{r}_{\text{sun}} - \boldsymbol{r}_{\text{cm}}$ to the radius of the Sun and to the distance from the Sun to the Earth. Repeat the calculation for the Sun–Jupiter system.

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5 CENTRAL FORCES

Our ultimate goal is to solve the equations of motion for two masses m_1 and m_2 subject to a central force acting between them. When you considered this problem in introductory physics, you assumed that one of the masses was so large that it effectively remained at rest while all of the motion belonged to the other object. This assumption works fairly well for the Earth orbiting around the Sun or for a satellite orbiting around the Earth, but in general we are going to have to solve for the motion of *both* objects.

In the introduction, we defined a central force to satisfy two characteristics. We can now write turn these descriptions of the characteristics into equations:

(a) a central force depends only on the separation between the two bodies

$$\boldsymbol{f}_{21} = -\boldsymbol{f}_{12} = \boldsymbol{f}(\boldsymbol{r}_2 - \boldsymbol{r}_1) \tag{17}$$

(b) it points along the line connecting the two bodies

$$\boldsymbol{f}_{21} = -\boldsymbol{f}_{12} = \boldsymbol{f}(\boldsymbol{r}_2 - \boldsymbol{r}_1) = f(\boldsymbol{r})\,\hat{\boldsymbol{r}} \tag{18}$$

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1 Problems

1. If a central force is the only force acting on a system of two masses (i.e. no external forces), what will the motion of the center of mass be?

2. Which of the forces which we found in the Static Fields Paradigm (*i.e.* $\vec{g}$, $q\vec{E}$, $q\vec{v} \times \vec{B}$) can be central forces? which cannot?

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6 ANGULAR MOMENTUM

Consider the angular momentum of the reduced mass system $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times \mu\boldsymbol{v}$. How does $\boldsymbol{L}$ change with time? We have:

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= \frac{d}{dt}(\boldsymbol{r} \times \mu\boldsymbol{v}) && (19) \\ &= \boldsymbol{r} \times \mu\dot{\boldsymbol{v}} + \boldsymbol{v} \times \mu\boldsymbol{v} && (20) \\ &= \boldsymbol{r} \times \mu\boldsymbol{a} && (21) \\ &= \boldsymbol{r} \times \boldsymbol{F} && (22) \\ &= r\hat{\boldsymbol{r}} \times f(r)\hat{\boldsymbol{r}} && (23) \\ &= 0 && (24) \end{aligned}$$

(To get from (19) to (20), use the product rule, which is valid for cross products as long as you don't change the order of the factors. The second term in (20) is zero since $\boldsymbol{v} \times \boldsymbol{v} = 0$.) Recall that $\boldsymbol{r} \times \boldsymbol{F}$ which occurs in (22) is called the torque τ. We have shown that in the case of central forces the time derivative of the angular momentum, and hence the torque, are zero. Therefore:

$$\tau = \frac{d\boldsymbol{L}}{dt} = 0 \qquad \Rightarrow \qquad \boldsymbol{L} = \text{constant} \tag{25}$$

i.e. the angular momentum is conserved.

The force $\boldsymbol{F}(r)$ depends only on the distance of the reduced mass from the center of mass and not on the orientation of the system in space. Therefore, this system is spherically symmetric; it is invariant (unchanged) under rotations. Noether's theorem states that whenever the laws of physics are invariant under a particular motion or other operation, there will be a corresponding conserved quantity. In this case, we see that the conservation of angular momentum is related to the invariance of the physical system under rotations. Noether's theorem, in general, is most easily discussed using Lagrangian techniques. You will see this again the Classical Mechanics Capstone.

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1 Problems

1. Which of the equations in the derivation of (19)–(24) are valid only for central forces, and which are true more generally?

2. (Challenging) What invariances of physics are related to conservation of linear momentum and conservation of energy?

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7 COORDINATES

The time has come to choose a coordinate system. We have argued that the problem is spherically symmetric in nature. Therefore, it will be to our advantage to use spherical coordinates, defined by:

$$x = r\sin\theta\cos\phi \tag{26}$$

$$y = r\sin\theta\sin\phi \tag{27}$$

$$z = r\cos\theta \tag{28}$$

(see Figure 2), rather than the more comfortable Cartesian coordinates x, y, and z.

In fact, in the present classical mechanics context, we can do even better. For a central force:

$$\boldsymbol{F} = f(r)\,\hat{\boldsymbol{r}} \tag{29}$$

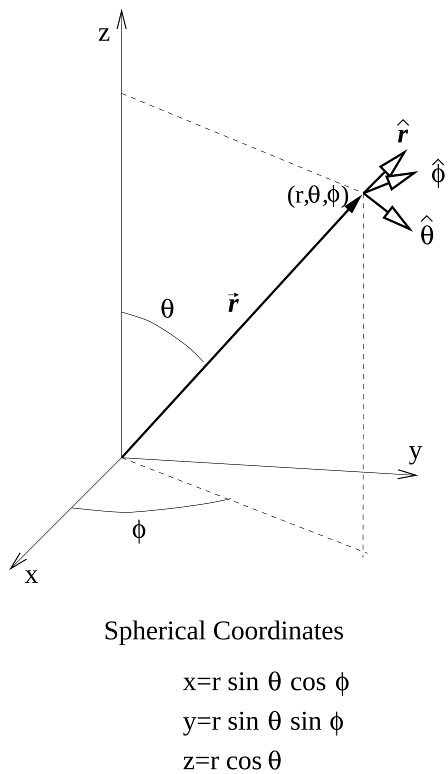

Figure 2: Spherical Coordinates.

the force, and hence the acceleration, are in the radial direction. Therefore, the path of the motion (orbit) will be in the plane determined by the position vector $\boldsymbol{r}$ and velocity vector $\boldsymbol{v}$ of the reduced mass at any one moment of time. Since there is never a component of force out of this plane, the subsequent motion must remain in the plane. In this plane, choose plane polar coordinates:

$$x = r\cos\phi \tag{30}$$

$$y = r\sin\phi \tag{31}$$

Notice that many textbooks choose to call the angle of plane polar coordinates θ. See Practice Problem 1.3 for the reason that we choose to call the angle ϕ.

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1 Problems

1. Convince yourself that the plane of the orbit is perpendicular to the angular momentum vector $\boldsymbol{L}$.

2. Show that a central force is **always** conservative. Find the scalar potential U corresponding to the central force $\boldsymbol{F} = f(r)\,\hat{r}$ and show that it depends only on the distance from the center of mass $U = U(r)$.

3. Show that the plane polar coordinates we have chosen are equivalent to spherical coordinates if we make the choices:

   (a) The direction of z in spherical coordinates is the same as the direction of $\boldsymbol{L}$.

   (b) The θ of spherical coordinates is chosen to be $\pi/2$, so that the orbit is in the equatorial plane of spherical coordinates.

   Some textbooks argue that you can obtain plane polar coordinates in terms of r and the polar angle θ by taking spherical coordinates (26)–(28) and making the choice $d\phi = 0$. Why is this choice actually misleading? Hint: In spherical coordinates, what is the range of θ? These textbooks label the angle θ because this is the most common convention for polar coordinates alone. However, if you do this, polar coordinates do **not** correspond in any nice way to spherical coordinates. Because I want you to see the relationship between classical and quantum mechanics and because the quantum version of central forces will require the use of spherical coordinates, we will call the polar coordinate angle ϕ.)

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8 VELOCITY & ACCELERATION

Newton's Laws require a knowledge of velocity and acceleration. With our choice of polar coordinates:

$$x = r\cos\phi \tag{32}$$
$$y = r\sin\phi \tag{33}$$

we must deal with the problem of how to compute velocity and acceleration as time derivatives of the position vector $\boldsymbol{r}$ in terms of the coordinates r and ϕ. A difficulty arises because $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are not independent of position and therefore are not independent of time. This problem does not present itself in Cartesian coordinates because $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are independent of position. We can exploit this Cartesian independence to help us in polar coordinates. $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are given, in terms of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, by

$$\hat{\boldsymbol{r}} = \cos\phi\,\hat{\boldsymbol{\imath}} + \sin\phi\,\hat{\boldsymbol{\jmath}} \tag{34}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}} \tag{35}$$

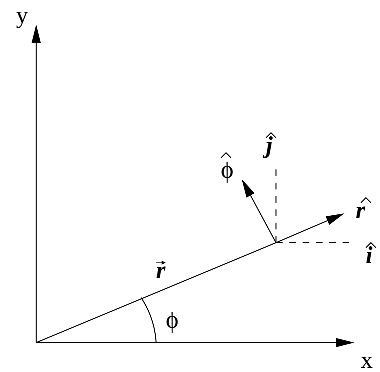

Figure 3: The relationship between unit vectors in polar coordinates ($\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\phi}}$) and unit vectors in Cartesian coordinates ($\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$).

You should recognize this basis change as a rotation performed on the $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ basis. As Figure 3 shows:

$$\begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \end{pmatrix} = R(\phi) \begin{pmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \end{pmatrix} \tag{36}$$

Using the chain rule, the general velocity vector is given by:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\hat{\boldsymbol{r}}}{dt} \tag{37}$$

To evaluate (37), we need the derivatives of $\hat{\boldsymbol{r}}$ (and $\hat{\boldsymbol{\phi}}$) with respect to time. Using the definitions in (36) above, we obtain:

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{d}{dt}(\cos\phi\hat{\boldsymbol{\imath}} + \sin\phi\hat{\boldsymbol{\jmath}}) = -\sin\phi\frac{d\phi}{dt}\hat{\boldsymbol{\imath}} + \cos\phi\frac{d\phi}{dt}\hat{\boldsymbol{\jmath}} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}} \tag{38}$$

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = \frac{d}{dt}\left(-\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}}\right) = -\cos\phi\frac{d\phi}{dt}\hat{\boldsymbol{\imath}} - \sin\phi\frac{d\phi}{dt}\hat{\boldsymbol{\jmath}} = -\frac{d\phi}{dt}\hat{\boldsymbol{r}} \quad (39)$$

Combining this with equation (37) gives:

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}} \quad (40)$$

Notice that we have used the convenient notation of putting a dot over a symbol to denote time derivative.

Taking another derivative of (40) with respect to time shows that the acceleration is given by:

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\phi}} \quad (41)$$

(40) can be used to show that the kinetic energy T of the reduced mass in polar coordinates is given by:

$$T = \frac{1}{2}\mu\,v^2 = \frac{1}{2}\mu\,\boldsymbol{v}\cdot\boldsymbol{v} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \quad (42)$$

Similarly, the magnitude of the angular momentum $\boldsymbol{L}$ of the reduced mass μ is given in polar coordinates by:

$$|\boldsymbol{L}| = |\boldsymbol{r}\times\mu\boldsymbol{v}| = l = \mu r^2\dot{\phi} \quad (43)$$

Since the angular momentum is a constant in central force problems, it's magnitude l is also constant. Therefore (43) can be used to rewrite differential equations, getting rid of $\dot{\phi}$'s in favor of the variable r and the constant l.

Kepler's second law says that the areal velocity of a planet in orbit is constant in time. This is equivalent to equation (43). To see why, read in section 8.3 of Marion and Thornton, page 294, from equation 8.10 to the bottom of the page.

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1 Practice Problems

1. Work through the steps deriving equations (41), (42), and (43) from (40).

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9 EQUATIONS OF MOTION: $\boldsymbol{F} = \mu \boldsymbol{a}$

The problem is now to the point where we can write the equations of motion in a form we can solve. However, the importance of the preceding sections cannot be stressed enough. The strategies that we used are important to the success of problem solving in many complicated physics situations. Drawing a picture, exploiting symmetries, choosing a convenient origin, and using the most appropriate coordinate system all combine to make the analysis as easy as possible. These and other tricks should always be regarded as a good beginning to any problem.

Newton's second law, reduced and modified for our specific problem is:

$$f(r)\hat{\boldsymbol{r}} = \mu\,\ddot{\boldsymbol{r}} = \mu\,\left((\ddot{r} - r\dot{\phi}^2)\hat{\boldsymbol{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}\right) \tag{44}$$

The vector equation breaks up, in polar coordinates, into two coupled differential equations for $r(t)$ and $\phi(t)$:

$$f(r) \;=\; \mu\,(\ddot{r} - r\dot{\phi}^2) \tag{45}$$

$$0 \;=\; \mu\,(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \tag{46}$$

Equation (46) is just the polar coordinate statement of angular momentum conservation, which we have already discussed, i.e.:

$$0 = r\,\mu\,(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = \frac{d}{dt}\left(\mu\,r^2\dot{\phi}\right) = \frac{dl}{dt} \tag{47}$$

(To derive verify the equalities in (47) it is easiest to work from right to left!) Therefore

$$\mu r^2\dot{\phi} = l = \text{constant} \tag{48}$$

(48) can be solved for $\dot{\phi}$ and used in (45) to obtain a messy, second order ODE for $r(t)$:

$$\ddot{r} = \frac{l^2}{\mu^2 r^3} + \frac{1}{\mu}f(r) \tag{49}$$

In principle, we could now insert the particular form of $f(r)$ we are concerned with, solve equation (49) for r as a function of t, and insert this value in (48) and solve for $\phi(t)$. We would then have solved the equations of motion for r, and ϕ, parameterized by the time t. In practice, for any but the simplest forms of $f(r)$, it is impossible to solve the differential equations analytically. Computers to the rescue! On Day 4, you will use a Maple worksheet which will allow you to explore numerical solutions for some important physical examples.

10 SHAPE OF THE ORBIT

If we are only interested in the shape of the orbit, we can do something simpler than solving the equations of motion for r and ϕ as functions of t; we can solve for the shape of the orbit, i.e. instead of using the variable t as a parameter in (49), we will use the variable ϕ and solve for $r(\phi)$. To do this, we need to change the time derivatives into ϕ derivatives.

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} \tag{50}$$

It turns out that the differential equation which we obtain will be much easier to solve if we also change independent variables from r to

$$u = r^{-1} \tag{51}$$

(There is no way that you could guess this, yourself.) Therefore,

$$\frac{dr}{dt} = \frac{\ell}{\mu r^2}\frac{dr}{d\phi} = -\frac{\ell}{\mu}\frac{d\,r^{-1}}{d\phi} = -\frac{\ell}{\mu}\frac{du}{d\phi} \tag{52}$$

(To verify the second equality, work from right to left.) Then the second derivative is given by

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\frac{dr}{dt} = \frac{\ell}{\mu}\,u^2\,\frac{d}{d\phi}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell^2}{\mu^2}\,u^2\,\frac{d^2u}{d\phi^2} \tag{53}$$

Plugging (51) and (53) into (49), dividing through by u^2, and rearranging, we obtain the orbit equation

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{\ell^2}\,\frac{1}{u^2}\,f\left(\frac{1}{u}\right) \tag{54}$$

For the special case of inverse square forces $f(r) = -k/r^2$ (spherical gravitational and electric sources), it turns out that the right-hand side of (54) is constant so that the equation is particularly easy to solve. First solve the homogeneous equation (with $f(r) = 0$), which is just the harmonic oscillator equation with general solution

$$u_{\mathrm{h}} = A\cos(\phi + \delta) \tag{55}$$

Add to this any particular solution of the inhomogeneous equation (with $f(r) = -k/r^2$). By inspection, such a solution is just

$$u_{\mathrm{p}} = \frac{\mu\, k}{\ell^2} \tag{56}$$

so that the general solution of (54) for an inverse square force is

$$r^{-1} = u = u_{\mathrm{h}} + u_{\mathrm{p}} = A\cos(\phi + \delta) + \frac{\mu\, k}{\ell^2} \tag{57}$$

Then solving for r in (57) we obtain

$$r = \frac{1}{\frac{\mu k}{\ell^2} + A\cos(\phi + \delta)} = \frac{\frac{\ell^2}{\mu k}}{1 + A'\cos(\phi + \delta)} \tag{58}$$

You can explore how the graph of this equation depends on the various parameters using the Maple worksheet *conics.mws*

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1 Practice Problems

1. Go through all the steps in the derivation of (54) from (49). (49) is the same as equation 8.18 in section 8.4 on page 296 of Marion and Thornton; an alternative derivation of (54) can be found following equation 8.18. Use whichever technique is easiest for you to follow, but make sure you understand at least one. This kind of change of variables is very common in physics.

2. How do the physical constants in (58) correspond to the mathematical constants: amplitude α, phase δ, and the eccentricity ϵ, from the Maple worksheet *conics.mws*?

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11 EQUATIONS OF MOTION: $E = T + U$

Another theoretical tool we can use to arrive at an equation for the orbit is conservation of energy. The central force $\boldsymbol{F}$ is conservative and can be derived from a potential $U(r)$ which depends only on the distance from the center of mass (see practice problem 1.2):

$$\boldsymbol{F} = -\nabla U = -\frac{\partial U(r)}{\partial r} \tag{59}$$

The statement of energy conservation:

$$E = T + U \tag{60}$$

becomes, using (42), (43), and (59):

$$E = \frac{1}{2}\mu\,\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r) \tag{61}$$

(61) can be solved for $\dot{r}$ to give:

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E - U(r)) - \frac{l^2}{\mu^2 r^2}} \tag{62}$$

(62) is an equivalent alternative to (49) as an equation of motion for $r(t)$. You might be surprised that (62) is a first order differential equation, whereas (49) is second order. This means that only one initial condition is required for the solution of (62) whereas two are needed for the solution of (49). There is nothing surprising going on here. We have already provided the extra information (the extra initial condition) by specifying the constant total energy E.

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1 Practice Problems

1. (Challenging) Show that the equation of motion derived from Newton's Law (49) is equivalent to the equation of motion derived from energy conservation (62). Hint: Multiply (49) by $2\dot{r}\,dt$ and integrate both sides.

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12 EVERYTHING ELSE

You should now work through sections 8.4-8.7 of Taylor. Pay particular attention to the concept of the effective potential.

There are many areas left to explore if you are interested: questions of the stability of orbits under perturbations, the precession of the orbit, and whether it is open or closed. There are many interesting examples, even within our solar system, that show the varied and unique outcomes of central force interactions: Lagrange points, resonant orbits, horseshoe orbits, to name a few. There are also other types of central forces. The repulsive inverse square force was very important to early atomic experiments. Rutherford bombarded a lattice of gold with alpha particles (helium nuclei). The repulsive electrostatic interaction can be handled easily by our preceding analysis. The theory fit experiment well until the alpha particle energies became high enough to overcome the effective potential and hit the nucleus head-on.

Many of the ideas in our analysis are handled nicely by the Lagrangian formalism which you will study in the Classical Mechanics Capstone. Lagrangian mechanics provides yet another starting point for obtaining the equations of motion. The ideas of symmetry and conservation are more easily recognized and handled within that context, which proves to be very powerful in more complicated situations. When you reach that point, remember some of the techniques we used here and then appreciate the simplicity and beauty provided by the new viewpoint.