Central Forces Homework 5

Due 3/11/09

PRACTICE:

1. In each of the following sums, shift the index $n \to n-2$. Don't forget to shift the limits of the sum as well. Then write out all of the terms in the sum (if the sum has a finite number of terms) or the first five terms in the sum (if the sum has an infinite number of terms) and convince yourself that the two different expressions for each sum are the same:

   (a)
   $$\sum_{n=0}^{3} n$$

   (b)
   $$\sum_{n=1}^{5} e^{in\phi}$$

   (c)
   $$\sum_{n=0}^{\infty} a_n n(n-1) z^n$$

2. Find a list of the first few spherical harmonics in Griffiths or some other book. Remember where you found it! Look at the functional forms of the spherical harmonics to get familiar with them. What patterns do you notice? Calculate a few from the formulas given in class.

3. In spherical coordinates, the square of the angular momentum vector L^2 and the z-component of the angular momentum vector L_z are given by:

   $$L^2 = \vec{L} \cdot \vec{L} = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right)$$

   $$L_z = -i\hbar \frac{\partial}{\partial\phi}$$

   (a) Show explicitly that the spherical harmonics are eigenvectors of L^2. What are the eigenvalues?

   (b) Show explicitly that the spherical harmonics are eigenvectors of L_z. What are the eigenvalues?

REQUIRED:

4. (a) By hand, find the recurrence relation for a power series solution $H(\rho)$ of the equation:

$$\rho\frac{d^2H}{d\rho^2} + (2\ell + 2 - \rho)\frac{dH}{d\rho} + (\lambda - \ell - 1)H = 0 \tag{1}$$

where ℓ is a known positive integer, and λ is an unknown constant.

   (b) Suppose that you want a solution to (a) which is a polynomial of degree 4. Assume that $\ell = 2$. What does that tell you about the unknown constant λ?

   (c) Find the polynomial of degree 4 solution to (1) assuming $\ell = 2$. Assume anything you need to about λ.

5. Use the Maple worksheet legseries.mws (see the link on the course syllabus) to explore the Legendre polynomial expansion of the function $f(z) = \sin(\pi z)$. How many terms do you need to include in a partial sum to get a "good" approximation to $f(z)$ for $-1 < z < 1$? What do you mean by a "good" approximation? How about the interval $-2 < z < 2$? How good is your approximation? Discuss your answers. Without using the worksheet, answer the same set of questions for the function $g(z) = \sin(3\pi z)$

6. Consider the normalized state given by:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|1,-1\rangle + \frac{1}{\sqrt{3}}\,|1,0\rangle + \frac{i}{\sqrt{6}}\,|0,0\rangle$$

   (a) What is the probability that a measurement of L_z will yield $2\hbar$? $-\hbar$? $0\hbar$?

   (b) What is the expectation value of L_z in this state?

   (c) What is the expectation value of L^2 in this state?

   (d) What is the expectation value of the energy in this state?

   (e) (Challenge: Required for PH 526 students) What is the expectation value of L_y in this state? L_y in spherical coordinates is given by:

$$L_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

7. Make a table, similar to the one passed out in class, for a rigid rotor. Include information about the operators $\hat{H}$, $\hat{L}_z$, and $\hat{L}^2$.