PH422: Static Fields

Quiz 4

For the quiz on Monday, February 3, you will be provided formulas for gradient, divergence, and curl in the standard three coordinate systems. The questions will be something similar to:

1. Find the gradient of each of the following functions:

   (a)
   $$f(x, y, z) = e^{(x+y)} + x^2 y^3 \ln \frac{x}{z}$$

   Solution:
   $\vec{\nabla} f = (e^{x+y} + 2xy^3 \ln \frac{x}{z} + xy^3)\hat{x} + 3x^2y^2 \ln \frac{x}{z}\hat{y} - \frac{x^2y^3}{z}\hat{z}$

   (b)
   $$\sigma(\theta, \phi) = \cos\theta \sin^2\phi$$

   Solution:
   $\vec{\nabla}\sigma = -\frac{\sin\theta\sin^2\phi}{r}\hat{\phi} + \frac{2\cot\theta\sin\phi\cos\phi}{r}\hat{\theta}$

   (c)
   $$\rho(s, \phi, z) = (s + 3z)^2 \cos\phi$$

   Solution:
   $\vec{\nabla}\rho = 2(s + 3z)\cos\phi\hat{s} - \frac{(s+3z)^2\sin\phi}{s}\hat{\phi} + 6(s + 3z)\cos\phi\hat{z}$

2. Calculate the divergence of each of the following vector fields. You may look up the formulas for divergence in curvilinear coordinates.

   (a) $\vec{\boldsymbol{F}} = z^2\,\hat{x} + x^2\,\hat{y} - y^2\,\hat{z}$

   (b) $\vec{\boldsymbol{G}} = e^{-x}\,\hat{x} + e^{-y}\,\hat{y} + e^{-z}\,\hat{z}$

   (c) $\vec{\boldsymbol{H}} = yz\,\hat{x} + zx\,\hat{y} + xy\,\hat{z}$

   (d) $\vec{\boldsymbol{I}} = x^2\,\hat{x} + z^2\,\hat{y} + y^2\,\hat{z}$

   (e) $\vec{\boldsymbol{J}} = xy\,\hat{x} + xz\,\hat{y} + yz\,\hat{z}$

   (f) $\vec{\boldsymbol{K}} = s^2\,\hat{s}$

   (g) $\vec{\boldsymbol{L}} = r^3\,\hat{\phi}$

   Solution:
   $\vec{\nabla}\cdot\vec{\boldsymbol{F}} = 0$
   $\vec{\nabla}\cdot\vec{\boldsymbol{G}} = -e^{-x} - e^{-y} - e^{-z}$
   $\vec{\nabla}\cdot\vec{\boldsymbol{H}} = 0$

$\vec{\nabla} \cdot \vec{\boldsymbol{I}} = 2x$
$\vec{\nabla} \cdot \vec{\boldsymbol{J}} = 2y$
$\vec{\nabla} \cdot \vec{\boldsymbol{K}} = 3s$
$\vec{\nabla} \cdot \vec{\boldsymbol{L}} = 0$

3. Calculate the curl of each of the following vector fields. You may look up the formulas for curl in curvilinear coordinates.

   (a) $\vec{\boldsymbol{F}} = z^2\,\hat{x} + x^2\,\hat{y} - y^2\,\hat{z}$

   (b) $\vec{\boldsymbol{G}} = e^{-x}\,\hat{x} + e^{-y}\,\hat{y} + e^{-z}\,\hat{z}$

   (c) $\vec{\boldsymbol{H}} = yz\,\hat{x} + zx\,\hat{y} + xy\,\hat{z}$

   (d) $\vec{\boldsymbol{I}} = x^2\,\hat{x} + z^2\,\hat{y} + y^2\,\hat{z}$

   (e) $\vec{\boldsymbol{J}} = xy\,\hat{x} + xz\,\hat{y} + yz\,\hat{z}$

   (f) $\vec{\boldsymbol{K}} = s^2\,\hat{s}$

   (g) $\vec{\boldsymbol{L}} = r^3\,\hat{\phi}$

   Solution:
   $\vec{\nabla} \times \vec{\boldsymbol{F}} = -2y\,\hat{x} + 2z\,\hat{y} + 2x\,\hat{z}$
   $\vec{\nabla} \times \vec{\boldsymbol{G}} = 0$
   $\vec{\nabla} \times \vec{\boldsymbol{H}} = 0$
   $\vec{\nabla} \times \vec{\boldsymbol{I}} = (2y - 2z)\,\hat{x}$
   $\vec{\nabla} \times \vec{\boldsymbol{J}} = (z - x)\,(\hat{x} + \hat{z})$
   $\vec{\nabla} \times \vec{\boldsymbol{K}} = 0$
   $\vec{\nabla} \times \vec{\boldsymbol{L}} = r^2(\cot\theta\,\hat{r} - 4\,\hat{\theta})$

4. Which of the following are valid operations? How do you know?

   (a)
   $$\vec{\nabla} \cdot \left(\vec{\nabla} F\right)$$

   Solution:

   True. The gradient of a scalar is a vector. It is possible to take the divergence of a vector. (The divergence of a gradient goes by the special name of of a the Laplacian, denoted $\nabla^2 F$.)

   (b)
   $$\vec{\nabla}\left(\vec{\nabla} \times \vec{F}\right)$$

   Solution:

   False. The curl produces a vector and the gradient only acts on scalars.

(c)

$$\vec{\nabla} \times \left(\vec{\nabla} \cdot \vec{F}\right)$$

Solution:

False. The divergence produces a scalar and one can only take the curl of a vector.

(d)

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{F}\right)$$

Solution:

True. This is a valid operation since divergence can act on the vector produced by the curl of a vector field. (It turns out that the divergence of a curl is identically zero, which can be shown by combining Stokes Theorem and the Divergence Theorem (see physics.oregonstate.edu/mathbook/GSF/second.html). But it is not possible to see this fact from looking at the form of the formula alone.)

(e)

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right)$$

Solution:

True. The curl of a vector field is a vector field. You can tak the curl of a vector field. (Using product rules, one can "simplify" this expression via:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$$

Again, this result is not possible to see by looking at the form of the equation alone.)