Calculating Line Elements in Cylindrical and Spherical Coordinates
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Rectangular Coordinates:
The arbitrary infinitesimal displacement vector in Cartesian coordinates is:
\[ d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k} \]

Given the cube shown below, find \(d\vec{r}\) on each of the three paths, leading from \(a\) to \(b\).

Path 1: \(d\vec{r} = \)
Path 2: \(d\vec{r} = \)
Path 3: \(d\vec{r} = \)

The first expression above for \(d\vec{r}\) is valid for any path in rectangular coordinates. Find the appropriate expression for \(d\vec{r}\) for the path which goes directly from \(a\) to \(c\) as drawn below.

Path 4: \(d\vec{r} = \)

However, Cartesian coordinates would be a poor choice to describe a path on a cylindrically or spherically shaped surface. Next we will find an appropriate expression in these cases.
Cylindrical Coordinates:
You will now derive the general form for $d\vec{r}$ in cylindrical coordinates by determining $d\vec{r}$ along the specific paths below.

Note that an infinitesimal element of length in the $\hat{r}$ direction is simply $dr$, just as an infinitesimal element of length in the $\hat{i}$ direction is $dx$. But, an infinitesimal element of length in the $\hat{\phi}$ direction is not just $d\phi$, since this would be an angle and does not even have the units of length.

Geometrically determine the length of the three paths leading from $a$ to $b$ and write these lengths in the corresponding boxes on the diagram.

Now, remembering that $d\vec{r}$ has both magnitude and direction, write down below the infinitesimal displacement vector $d\vec{r}$ along the three paths from $a$ to $b$. Notice that, along any of these three paths, only one coordinate $r$, $\phi$, or $z$ is changing at a time. (i.e. along path 1, $dz \neq 0$, but $d\phi = 0$ and $dr = 0$).

Path 1: $d\vec{r} =$
Path 2: $d\vec{r} =$
Path 3: $d\vec{r} =$

If all three coordinates are allowed to change simultaneously, by an infinitesimal amount, we could write this $d\vec{r}$ for any path as:

$$d\vec{r} =$$

This is the general line element in cylindrical coordinates.
Spherical Coordinates

You will now derive the general form for $d\vec{r}$ in spherical coordinates by determining $d\vec{r}$ along the specific paths below. As in the cylindrical case, note that an infinitesimal element of length in the $\theta$ or $\phi$ direction is not just $d\theta$ or $d\phi$. You will need to be more careful. Geometrically determine the length of the three paths leading from $a$ to $b$ and write these lengths in the corresponding boxes on the diagram. Now, remembering that $d\vec{r}$ has both magnitude and direction, write down below the infinitesimal displacement vector $d\vec{r}$ along the three paths from $a$ to $b$. Notice that, along any of these three paths, only one coordinate $r$, $\theta$, or $\phi$ is changing at a time. (i.e. along path 1, $d\theta \neq 0$, but $dr = 0$ and $d\phi = 0$).

Path 1: $d\vec{r}$ =
Path 2: $d\vec{r}$ = (Be careful, this is the tricky one.)
Path 3: $d\vec{r}$ =

If all 3 coordinates are allowed to change simultaneously, by an infinitesimal amount, we could write this $d\vec{r}$ for any path as:

$\vec{dr}$ =

This is the general line element in spherical coordinates.