Prep 3

Recommended finish date: Friday, January 25

The formats (type, length, scope) of these Prep problems have been purposely created to closely parallel those of a typical exam (indeed, these problems were taken from past exams). To get an idea of how best to approach various problem types (there are three basic types), refer to these Sample Problems.
1. a. Evaluate the following statements (T/F/N). As always, you must explain your reasoning with a valid mix of words, diagrams and/or calculations.

(i) If a steady net torque τ (a known value) is applied to an object (initially at rest) about some fixed axis of rotation, for a known time interval, Δt, and its resulting angular speed is measured as ω at the end of that time interval, then this is enough information to calculate the object’s moment of inertia about that axis of rotation.
   True. We can calculate α via \( \omega_f = \omega_i + \alpha \Delta t \) (where \( \omega_i = 0 \)); then we get \( I \) via \( \Sigma \tau = I \alpha \).

(ii) For a moving object, it is possible that \( \Sigma F = 0 \) and \( \Sigma \tau \neq 0 \).
   True. For example:

   (iii) For a moving object, it is possible that \( \Sigma F = 0 \) and \( \Sigma \tau = 0 \).
   True. An object could be rotating and traveling (translating) through the universe with absolutely no net forces or torques on it. *Equilibrium does not necessarily mean static equilibrium.*

(iv) It is possible that an object moving (translating) in a circle might have no forces acting on it.
   False. Only straight-line translation is possible without a net force to cause a change of direction.

(v) A rotating object can be in total mechanical equilibrium.
   True. See item (iii) above.

(vi) If the gravitational force on an object produces zero torque about a certain axis, the object’s center of gravity is located on the same vertical line as that axis.
   True. \( \tau = |F_G| |l_G| \sin \theta \), where \( \theta \) is the angle between the direction of \( F_G \) and the direction of \( l_G \). So if \( \tau = 0 \), but if neither \( F_G \) nor \( l_G \) is zero in magnitude, \textit{then sin} \( \theta \text{ must be zero; that is, } F_G \text{ and } l \text{ are collinear:} \)

   (vii) Torque is a vector quantity.
   True. It has both a magnitude (\( |F| |l| \sin \theta \)) and direction (right-hand cross-product: ccw = +, conventionally).

(viii) A radial force is a constant vector value.
   False. A radial force has a constantly changing direction, as the object turns on its circular path.

b. Express torque in fundamental (base) SI units. \( \text{kg} \cdot \text{m}^2/\text{s}^2 \)

c. You are facing a wheel initially at rest. A force is applied to the right edge of the wheel, accelerating the wheel in a clockwise direction. The direction of the torque vector here is... away from you—along the axis of rotation of the wheel—indicated by your right thumb when you curl the fingers of your right hand in the direction of angular acceleration.

d. Exercise 20, page 349.
   The disk shown here is 20 cm in diameter and is free to rotate on a frictionless axle at its center (perpendicular to the page). Points A and B are each 5.0 cm from the axle.
   Find the net torque on the disk.
   \[
   \Delta \tau = (20)(0.05)\sin90^\circ + (30)(0.05)\sin135^\circ + (20)(0.10)\sin0^\circ - (30)(0.10)\sin90^\circ \\
   = -0.939 \text{ N} \cdot \text{m}
   \]
2. a. The second hand on a certain standard clock has a length of 12.0 cm, measured from the central hub to its tip. Normally, that second hand sweeps smoothly and steadily around one full revolution every minute (60 seconds). Suppose that it is indeed operating normally—until a certain moment, when it begins to speed up.

(i) What angular acceleration would it need to have—at that first moment of speeding up—so that the tip of the second hand then has equal magnitudes of tangential and radial acceleration?

At that first speeding up moment, the second hand is still moving at its normal angular velocity.

Normal \( \omega = -2\pi/60 = -0.10472 \text{ rad/s} \) (negative because it’s clockwise, of course)
|\(a_R| = r|\omega|^2 \) and |\(a_T| = r|\alpha|, so if these two magnitudes are to be equal, then: \(r|\omega|^2 = r|\alpha|
Thus: \(|\alpha| = |\omega|^2 = 0.10472^2 = 1.0966 \times 10^{-2} \text{ rad/s}^2\)
But \(\alpha\) is speeding up \(\omega\), so it’s in the same direction as \(\omega\). Therefore: \(\alpha = -1.10 \times 10^{-2} \text{ rad/s}^2\)

(ii) Assuming the behavior described in part (i), find the total acceleration magnitude of the tip at that first moment of speeding up.

\[|a| = \sqrt{|a_R|^2 + |a_T|^2} = \sqrt{(r|\omega|^2)^2 + (r|\alpha|)^2} = \sqrt{[(0.120)(0.10472)^2]^2 + [(0.120)(1.0966 \times 10^{-2})]^2} = 1.86 \times 10^{-3} \text{ m/s}^2\]

(iii) Assuming the behavior described in part (i), if the tip of the second hand is located at the “4 o’clock” position at that first moment of speeding up, find the direction of its total acceleration, \(a\), at that moment.

The “4 o’clock” position is \(-30^\circ\). So the direction of \(a_R\) is given by \(\theta_R = -30 + 180 = 150^\circ\)

And \(a_T\) is perpendicular to the position and in the direction of \(\alpha\) (clockwise), so that’s \(90^\circ\) more negative than the position.

So the direction of \(a_T\) is given by \(\theta_T = -30 - 90 = -120^\circ\)

The acceleration vector triangle is a right triangle with equal legs (since \(|a_R| = |a_T|\).
So the angle \(\theta = 45^\circ\), and \(\theta_a = \theta_T - 45^\circ = -120 - 45 = -165^\circ\)
2. At the moment depicted here, this roller coaster car of mass \( m \) is at the angular position \( \theta \), sliding clockwise at speed \( v \), in a vertical “loop-the-loop” maneuver on this circular track of radius \( r \), which has a kinetic friction coefficient \( \mu_k \) with the car.

Find an expression for the net force (both magnitude and direction) acting on the car.

You may consider these values as known: \( m, \theta, v, r, \mu_k, g \).

### Net Radial Force

\[
\Sigma F_R = ma_R
\]
\[
F_{G,R} - F_K = ma_R
\]
\[
m\cos\theta - \mu_k F_N = ma_R
\]

**I.** Find the magnitude of the net radial force, \( F_{R,net} \), acting on the car:

\[
F_{R,net} = \frac{mv^2}{r}
\]

**II.** Solve for the normal force, \( F_N \), acting on the car:

\[
F_N = (\frac{mv^2}{r}) - mgsin\theta
\]

**III.** Find the net tangential force, \( F_{T,net} \), acting on the car:

\[
F_{T,net} = mc\cos\theta - \mu_k(\frac{mv^2}{r}) - mgsin\theta
\]

**IV.** Find \( |F_{net}| \), the magnitude of the total net force acting on the car:

\[
|F_{net}| = \sqrt{|F_{R,net}|^2 + |F_{T,net}|^2}
= \sqrt{\left(\frac{mv^2}{r}\right)^2 + \left( mc\cos\theta - \mu_k(\frac{mv^2}{r}) - mgsin\theta \right)^2}
\]

**V.** Find the direction, \( \theta_R \), of \( F_{R,net} \):

\[
\theta_R = \theta + 180
\]

**VI.** Find the angle, \( \beta \), between \( F_{R,net} \) and \( F_{net} \):

\[
\tan \beta = \frac{|F_{T,net}|}{|F_{R,net}|}
\]

Or:

\[
\beta = \tan^{-1}(\frac{mc\cos\theta - \mu_k(\frac{mv^2}{r}) - mgsin\theta}{\frac{mv^2}{r}})
\]

**VII.** Find the direction angle, \( \theta_{net} \), of \( F_{net} \):

\[
\theta_{net} = \theta_R + \beta
= \theta + 180 + \tan^{-1}(\frac{mc\cos\theta - \mu_k(\frac{mv^2}{r}) - mgsin\theta}{\frac{mv^2}{r}})
\]
2. c. A record with a 10 cm radius and 200 g mass is dropped vertically (and without rotation) onto a turntable that is rotating but not being driven by a motor; rather, it is gradually slowing, due to a constant net torque of 0.01 N·m exerted by the friction in its hub. When the record lands on the turntable, the two objects stick together and rotate around the turntable’s hub. The radius of the turntable is 10 cm; its mass is 2 kg; its angular velocity just before the record lands on it is 5 rad/s.

(i) What is earliest common angular speed achieved by the record and the turntable?

Model both objects as solid, uniform disks:

Data:

\[ \omega_{i,r} = 5 \quad I_r = \frac{1}{2} M_r R_r^2 = \frac{1}{2}(0.200)(0.10)^2 = 0.001 \quad I_t = \frac{1}{2} M_t R_t^2 = \frac{1}{2}(2)(0.10)^2 = 0.01 \]

This is a collision between two objects that can rotate around a common axis.

Use conservation of angular momentum:

\[ I_r \omega_{i,r} + I_t \omega_{i,t} = (I_r + I_t) \omega_f \]

And \( \omega_{i,r} = 0 \):

\[ I_r \omega_{i,t} = (I_r + I_t) \omega_f \]

Solve for \( \omega_f \):

\[ \omega_f = \frac{I_r \omega_{i,t}}{I_r + I_t} = \frac{(0.01)(5)}{0.001 + 0.01} = 4.55 \text{ rad/s} \]

(ii) How long will it take for the record and turntable to stop rotating altogether?

Newton’s 2nd Law:

\[ \alpha = \frac{\tau_{net}}{I_{total}} \]

Kinematics:

\[ \Delta t = \frac{(\omega_f - \omega_i)}{\alpha} = \frac{I_{total} (\omega_f - \omega_v)}{\tau_{net}} = \frac{(0.11)(0 - 4.54545)}{-0.01} = 5.00 \text{ s} \]

d. A uniform rod (2.50 m long, 41.3 kg) is rotating in a horizontal circle about one end, as shown in this overhead view. When the rod is in the position shown, it is rotating at –6.7 rad/s, and a torque of 89.0 N·m is being applied to it. What is the direction of the net force being exerted on the outer tip of the rod at that instant?

The net force is in the same direction as the total (net) acceleration, \( \mathbf{a} \). And the total (net) acceleration is the vector sum of the tangential and centripetal acceleration components:

\[ \mathbf{a} = a_T + a_C \]

The mass is slowing, so that means the tangential acceleration \( a_T \) is in the opposite direction to the tangential velocity:

\[ \Sigma \tau = I \alpha \]

\[ \Sigma \tau = 89 \]

\[ I = \frac{1}{2} M L^2 = \frac{1}{2}(41.3)(2.5^2) = 86.042 \text{ kg·m}^2 \]

\[ \alpha = \Sigma \tau / I = \frac{89}{86.042} = 1.0344 \text{ rad/s}^2 \]

\[ |a_T| = r \alpha = 2.5(1.0344) = 2.586 \text{ m/s}^2 \]

\[ |a_C| = r \omega^2 = 2.5(6.7^2) = 112.225 \text{ m/s}^2 \]

\[ \theta_a = \tan^{-1}(|a_T|/|a_C|) + 90^\circ = 179^\circ \]
2. e A horizontal turntable is a disk free to rotate frictionlessly around its central axis. Initially it is at rest, and a coin is resting on it at position A (a distance $r_A$ from the axis), as shown here. Then a constant torque $\tau$ is applied to the turntable (around its central axis) for a certain time interval, $\Delta t$. At the end of $\Delta t$, the torque ceases.

At that very same moment (the end of $\Delta t$), the coin slips and begins to slide across the surface of the turntable. It stops slipping at position B (a radial distance $d$ from A), as shown, because it encounters an obstruction on the turntable surface.

The mass of the coin is $m$. The moment of inertia of the turntable (without the coin), around its central axis, is $I$. After the coin has arrived at position B (after it has stopped slipping), the friction force on the coin is half the maximum possible. Find the steady force magnitude exerted by the obstruction on the coin after it has arrived at position B (after it has stopped slipping).

You may consider these values as known: $r_A$, $\tau$, $\Delta t$, $d$, $m$, $I$, $g$.

This is an ODAVEST item—use the full seven-step problem-solving protocol—but keep in mind that you’re not being asked to actually solve for the final expression. In fact, you’re not being asked to do any math at all—not even any algebra. Rather, for the Solve step, you are to write a series of succinct instructions on how to solve this problem. Pretend that your instructions will be given to someone who knows math but not physics. And for the Test step, you should consider the situation and predict how the solution would change if the data were different—changing one data value at a time.

**Objective:** A horizontal turntable, with a known moment of inertia, is free to rotate on a frictionless central axle. A coin of known mass is sitting at rest at a known location on the turntable (also initially at rest).

A steady known torque is applied to the turntable (about its central axle) for a known time interval.

At the end of that time interval, two events occur simultaneously: The torque ceases, and the coin slips from its initial position and begins to slide across the surface of the turntable.

The coin stops against an obstruction after sliding a known distance—measured with respect to the turntable—radially outward from the axle.

At that final position, the static friction force exerted on the coin by the turntable is half the maximum possible.

We need to find the magnitude of the force exerted on the coin by the obstruction.

**Data:**

- $r_A$ The radial distance from the axle of the coin’s initial position.
- $\tau$ The magnitude of the steady torque (about the axle) exerted on the turntable.
- $\Delta t$ The time interval during which the torque, $\tau$, is applied to the turntable.
- $d$ The distance across the surface of the carousel that the coin slides.
- $m$ The mass of the coin.
- $I$ The moment of inertia of the turntable (without the coin).
- $g$ The local value of free-fall acceleration.

**Assumptions:**

- **Surfaces** We assume the surfaces of both the turntable and coin are parallel and uniform (except for the obstruction stipulated on the surface of the turntable), so that the coefficients of static and kinetic friction between these surfaces remains constant throughout this scenario.

- **Alignment** We assume that the coin remains motionless with respect to the turntable all during the time interval when the torque is being applied.

- **Coin** We model the coin as point mass, for the purposes of computing its contribution to the overall moment of inertia.

- **Air** We disregard any effects of wind or air drag.

- **Gravity** We assume the local value of $g$ is constant throughout this scenario.
Solving:

Solve I for $I_A$. Substitute that result into II.

Solve II for $\alpha$. Substitute that result into III.

Solve III for $\omega_A$ (noting that $\omega_i = 0$). Substitute that result into VI.

Solve IV for $F_s^{\text{max}}$. Substitute that result into VII.

Solve V for $I_B$. Substitute that result into VI.

Solve VI for $\omega_B$. Substitute that result into VII.

Solve VII for $F_{\text{obstr}}$.

Testing:

Dimensions: $F_{\text{obstr}}$ should have units of force (dimensions of mass·length/time²).

Dependencies:

If the coin starts farther from the axle (a greater $r_A$), then for the given torque, $\omega_A$ will be less, due to the larger moment of inertia. That implies a lesser value for $F_s^{\text{max}}$, but since $\omega_B$ would also be less, it’s unclear without an explicit solution how all this would affect $F_{\text{obstr}}$.

If the torque magnitude is greater, then $\omega_A$ will be greater, which would imply a stronger $F_s^{\text{max}}$. But since $\omega_B$ would also be greater, it’s unclear without an explicit solution how all this would affect $F_{\text{obstr}}$.

If the torque interval, $\Delta t$, is greater, then $\omega_A$ will be greater, which would imply a stronger $F_s^{\text{max}}$. But since $\omega_B$ would also be greater, it’s unclear without an explicit solution how all this would affect $F_{\text{obstr}}$.

If the coin’s sliding distance, $d$ is greater, this will reduce $\omega_B$, and therefore the net centripetal force acting on the coin at point B. That would imply a lesser role necessary for $F_{\text{obstr}}$.

If the coin’s mass, $m$, were greater, then for the given torque, $\omega_A$ will be less, due to the larger moment of inertia. That implies a lesser value for $F_s^{\text{max}}$, but since $\omega_B$ would also be less, it’s unclear without an explicit solution how all this would affect $F_{\text{obstr}}$.

If the carousel has a greater moment of inertia, $I$, then for the given torque, $\omega_A$ will be less. That implies a lesser value for $F_s^{\text{max}}$, but since $\omega_B$ would also be less, it’s unclear without an explicit solution how all this would affect $F_{\text{obstr}}$.

If the local g value were greater, this would increase the value of $F_s^{\text{max}}$, so the obstruction force, $F_{\text{obstr}}$, would play a lesser role at point B.

Visual Rep(s): **OVERHEAD VIEW**

Equations:

I. $I_A = I + mr_A^2$

II. $\tau = I_A \alpha$

III. $\omega_A = \omega_i + \alpha \Delta t$

IV. $F_s^{\text{max}} = m(r_A \omega_A^2)$

V. $I_B = I + m(r_A + d)^2$

VI. $I_A \omega_A = I_B \omega_B$

VII. $F_{\text{obstr}} + F_s^{\text{max}}/2 = m[(r_A + d) \omega_B^2]$
2. f. Refer to the overhead view here. A large, uniform disk (mass = $m_1$, outer radius = $r_1$) is rotating freely on a frictionless central axis. A small coin (mass = $m_2$) is “riding” on the disk at a distance $d$ from the axis. The coin’s speed (relative to the ground) is $v$, and it does not slip across the disk’s surface. The static friction coefficient between the coin and disk surfaces is $\mu_s$. Ignore any effects of wind/air drag.

(i) What is the maximum steady torque magnitude you could apply to the disk (without touching the coin) in order to bring the disk (and coin) to rest without the coin ever slipping across the disk’s surface?

(ii) How much time would this slowing-to-a-stop process require?

Here is a summary of all known values: $m_1$, $r_1$, $m_2$, $d$, $\mu_s$, $v$, $g$

For this item, the full seven-step ODAVEST problem-solving procedure is required. Note that each part will be awarded with points, so even if you don’t get all the way through the problem, there are many ways to earn partial credit for parts that are valid. Keep in mind that you’re not being asked to actually solve for the final expression. In fact, you’re not being asked to do any math at all—not even any algebra. Rather, for the Solve step, you are to write a series of succinct instructions on how to solve this problem. Pretend that your instructions will be given to someone who knows math but not physics. And for the Test step, you should consider the situation and predict how the solution would change if the data were different—changing one data value at a time.

Objective: A large, uniform disk of known mass is rotating about its center. Its axis of rotation is vertical and the disk’s large surface is horizontal. The rotation occurs without any friction in the axle (axis of rotation). A small coin of known mass is lying on the disk surface, at rest with respect to the disk. The coin’s center is a known distance from the axis of the disk’s rotation. The speed of the coin with respect to the ground is known. The coefficient of static friction between the coin and disk is known. Effects of air drag and/or wind are negligible.

We want to estimate the maximum steady torque that could be applied to the disk (without touching the coin), so that the coin does not move relative to the disk.

We also want to calculate the time that torque would require to slow the disk and coin to a complete stop.

Data: $m_1$ The mass of the large, uniform disk.
$r_1$ The outer radius of the large uniform disk.
$m_2$ The mass of the coin.
$d$ The radial distance from the coin center to the axis of rotation.
$\mu_s$ The coefficient of static friction between the surfaces of the disk and the coin.
$v$ The (translational) speed of the coin relative to the ground, before any torque is applied.
$g$ The local gravitational free-fall acceleration magnitude.
**Assumptions:**

**Disk** We assume that the disk is perfectly balanced on its axle, so that there is no “wobble” or vibration as it rotates.

**Coin** We assume that the coin is small enough to model as a point mass on the surface of the large disk.

**Surfaces** We assume that the given coefficient of static friction applies equally in all horizontal directions for the disk—that its surface has no “grain” or bias. We assume that the given coefficient of static friction applies equally in all horizontal directions for the coin—that its surface has no “grain” or bias.

**Torque** We assume that the sudden application of the steady torque causes no wobble or vibration.

**Speed** We assume that the given speed \( v \) is less than the maximum steady speed at which the coin could ride without slipping.

**Visual Rep(s):**

![Diagram](side_view.png)

**Equations:**

I. \[ \tau_{\text{max}} = I_{\text{total}} a_{\text{max}} \]

II. \[ I_{\text{total}} = (1/2)m_1 r_1^2 + m_2 d^2 \]

III. \[ v = d\omega \]

IV. \[ |a_R|_{\text{max}} = d\omega^2 \]

V. \[ |a_T|_{\text{max}} = d\alpha_{\text{max}} \]

VI. \[ a_{\text{max}} = \sqrt{(|a_R|_{\text{max}}^2 + |a_T|_{\text{max}}^2)} \]

VII. \[ F_{S,12}^{\text{max}} = m_2 a_{\text{max}} \]

VIII. \[ F_{S,12} = \mu_{S} F_{N,12} \]

IX. \[ F_{N,12} = m_2 g \]

X. \[ \omega = (0) + \alpha_{\text{max}} (\Delta t_{\text{stop,min}}) \]
Solving:

Solve II for $I_{total}$. Substitute that result into I.
Solve III for $\omega$. Substitute that result into IV and X.
Solve IV for $a_{R\text{max}}$. Substitute that result into VI.
Solve IX for $F_{N,12}$. Substitute that result into VIII.
Solve VIII for $F_{S\text{max}}$. Substitute that result into VII.
Solve VII for $a_{max}$. Substitute that result into VI.
Solve V for $\alpha_{max}$. Substitute that result into I and X.
Solve I for $t_{max}$.
Solve X for $D_{t\text{stop.min}}$.

Testing:

Dimensions: $\tau_{\text{max}}$ should have dimensions of force-length.
$\Delta t_{\text{stop.min}}$ should have dimensions of time.

Dependencies:

If $m_1$ were greater, then with all other variables the same, this would not change $a_{max}$, thus not affect $\alpha_{max}$. But it would produce a larger $I_{total}$.
So: $\tau_{\text{max}}$ would be greater, but $\Delta t_{\text{stop.min}}$ would not change.

If $r_1$ were greater, then with all other variables the same, this would not change $a_{max}$, thus not affect $\alpha_{max}$. But it would produce a larger $I_{total}$.
So: $\tau_{\text{max}}$ would be greater, but $\Delta t_{\text{stop.min}}$ would not change.

If $m_2$ were greater, then with all other variables the same, this would not change $a_{max}$, thus not affect $\alpha_{max}$. But it would produce a larger $I_{total}$.
So: $\tau_{\text{max}}$ would be greater, but $\Delta t_{\text{stop.min}}$ would not change.

If $d$ were greater, then with all other variables the same, this would produce smaller values for $\omega$ and for $|a_{T\text{max}}|$, which would allow a greater value for $|a_{T\text{max}}|$ (since $F_{S\text{max}}$ is unchanged), thus a greater $\alpha_{max}$. So:
$\tau_{\text{max}}$ would be greater, and $\Delta t_{\text{stop.min}}$ would be smaller.

If $\mu_S$ were greater, then with all other variables the same, this would mean $F_{S\text{max}}$ would be greater, allowing a greater $a_{max}$. This would allow a greater $|a_{T\text{max}}|$, thus a greater $\alpha_{max}$. So:
$\tau_{\text{max}}$ would be greater, and $\Delta t_{\text{stop.min}}$ would be smaller.

If $v$ were greater, then with all other variables the same, this would produce greater values for $\omega$ and for $|a_{T\text{max}}|$, which would require a smaller “allowance” for $|a_{T\text{max}}|$ (since $F_{S\text{max}}$ is unchanged), thus a lesser $\alpha_{max}$. So:
$\tau_{\text{max}}$ would be smaller, and $\Delta t_{\text{stop.min}}$ would be greater.

If $g$ were greater, then with all other variables the same, this would mean $F_{S\text{max}}$ would be greater, allowing a greater $a_{max}$. This would allow a greater $|a_{T\text{max}}|$, thus a greater $\alpha_{max}$. So:
$\tau_{\text{max}}$ would be greater, and $\Delta t_{\text{stop.min}}$ would be smaller.
3. a. Early peoples exploited the effects of Newton’s laws when they discovered a fulcrum could be used for mechanical advantage. To lift a large 200-kg-boulder, a 3-m-long board is setup with a fulcrum. If a person weighing 750 N stands on the opposite end, where along the board (d) must the fulcrum be placed to lift the boulder. Assume the mass of the board is negligible and the maximum force the person can apply is simply his (resting) weight.

\[ \Sigma \tau = I \alpha = 0 \]

\[ [(200)(9.80)(d) \sin 90^\circ] - [(750)(3 - d) \sin 90^\circ] = 0 \]

\[ 1960d - (2250 - 750d) = 0 \]

\[ 2710d - 2250 = 0 \]

\[ d = 2250/2710 = 0.830 \text{ m} \]

(That’s 83.0 cm from the end holding the boulder.)

b. A long, narrow, heavy but uniform board rests on the ground. To just lift one end off the ground with a vertically-directed force while the other end stays on the ground, you effectively have to lift half the board’s weight. You continue to lift the one end of the board until it makes a 40° angle with the ground. If your force at this point is perpendicular to the board itself, how much force (expressed as a fraction of the board’s weight) must you now supply to hold the board at that angle?

\[ \Sigma \tau = I \alpha = 0 \]

\[ (F)(L) \sin 90^\circ - (mg)(L/2) \sin 90^\circ = 0 \]

\[ FL - mgL/2 = 0 \]

\[ F = mg/2 \]

(c. Exercise 30, page 349.

The 1.00-kg board is 2.00 m long. The 4.00-kg block is 1.00 m long. Find the distance d locating the fulcrum to balance the situation shown.

The center of mass of the board is located 1.00 m from the left end; that’s (d – 1.00) m from the fulcrum. The center of mass of the block is located 1.50 m from the left end; that’s (1.50 – d) m from the fulcrum.

\[ \Sigma \tau = I \alpha = 0 \]

\[ [(1.00)(9.80)(d - 1.00) \sin 90^\circ] - [(4.00)(9.80)(1.50 - d) \sin 90^\circ] = 0 \]

\[ 9.80d - 9.80 - (58.8 - 39.2d) = 0 \]

\[ 49d - 68.6 = 0 \]

\[ d = 68.6/49 = 1.40 \text{ m} \]
3. **d.** This uniform, 10-kg board is 4.00 m long and is at rest. Find the distance from its fulcrum to its right end.

The board is uniform, so its center of gravity is the geometric center of the board (midway between ends). The fulcrum is located at an unknown distance \( l \) from the board’s right end. And the board is in (static) mechanical equilibrium: \( \Sigma \tau = 0 \)

Sum the torques about the fulcrum axis:

\[
\Sigma \tau = l \alpha \\
F_G(2-\ell) - 49\sin30^\circ(l) = l \alpha \\
mg(2-\ell) - 49\sin30^\circ(l) = 0 \\
2mg - lmg - 49\sin30^\circ(l) = 0 \\
2mg = lmg + 49\sin30^\circ(l) \\
2mg/(mg + 49\sin30^\circ) = l = 1.60 \text{ m}
\]

\[
\text{e. A uniform board has a mass of 35 kg and sits horizontally at rest, supported as shown, by two scales that measure vertical force only (but the scales can exert both vertical and horizontal forces). The force shown is 107 N and is applied directly over scale B. The reading on scale B is 294 N. How long is the entire board?}
\]

Find center of board (i.e. cg): sum torques about the point above scale B.

\[
\Sigma F_y = ma_y \\
F_{A_y} + F_{B_y} - F_y - F_G = ma_y \\
F_{A_y} + 294 - 107\sin\theta - mg = 0 \\
F_{A_y} = 141.66 \text{ N}
\]

\[
\Sigma \tau_B = l_B \alpha_B \\
F_G(\sin90^\circ)l_G - F_{A_y}(\sin90^\circ)l_A = l_B \alpha_B \\
mg(l_G) - 141.66(4.26) = 0 \\
l_G = 1.759 \text{ m}
\]

Board length: \( L = 2(4.26 + 1.35 - l_G) = 7.70 \text{ m} \)

\[
\text{f. A uniform beam, sitting at rest, has a mass of 53 kg and length of 8.6 m. It is supported as shown, by two scales that measure vertical force only (but the scales can exert both vertical and horizontal forces). The force shown is 170 N and is applied at the point shown. The reading on scale A is 294 N. How far is scale B from the right end?}
\]

Sum torques about \( A \):

\[
\Sigma F_y = ma_y \\
F_{A_y} + F_{B_y} - F_y - F_G = ma_y \\
F_{B_y} + 294 - 170\sin\theta - mg = 0 \\
F_{B_y} = 364.66 \text{ N}
\]

\[
\Sigma \tau_A = l_A \alpha_A \\
-F_G[(8.6/2) - 1.53] - 170(\sin55^\circ)(4.62) + F_{B_y}l_B = 0 \\
-mg(2.77) - 170(\sin55^\circ)(4.62) + 364.66l_B = 0 \\
l_B = 5.710 \text{ m} \\
d_B = 8.6 - (l_B + 1.53) = 1.36 \text{ m}
\]
3. g. A uniform board of mass $m$ is propped between two parallel, vertical walls that are separated by a distance $d$, as shown here in this “edge-on” view. Wall 1 is frictionless, but Wall 2 has static friction (coefficient $\mu_s$) with the board. The board length, $L$, is the maximum length possible so that the board does not slip. Let $F_{N1}$ and $F_{N2}$ denote the normal forces exerted by each wall, respectively, on the board.

Evaluate (T/F/N) each statement. Justify your answers fully with any valid mix of words, drawings and calculations.

(i) $F_{N1} = F_{N2}$

True. FBD analysis of the board...

But:

$F_{N1} = F_{N2}$ (from above)

So:

$\mu_s F_{N1} = mg$

(ii) $\mu_s F_{N1} = mg$

True. FBD analysis of the board...

But:

$F_{N1} = F_{N2}$ (from above)

So:

$\mu_s F_{N1} = mg$

(iii) $mgd/2 = (F_{N1})\sqrt{(L^2 - d^2)}$

True. Sum the torques around the lower end of the board:

But:

$(L/2)\cos \theta = d/2$

And:

$(L)\sin \theta = \sqrt{(L^2 - d^2)}$

So:

$mgd/2 - (F_{N1})\sqrt{(L^2 - d^2)} = 0$
4. a. Blocks with masses $m_1 = 30.0 \text{ kg}$ and $m_2 = 40.0 \text{ kg}$ are connected by a string (with negligible mass) that passes over a pulley. The pulley is a solid disk that has a radius of 6.00 cm and a mass of 10.0 kg.

The string is taut at all times, and it does not slip as it passes over the pulley. The inclined ramp is frictionless.

Find the magnitude of the acceleration of $m_1$.

*Be sure to do a FBD and use Newton's 2nd Law for each of the three masses!* 

Before starting the problem, notice: Since the string links all three objects, for convenience, you should choose the positive direction for each object's presumed acceleration to correspond to the assumed acceleration of the string. For example, if you assume that $m_1$ will be accelerating downward, choose that direction to be positive for $m_1$; and so the direction of positive acceleration for $m_2$ should be up the slope; and the direction of positive angular acceleration for the pulley ($M$) should be counter-clockwise.

For $m_1$: 
\[ \Sigma F_y = m_1 \alpha \]
\[ m_1 g - F_{T1} = m_1 \alpha \]

For $m_2$: 
\[ \Sigma F_x = m_2 \alpha \]
\[ F_{T2} - m_2 g \sin(15°) = m_2 \alpha \]

For $M$: 
\[ \Sigma \tau = I \alpha \]
\[ (\text{but } \alpha = \alpha_y/R) \]
\[ F_{T1}(R) - F_{T2}(R) = (MR^2/2)(\alpha_y/R) = (MR \alpha_y/2) \]

Or:
\[ F_{T1} - F_{T2} = Ma/2 \]

Now, because you chose the axes carefully, $\alpha_y = a_x = a_T = a$. If you substitute this fact into the above three equations, you’ll have three equations (shown at right), ready to solve for $a$.

Solve the first two equations for $F_{T1}$ and $F_{T2}$, respectively, (shown at right), and substitute those results into the third equation.
\[ m_1 g - m_1 a - [m_2 a + m_2 g \sin(15°)] = Ma/2 \]

Solve for $a$:
\[ a = g[m_2\sin(15°) - m_1]/(-m_1 - m_2 - M/2) \]
\[ = (9.8)[(40)\sin(15°) - (30)]/(-30 - 40 - 10/2) = 2.57 \text{ m/s}^2 \]
4. b. A fish is hooked on a fishing line and is pulling on it with a constant force, trying to escape. The line is wound around a reel, which is a solid disk (r = 0.213 m; m = 654 kg). At first, the reel is held motionless by the fisherman, who exerts a perpendicular force, \( F \), of 98.7 N on the 1.02 m (massless) handle attached to the hub of the reel, as shown. Suddenly, the entire handle breaks off of the reel, allowing it to spin freely in response to the fish’s pull. Assuming the fish maintains the same steady tension in the line as it swims away, how far will it have swum (directly away from the reel) when the reel’s angular speed is 54.3 rad/sec?

This problem has two parts: before and after the handle breaks. In each part, you analyze the reel as the free body. Notice that at all times, the reel is in translational equilibrium—it’s not accelerating up/down or sideways. But only in the first part is the reel in rotational equilibrium; after the handle breaks, the tension in the fishing line (which is then the only force causing any torque about the reel’s axis—all other forces go through that axis). This causes an angular acceleration of the reel.

**Strategy**: Use the first part to determine the tension, \( F_T \), caused by the fish. Then use that tension (maintained constantly by the fish after the handle breaks) to find the resulting angular acceleration, \( \alpha \). Finally, use that constant \( \alpha \) value in a kinematics calculation to find out how much line has been paid out as the fish swims away.

**Part 1**: \( \Sigma \tau = 0 \)

\[
F \cdot d_{\text{handle}} \sin(90^\circ) - F_T R \sin(90^\circ) = 0 \quad \text{(where R is the radius of the reel).}
\]

Solve this for \( F_T \)

\[
F_T = \frac{(F \cdot d_{\text{handle}})}{R}
\]

**Part 2**: \( \Sigma \tau = I \alpha \)

\[
- F_T R = \frac{(MR^2)}{2} \alpha \quad \text{Solve this for } \alpha ....
\]

\[
\alpha = \frac{-2F_T}{MR} \quad \text{(Keep just the magnitude—let the outbound direction be positive.)}
\]

Substitute from Part 1:

\[
\alpha = \frac{2(F \cdot d_{\text{handle}})}{(MR^2)}
\]

Now do kinematics. You know 3 values: \( \omega_0 (= 0) \), \( \omega (= 54.3) \) and \( \alpha \) (just solved for).

Use \( \omega^2 = \omega_0^2 + 2\alpha \Delta \theta \), rearranged to solve for \( \Delta \theta \):

\[
\Delta \theta = \frac{(\omega^2 - \omega_0^2)}{2\alpha}
\]

And the amount of line paid out is just \( \Delta s = R \Delta \theta = \frac{R(\omega^2 - \omega_0^2)}{2\alpha} \).

Substitute for \( \alpha \):

\[
\Delta s = R(\omega^2 - \omega_0^2)/[2(2(F \cdot d_{\text{handle}})/(MR^2))]
\]

\[
= \frac{MR^2(\omega^2 - \omega_0^2)}{2(4(F \cdot d_{\text{handle}}))}
\]

\[
= (654)(.213^3)[(54.3)^2 - (0)^2]/[4(98.7)(1.02)] = 46.3 \text{ m}
\]
4. c. A small child amuses himself by repeatedly slamming his bedroom door, which has a total moment of inertia (about its hinged edge) of \( I \). Each time, he starts with the door at rest—open at the same angle \( \theta \), as shown—then exerts the same steady push \( F \) at the same point, a distance \( d \) from the door’s hinged edge (and he follows the swinging door, so his push is always at right angles to the door) all the way until it slams shut.

Of course, this delights him but annoys everyone else—and it soon damages the door. So now the knobs/lock set (of mass \( m \), located at a distance \( L \) from the hinged edge), must be removed from the door.

(i) How far from the hinged edge must the child now push on the door (again, always perpendicularly) with the same force \( F \) as before (again, starting with the door at rest at angle \( \theta \)) to get the same impact speed?

(ii) Comparing the two scenarios (whole door vs. door without the knobs/lock):

What is the difference in the speed of the child’s hand (which is still pushing on the door) at the moment of impact?

Ignore air drag and hinge friction.

Here are the data: \( F = 20.0 \) N; \( I = 5.00 \) kg·m\(^2\); \( \theta = 60^\circ \); \( d = 65.0 \) cm; \( m = 2.00 \) kg; \( L = 80.0 \) cm.

(i) If the kid wants the same impact speed (\( \omega_f \)) in either scenario (call them 1 and 2), and in both cases he starts the door at rest (\( \omega_i = 0 \)) and swings it through the same angular displacement (\( \Delta \theta \)), then by simple kinematics (\( \omega_f^2 = \omega_i^2 + 2\alpha\Delta \theta \)), he needs to accomplish the same angular acceleration: \( \alpha_1 = \alpha_2 \)

But for the door as a rotating object, \( \tau_{net} = \tau \alpha \). Therefore: \( \tau_{net,1}/I_1 = \tau_{net,2}/I_2 \)

Since the child’s pushing force, \( F \), is the same in either case: \( (F)(d)\sin90^\circ/I_1 = (F)(x)\sin90^\circ/I_2 \)

Simplifying: \( x = d(I_2/I_1) \)

But: \( I_2 = I_1 - mL^2 \)

Therefore: \( x = d[(I_1 - mL^2)/I_1] = (0.65)[5 - (2)(0.80)^2)/5] = 0.484 \) m

(ii) At impact, the speed of the child’s hand is given by \( v_{rf} = r\omega_f \).

So now calculate \( \omega_f \) using the data from the starting scenario (and the resulting \( \omega_f \) must be the same in either case, as specified above): \( \omega_f^2 = \omega_i^2 + 2\alpha\Delta \theta \), where \( \alpha = Fd/I_1 \)

Therefore: \( \omega_f = \sqrt{2Fd\Delta \theta/I_1} = \sqrt{(2)(20)(0.65)(\pi/3)/5} = 2.334 \) rad/s

And now find the difference between the two impact speeds:

\( \Delta v_T = v_{rf2} - v_{rf1} = x\omega_f - d\omega_f = (0.4836)(2.334) - (0.650)(2.334) = -0.388 \) m/s

When slamming the door without the knobs/lock assembly, the child’s hands are moving 0.388 m/s more slowly at impact.
4. d. A solid, rigid, uniform disk of mass $M$ and outer radius $R$ is positioned horizontally over level ground and attached to a thin, frictionless, fixed vertical axle shaft at point $X$, as shown in this overhead view. (The axle shaft is planted vertically in the ground.)

Point A on the disk is located as shown, at a distance $d$ from the center of the disk. An additional point mass $m$ is located there.

Point B is located on the rim of the disk diametrically opposite to the rotation axis $X$. No extra mass is located there.

With the disk initially at rest, a [steady] torque $\tau$ is applied to it (about point $X$) for a time interval $\Delta t$. Then $\tau$ is removed and the disk is allowed to spin freely about its axle at point $X$.

You may ignore all friction and air resistance. (And the details of how the torque $\tau$ is applied are not needed. That is, you do not need to assume any particular point on the disk where an external force is being applied to produce the torque $\tau$. The value of the torque is sufficient.)

Here are the values you may treat as known: $M$, $R$, $d$, $m$, $\tau$, $\Delta t$

Your answers to each of the following should be an expression containing only known values. (Or, if you don’t want to do the algebra, you may instead list each equation you need—in order—and describe what to solve for in each (much like the Equations and Solve steps in the ODAVEST protocol).

(i) How many revolutions does the disk rotate in the time interval $\Delta t$?

$\tau$ is a steady (net) torque, so it will produce a constant acceleration, $\alpha$: $\Sigma \tau = I \alpha$

So constant-$\alpha$ angular kinematics applies here:

A useful equation: $\Delta \theta = \omega_i \Delta t + (1/2)\alpha(\Delta t)^2$

Where $\omega_i = 0$ (disk starts from rest)

And $\alpha = \tau/I_X$ ($I_X$ is the total $I$ about point $X$)

In other words: $\Delta \theta = \tau(\Delta t)^2/(2I)$

To find $I_X$:

$I_{\text{disk,X}} = I_{\text{disk,center}} + M_{\text{disk}} (R^2)$ (by the Parallel Axis Theorem)

$= (1/2)MR^2 + MR^2$

$= (3/2)MR^2$

$I_{\text{m,X}} = mR_m^2$ (what any point mass contributes to $I$)

$= m(R^2 + d^2)$

Thus: $I_X = (3/2)MR^2 + m(R^2 + d^2)$

Put it all together: $\Delta \theta = \tau(\Delta t)^2/(2[(3/2)MR^2 + m(R^2 + d^2)])$

Convert rad to rev: $\Delta \theta_{\text{rev}} = \Delta \theta/(2\pi) = \tau(\Delta t)^2/(4\pi[(3/2)MR^2 + m(R^2 + d^2)])$
(ii) Find the magnitude of the net force acting on the point mass \( m \) at \( A \) at the very end of the time interval \( \Delta t \) (while \( \tau \) is still being exerted).

A bit more kinematics to determine the final angular speed of the disk:

\[
\omega_f = \omega_i + \alpha \Delta t
\]

But \( \omega_i = 0 \), so:

\[
\omega_f = \alpha \Delta t
\]

where \( \alpha = \tau/[(3/2)MR^2 + m(R^2 + d^2)] \)

Newton’s 2nd Law:

\[
F_{net} = ma = m(a_T + a_R)
\]

Magnitudes:

\[
F_{net} = ma = m\sqrt{(a_T^2 + a_R^2)}
\]

But:

\[
a_T = r_m \alpha
\]

where \( \alpha = \tau/[(3/2)MR^2 + m(R^2 + d^2)] \)

And:

\[
a_R = r_m \omega_f^2 = \sqrt{(R^2 + d^2)}(\alpha \Delta t)^2
\]

Thus:

\[
F_{net} = m\sqrt{(r_m \alpha)^2 + (R^2 + d^2)(\alpha \Delta t)^4}
\]

Where \( \alpha = \tau/[(3/2)MR^2 + m(R^2 + d^2)] \)

(iii) After \( \tau \) is removed and the disk is spinning freely, suppose that \( m \) breaks loose from point \( A \) and rolls out to become stuck at point \( B \) instead. What is the speed of mass \( m \) now (relative to the ground)?

\[
v_{mf} = r_m \omega_f = 2R \omega_f
\]

But what is \( \omega_f \)? With no friction or other outside interference, the disk is an isolated system; conservation of angular momentum applies:

\[
L_i = L_f
\]

That is:

\[
I_i \omega_i = I_f \omega_f
\]

Or:

\[
\omega_f = I_i \omega_i / I_f
\]

So:

\[
v_{mf} = 2RI_i \omega_i / I_f
\]

From part ii:

\[
\omega_i = \alpha \Delta t = \tau \Delta t / I_i
\]

Therefore:

\[
I_i \omega_i = \tau \Delta t
\]

Thus:

\[
v_{mf} = 2R \tau \Delta t / I_f
\]

To find \( I_f \):

\[
I_f = I_{disk,x} + I_{mf,x}
\]

\[
= I_{disk,x} + mr_{mf}^2
\]

\[
= I_{disk,x} + m(2R)^2
\]

\[
= (3/2)MR^2 + 4mR^2
\]

All together:

\[
v_{mf} = 2R \tau \Delta t / [(3/2)MR^2 + 4mR^2]
\]
4. e. This is an ODAVEST item—use the full seven-step problem-solving protocol—but keep in mind that you’re not being asked to actually solve for the final expression. In fact, you’re not being asked to do any math at all—not even any algebra. Rather, for the Solve step, you are to write a series of succinct instructions on how to solve this problem. Pretend that your instructions will be given to someone who knows math but not physics. And for the Test step, you should consider the situation and predict how the solution would change if the data were different—changing one data value at a time.

A jet engine (mass \(m\)) that produces a constant thrust force is mounted for testing on the end of a pivoting arm (length = \(L\); moment of inertia = \(I_{\text{arm}}\)), so that the engine pushes itself around in a horizontal circle with its thrust force, which is perpendicular to the pivot arm, as shown here.

At first, there is a perpendicular braking force applied as shown, at the 3/4 position on the pivot arm, to allow the engine to warm up while still at rest. But when the engine has reached its full operating thrust, the brake is released, and the pivot arm and engine are allowed to rotate freely (with no friction or air resistance). If the pivot and engine turn \(n\) revolutions in the first \(t\) seconds of motion, what was the force applied by the brake?

Assume these are known values: \(m, L, I_{\text{arm}}, n, t\)

**Objective:** A jet engine of known mass is being tested by attaching it to a horizontal pivot arm of known length and moment of inertia.

The arm is held stationary by a brake, set 3/4 of the distance from the pivot to the engine, until the engine reaches its constant full thrust.

The brake is then released so that the engine can rotate the arm freely for a known time interval.

During that time interval, the arm rotates a known number of revolutions.

We must find the force exerted by the brake just before it was released.

**Data:**

\(m\) The mass of the jet engine.

\(L\) The length of the pivot arm.

\(I_{\text{arm}}\) The moment of inertia of the pivot arm, measured at its end (i.e. at the pivot point).

\(t\) The time interval the engine was freely pivoting the arm while at full thrust.

\(n\) The number of revolutions the arm and engine made during time interval \(t\).

**Assumptions:**

**Surfaces** We assume the pivot bearing is frictionless.

**Alignment** We assume that the pivot arm is exactly horizontal at all times—no bouncing or vibration.

We also assume that the brake force is exactly horizontal and exactly perpendicular to the pivot arm.

We also assume that the engine thrust force is exactly horizontal and exactly perpendicular to the pivot arm.

**Arm** We assume the pivot arm is uniform and that it pivots exactly at one end.

**Engine** We model the engine as point mass, mounted at the exact (other) end of the arm.

**Air** We disregard any effects of wind or air drag.
Visual Rep(s):  

Equations:

I. \( \Delta \theta = 2\pi n \)
II. \( \Delta \theta = \omega_0(t) + (1/2)\alpha(t)^2 \)
III. \( I_T = I_{arm} + mL^2 \)
IV. \( \tau_{engine} = I_T \alpha \)
V. \( \tau_{engine} - (F_{brake}\cdot\sin90^\circ)(0.75L) = 0 \)

Solving:  

Solve I for \( \Delta \theta \). Substitute that result into II.

Solve II for \( \alpha \) (noting that \( \omega_0 = 0 \)). Substitute that result into IV.

Solve III for \( I_T \). Substitute that result into IV.

Solve IV for \( \tau_{engine} \). Substitute that result into V.

Solve V for \( F_{brake} \).

Testing:  

Dimensions: \( F_{brake} \), should have dimensions of force (mass\cdot length/time^2).

Dependencies: If the mass, \( m \), of the engine were greater, that would imply a greater thrust capability overcoming more total moment of inertia in order to achieve the same number of revolutions in the same time. Thus, the braking force, \( F_{brake} \), would be need to be greater.

If the length, \( L \), of the arm were greater, that would imply a greater thrust capability overcoming more total moment of inertia in order to achieve the same number of revolutions in the same time. Thus, the braking force, \( F_{brake} \), would be need to be greater.

If the moment of inertia, \( I_{arm} \), were greater, that would imply a greater thrust capability overcoming more total moment of inertia in order to achieve the same number of revolutions in the same time. Thus, the braking force, \( F_{brake} \), would be need to be greater.

If the time, \( t \), spent rotating under full thrust were greater, that would imply a smaller thrust capability used to accomplish the same \( n \) revolutions. Thus, the necessary braking force, \( F_{brake} \), would also be smaller.

If the number, \( n \), of revolutions accomplished while rotating under full thrust were greater, that would imply a greater thrust capability in order to achieve this in the same time. Thus, the braking force, \( F_{brake} \), would be need to be greater.
5. a. Evaluate (T/F/N) this statement (and justify your answers fully with any valid mix of words, drawings and calculations): An object rotating at speed \( \omega \) about its center of mass has less rotational kinetic energy than if it’s rotating at that same \( \omega \) about any other parallel axis. 
   True. \( K_R \) is calculated as \((1/2)I\omega^2\), and \( I_{c.m.} < I_{axis} \) for any other parallel axis (evident by inspection of the Parallel Axis Theorem).

b. A rotating skater draws her arms in, changing her moment of inertia by \(-8.50\%\) (i.e. reducing it by \(8.50\%\)).
   (i) By what percentage does she change her rotational kinetic energy, \( K_R \)?
   (ii) Where does this extra energy come from?
   (i) First, use conservation of angular momentum \((L_i = L_f)\): \( I_1\omega_1 = I_2\omega_2 \), where \( I_1/I_2 = 0.915 \)
   Thus: \( \omega_2 = 0.915(\omega_1) \) or: \( (\omega_2/\omega_1) = 1/0.915 \)
   Then note: \( \Delta\%(K_R) = [(K_{R,f} - K_{R,i})/K_{R,i}]\cdot100 = [(K_{R,f}/K_{R,i}) - 1]\cdot100 \)
   In this case: \( K_{R,f}/K_{R,i} = [(1/2)I_2\omega_2^2]/[(1/2)I_1\omega_1^2] \)
   \[ = (I_1/I_2)(\omega_2/\omega_1)^2 = (0.915)(1/0.915)^2 = 1/0.915 \]
   Therefore: \( \Delta\%(K_R) = [(1/0.915) - 1]\cdot100 \)
   Answer: \(+9.29\%\)
   (ii) The skater uses chemical energy from her muscles to do work by drawing her arms in.

c. In a test laboratory, an electric engine is used to accelerate a wheel from rest. The center of mass of the wheel remains stationary at all times. The wheel’s angular acceleration is in the counter-clockwise direction. It rotates through an angular distance \( \Delta\theta \), achieves a final angular speed \( \omega \) and has a constant moment of inertia \( I \). If you assume constant angular acceleration and ignore friction and air resistance, what is the average power output of the engine, expressed in terms of \( \Delta\theta \), \( \omega \) and/or \( I \)?
   \( P_{mech.avg} = W/\Delta t \)
   There is no elastic source here, nor any change in altitude by the center of mass—nor any translational kinetic energy (since the c.m. is entirely stationary). So the work-energy equation for the wheel simplifies to this:
   \( K_{R,f} + W_{ext} = K_{R,i} \) And since the wheel starts from rest: \( W_{ext} = K_{R,f} \)
   So our calculation of the average power is:
   \( P_{mech.avg} = K_{R,f}/\Delta t = (1/2)I\omega^2/\Delta t = I\omega^2/(2\cdot\Delta t) \)
   A bit of kinematics reveals \( \Delta t \):
   \( \Delta\theta = (1/2)(\omega_i + \omega_f)\cdot\Delta t \)
   In this case, that reduces to:
   \( \Delta\theta = (1/2)(\omega)\cdot\Delta t \)
   Solving for \( \Delta t \):
   \( \Delta t = 2\Delta\theta/\omega \)
   Substituting into the above:
   \( P_{mech.avg} = I\omega^2/(2\cdot\Delta t) = I\omega^3/(4\cdot\Delta\theta) \)

d. One end of a thin, uniform rod, 1.40 m in length, is attached to a pivot. The rod is free to rotate about the pivot without friction or air resistance. Initially, it is hanging straight down (i.e. in the “6 o’clock” position). Then the lower tip of the rod is given an initial horizontal speed \( v_{T,i} \), so that the rod rotates upward, about its pivot. Find the value of \( v_{T,i} \) so that the rod just reaches the “12 o’clock” position (i.e. straight up), coming to a momentary halt there.
   Modeling this as a rod that is freely rotating about the fixed, frictionless axis at its end, only gravity does work on it, so \( W_{ext} = 0 \). Therefore (letting the initial position of the center of mass be \( h_i = 0 \), so that \( h_f = L \), we have:
   \( U_{G,f} = K_{R,end,i} \) or: \( MgL = (1/2)I_{pin}\omega^2 \)
   \( MgL = (1/6)ML^2\omega^2 \) or: \( MgL = (1/6)ML^2(v/L)^2 \)
   Thus: \( 6gL = v_i^2 \)
   And: \( v_i = \sqrt{6gL} = \sqrt{6(9.80)(1.40)} = 9.07 \text{ m/s} \)

e. A dead tree is initially stationary and there is a 30.0-degree angle between the tree and the vertical. The roots finally give way and the tree topples to the ground (rotating on an axis that is parallel to the level ground and intersecting the base of the tree perpendicularly through the base of the tree. The length of the tree is 30.0 m. How fast is the top of the tree moving (in m/s) just before it hits the ground? Model the tree as straight, rigid rod, and ignore friction and air resistance.
   Modeling this as a rod that is free-falling (rotating) from rest about a frictionless axis (the root ball), only gravity does work on the tree, so \( W_{ext} = 0 \). Therefore (letting \( h_f = 0 \) and noting that the center of mass of the tree is half-way along its leaning length), we have:
   \( U_{G,f} = K_{R,end,f} \) or: \( MgL/2\sin60^\circ = (1/2)I_{pin}\omega^2 \)
   \( MgL(2)\sin60^\circ = (1/6)ML^2\omega^2 \) or: \( MgL(2)\sin60^\circ = (1/6)ML^2(v_{top}/L)^2 \)
   Thus: \( 3gL\sin60^\circ = v_{top}^2 \)
   And: \( v_{top} = \sqrt{3gL\sin60^\circ} = \sqrt{3(9.80)(30)(\sin60^\circ)} = 27.6 \text{ m/s} \)
5. f. Two identical solid disks are rotating independently about the same thin, frictionless rod. That rod goes through the center of disk A, but in disk B, the rod goes through a point midway between that disk’s center and its outer edge (see overhead view here). Initially, the disks are rotating about the rod with the same angular speed but in opposite directions. Then they collide (without outside interference or any change in altitude) and they stick together as a result. Evaluate (T/F/N) each statement. Justify your answers fully with any valid mix of words, drawings and calculations.

(i) Disk B reverses its direction of rotation as a result of the collision.

False. The two disks are an isolated system, so their total angular momentum is conserved. This is a completely inelastic collision, where the two objects—always rotating on the same axis—become one. Thus:

\[ I_A w_{iA} + I_B w_{iB} = (I_A + I_B)w_f \]

Let \( w_{iA} = w_i \) (ccw) and \( w_{iB} = -w_i \) (cw)

Then:
\[ (I_A - I_B)w_i = (I_A + I_B)w_f \]

Or:
\[ w_f = w_i (I_A - I_B)/(I_A + I_B) \]

But \( I_B > I_A \), since \( I_A \) is the I_{cm} (the minimum I).

So \( w_f \) is negative—that’s in the same rotational direction as \( w_iB \).

(ii) As a result of the collision, \( \Delta L_A = -\Delta L_B \).

True. The two disks are an isolated system, so their total angular momentum is conserved.

Thus the sum of their two individual momentum changes must be zero: \( \Delta L_A + \Delta L_B = 0 \)

(iii) About 96.0% of the disks’ total initial mechanical energy is lost as a result of the collision.

True. There was only \( K_R \) in the system (no other \( E_{mech} \)).

And:
\[ \Delta % K_R = 100(K_{Rf}/K_{Ri} - 1) \]

From part 2A:
\[ w_f = w_i (I_A - I_B)/(I_A + I_B) \]

Then note:
\[ I_{cm} = I_A = (1/2)MR^2 \quad I_B = I_{cm} + M(R/2)^2 = (3/4)MR^2 \]

(Parallel Axis Theorem)

Therefore:
\[ w_f = w_i[-(1/4)MR^2]/[(5/4)MR^2] = -(1/5)w_i \]

\[ K_{Rf} = (1/2)I_A w_f^2 = (1/2)[(5/4)MR^2][(1/5)w_i]^2 = 0.025MR^2 w_i^2 \]

\[ K_{Ri} = (1/2)I_A w_i^2 + (1/2)I_B w_i^2 = (1/2)(5/4)MR^2 w_i^2 = 0.625MR^2 w_i^2 \]

\[ \Delta % K_R = 100(0.025/0.625 - 1) = -96.0\% \]
5. g. A merry-go-round (carousel) with a moment of inertia of 594 kg·m², is initially rotating horizontally at 2 rad/s about a frictionless axis through its center. Then a metal plate located at that center loosens and gradually slides outward along the floor of the merry-go-round. When the plate is 1.4 m from the center, the merry-go-round has lost 1% of its original rotational speed. Assuming that all mechanical energy losses are due to kinetic friction between the plate and floor, find that coefficient of kinetic friction. (Ignore the slow speed of the plate’s sliding.)

Find the reduced angular speed:
\[ \omega_f = 0.99\omega_i = 0.99(2) = 1.98 \text{ rad/s} \]

Angular momentum is conserved:
\[ I_1\omega_i = I_f\omega_f \]
Or:
\[ I_1/\omega_i = \omega_f/\omega_i = 0.99 \]
Or:
\[ I_1/0.99 = I_f = 594/0.99 = 600 \text{ kg·m}^2 \]
So the plate has contributed an extra 6 kg·m² in its new position (modeled as a point mass): \( mr^2 = 6 \)
The mass of the plate is thus given by:
\[ mr^2 = 6/r^2 = 6/(1.40)^2 = 3.061 \text{ kg} \]
The carousel floor is level and no other vertical forces act on the plate.

Therefore:
\[ F_N = F_G = mg \]
Now start with the basic equation:
\[ E_{mech,f} = E_{mech,i} + W_{ext} \]
There is no spring present, and gravity is not an energy factor.
The external work is by the floor on the plate:
\[ W_{ext} = |F_k\cos\theta|dL = -F_k(d) = -\mu_kF_N(d) = -\mu_kmgd \]
Include non-zero energy terms only:
\[ K_{R,i} = K_{R,f} + W_{ext} \]
The details:
\[ (1/2)I_f\omega_f^2 = (1/2)I_i\omega_i^2 - \mu_kmgd \]
Solve for \( \mu_k \):
\[ \mu_kmgd = (1/2)I_f\omega_f^2 - (1/2)I_i\omega_i^2 \]
\[ \mu_k = [(1/2)I_i\omega_i^2 - (1/2)I_f\omega_f^2] / (mgd) = \text{0.283} \]

h. A rigid, massless rod connects three point masses as shown here. Doing a total of 192 J of work on this object (which is initially at rest in your hand), you throw it across a level field, launching it so that its center of mass leaves your hand at an angle of 34.0° above the horizontal. At the peak of its arc as a projectile, the object’s center of mass has gained 7.00 m in altitude (i.e. above its altitude when released).

(i) Assuming that this projectile maintained its rotational speed throughout the upward portion of its arc, what was that \( \omega \) value?

Now suppose that, during the downward portion of the object’s arc, \( m_2 \) slides 10.0 cm toward \( m_1 \).

(ii) What is the rotational speed now—with \( m_2 \) in its new position?

(iii) And now what is the speed of \( m_1 \) as measured from the object’s axis of rotation (i.e. what is \( v_{r1} \) around the axis of rotation)?

Ignore any effects of air drag and wind, and assume a constant local g value of 9.80 m/s².

(i) Unconstrained, a rigid object will rotate around its center of mass. So, set an x-axis origin at the left end of the object (i.e. at \( m_1 \)), and calculate \( x_{cm} \) for this object:
\[ x_{cm} = [(500)(0) + (.200)(.350) + (.150)(.550)] / (.500+.200+.150) = 0.17941 \text{ m} \]
Next, calculate \( I_{cm} \), the moment of inertia for rotation about the center of mass:
\[ I_{cm} = [(500)(.17941)^2 + (.200)(.350–.17941)^2 + (.150)(.550–.17941)^2] = 0.042515 \text{ kg·m}^2 \]
Now analyze the energy of the projectile (i.e. after the work has been done) from launch to peak:

Only gravity acts on it, so its mechanical energy is unchanged. Setting \( U_{G,i} = 0 \) at launch level:
\[ (1/2)mv_i^2 + (1/2)I\omega_i^2 = (1/2)mv_f^2 + (1/2)I\omega_f^2 + mgh_f \]
But:
\[ \omega_f = \omega_i \quad \text{And:} \quad v_f = v_i\cos 34° \quad \text{So:} \quad (1/2)mv_f^2 = (1/2)m(v_i\cos 34°)^2 + mgh_f \]
Simplify:
\[ v_f^2 = v_i^2\cos^2 34° + 2gh_f \quad \text{So:} \quad v_f = \sqrt{2gh_f / [1 - \cos^2 34°]} = 20.947 \text{ m/s} \]
Now analyze the energy “loading” of the system—the object (at \( U_{G,i} = 0 \)) as launched:
\[ W_{ext} = (1/2)mv_i^2 + (1/2)I\omega_i^2 = \sqrt{(2W_{ext} - mv_i^2)/I} = \sqrt{(2(192) - (0.85)(20.947)^2)/0.042515} = \text{16.1 rad/s} \]
(ii) Re-calculate for after \( m_2 \) shifts, when the object looks like this:
\[ x_{cm} = [(200)(.450) + (.150)(.550)] / (.500+.200+.150) = 0.20294 \text{ m} \]
\[ I_{cm} = [(500)(.20294)^2 + (.200)(.450–.20294)^2 + (.150)(.550–.20294)^2] = 0.050868 \text{ kg·m}^2 \]
Conserve angular momentum:
\[ I_1\omega_i = I_f\omega_f \]
\[ \omega_f = \omega_i(I/I_f) = (16.1)(.042515/0.050868) = \text{13.5 rad/s} \]
(iii) \( v_{r1} = r_f\omega_f = (0.20294)(13.5) = 2.73 \text{ m/s} \]
A certain baton consists of a thin, rigid, massless rod, 1.00 m long with a different point mass attached to each end ($m_1 = 1.00 \text{ kg}$; $m_2 = 4.00 \text{ kg}$). You may assume normal local $g_{\text{earth}}$.

(i) Initially, the rod is pinned at its center to a fulcrum supported by level ground. A force $F$ is applied as shown, at a point midway between $m_1$ and the fulcrum, so that the rod stays horizontal and in static mechanical equilibrium. Find the magnitude of $F$.

Use a diagram, and sum the torques about the pin:

\[ \sum \tau = I \alpha \]

\[
(F_{G,1})(l_1)(\sin \theta_1) + (F)(l)(\sin \theta) - (F_{G,2})(l_2)(\sin \theta_2) = I \alpha
\]

\[
(m_1g)(l_1) + (F)(l)(\sin 70^\circ) - (m_2g)(l_2) = 0
\]

Now just solve for $F$:

\[
F = [(m_2g)(l_2) - (m_1g)(l_1)]/[(l)(\sin 70^\circ)]
\]

\[
= [(4.00)(9.80)(0.50) - (1.00)(9.80)(0.50)]/[(0.25)(\sin 70^\circ)] = 62.6 \text{ N}
\]

(ii) Now $F$ is removed, and the baton is removed from fulcrum. It is then hurled into the air so that it rotates freely at 300 rpm. Find its rotational kinetic energy.

An object rotating freely will do so about its center of mass.

First, compute that position along the rod. Define the origin ($x = 0$) at the rod’s left end:

\[
x_{c.m.} = (1/M) \sum m_i x_i
\]

\[
= [1/(m_1 + m_2)] \{(m_1)(x_1) + (m_2)(x_2)\}
\]

\[
= [1/(1.00 + 4.00)] \{(1.00)(0) + (4.00)(1.00)\} = 0.800 \text{ m}
\]

Now, the baton’s rotational kinetic energy, $K_{R,c.m.}$, computed around its center of mass is:

\[
K_{R,c.m.} = (1/2) I_{c.m.} \omega^2
\]

Where: \(
\omega = (300 \text{ rev/min})(1 \text{ min}/60 \text{ s})(2\pi \text{ rad/rev}) = 10\pi \text{ rad/s}
\)

And:

\[
I_{c.m.} = \sum m_i r_i^2
\]

\[
= [(m_1)(r_1)^2 + (m_2)(r_2)^2]\]

\[
= [(1.00)(0.800)^2 + (4.00)(0.200)^2] = 0.800 \text{ kg\cdot m}^2
\]

Thus:

\[
K_{R,c.m.} = (1/2)(0.800)(10\pi)^2
\]

\[
= 395 \text{ J}
\]
5. Rigid, massless rods connect three point masses as shown in this overhead view. This object is set at rest on level, frictionless ice, then pinned at $m_2$ to that ice so that it can rotate freely (without friction, in a horizontal plane) around $m_2$. Then a torque of 87.0 N·m is applied to the object for just 4.00 s (at which time the torque ceases). Ignore wind/air drag.

(i) At what translational speed does the object’s center of mass now move?

(ii) If the pin could then somehow pull in (shorten—like an antenna or telescope) the 3-m rod so that its new length was 1.50 m, how much work would need to be done by the pin to accomplish this?

(i) First, find the moment of inertia around the fixed pin: 

$$I_{pin} = \sum m_i r_i^2 = (6)2^2 + (9)3^2 = 105 \text{ kg} \cdot \text{m}^2$$

Now find the rotational speed after the torque is finished—using either 

$$\sum \tau_{pin} = I_{pin} \omega_{pin} \quad \text{or} \quad \Delta L_{pin} = \tau_{pin, net} (\Delta t)$$

In this case, $\omega_i = 0$, so we have: 

$$\omega_f = \frac{\tau_{pin, net} (\Delta t)}{I_{pin}} = \frac{(87)(4)}{105} = 3.314 \text{ rad/s}$$

Now find the center of mass (using the pin as the origin):

$$x_{c.m.} = \frac{(\sum m_i x_i)}{M} = \frac{[(6)(0) + (10)(0) + (9)(3)]}{(6 + 10 + 9)} = 1.08$$

$$y_{c.m.} = \frac{(\sum m_i y_i)}{M} = \frac{[(6)(2) + (10)(0) + (9)(0)]}{(6 + 10 + 9)} = 0.48$$

So: The distance $r_{c.m.}$ from the pin to the c.m. is:

$$r_{c.m.} = \sqrt{(1.08)^2 + (0.48)^2} = 1.182 \text{ m}$$

Therefore:

$$v_{c.m.} = r_{c.m.} \omega_f = (1.182)(3.314) = 3.92 \text{ m/s}$$

(ii) Find the new moment of inertia around the fixed pin:

$$I_{pin, new} = \sum m_i r_i^2 = (6)2^2 + (9)(1.50)^2 = 44.25 \text{ kg} \cdot \text{m}^2$$

Now find the new rotational speed, using conservation of angular momentum as the pin reels in the mass:

$$I_{pin, old} \omega_{old} = I_{pin, new} \omega_{new} \quad \text{That is:} \quad \omega_{new} = \frac{I_{pin, old} \omega_{old}}{I_{pin, new}} = \frac{(105)(3.314)}{44.25} = 7.864 \text{ rad/s}$$

The change in $K_R$ here is the work done in pulling the mass inward:

$$W_{ext} = K_{R_f} - K_{R_i} = \frac{1}{2} I_{pin, new} \omega_{new}^2 - \frac{1}{2} I_{pin, old} \omega_{old}^2$$

$$= \frac{1}{2}(44.25)(7.864)^2 - \frac{1}{2}(105)(3.314)^2 = 792 \text{ J}$$
5. k. A long thin, rigid, uniform rod of mass $M$ and length $L$ has one end attached at ground level to a frictionless pivot. The rod is initially upright and at rest, as shown. Then it falls freely (with no push) onto level ground, as shown.

You may consider $M$, $L$, and local $g$ to be known values.

Evaluate (T/F/N) each statement. Justify your answers fully with any valid mix of words, drawings and calculations.

(i) During its fall, the rod’s center of mass has the same angular speed, $\omega$, as its free end.  

True. The angular motion (position $q$, velocity $w$, and acceleration $a$) is the same for every non-axis point on any rigid rotating body.

(ii) When the rod hits the ground, the impact speed ($v_{\text{impact,end}}$) of the free end is:  

$v_{\text{impact,end}} = \sqrt{2gL}$

False. An energy analysis is easiest here.

The basic equation: $E_{\text{mech,f}} = E_{\text{mech,i}} + W_{\text{ext}}$

Itemized: $K_{T,f} + K_{R,f} + U_{G,f} + U_{S,f} = K_{T,i} + K_{R,i} + U_{G,i} + U_{S,i} + W_{\text{ext}}$

Simplifications:

$W_{\text{ext}} = 0$ (No friction or push done on rod.)
$U_{S,f} = U_{S,i} = 0$ (No ideal elastic source in this problem.)
$U_{G,f} = 0$ (Assign ground level as $h = 0$ for $U_{G}$.)
$K_{T,f} = K_{T,i} = 0$ (The axis is fixed, so this motion can be calculated as purely rotational energy.)
$K_{R,i} = 0$ (Rod starts at rest.)

Therefore: $K_{R,f} = U_{G,i}$

The details: $(1/2)I_{\text{end}}\omega_f^2 = Mgh_i$

Known values: $h_i = L/2$ (Gravity acts at the rod’s center of mass, located at its geometric center—it’s uniform.)

$I_{\text{end}} = (1/3)ML^2$ (Given—a known formula.)

$v_{\text{impact,end}}/L = \omega_f$ ($v_T = r\omega$ gives the speed of any moving point on a rigid rotating body.)

Substituting: $(1/2)\left[(1/3)ML^2\right](v_{\text{impact,end}}/L)^2 = MgL/2$

Simplifying: $[(1/3)](v_{\text{impact,end}})^2 = gL$

Solve for $v_{\text{impact,end}}$: $v_{\text{impact,end}} = \sqrt{3gL}$
5. 1. A rigid beam (mass = 400 kg; length = 12.0 m) has a pulley (mass = 30.0 kg; outer radius = 1.50 m) mounted at its upper end; and the beam’s lower end is anchored with a pin to a wall. The beam is also supported by a horizontal tension cable connected to the wall, as shown here. The distance along the beam from the pulley axle to the cable attach point is 3.50 m.

A strong, thin cord has been wrapped several times around the pulley and then connected to a block ($m_3 = 100$ kg)—which hangs at rest so long as the pulley is secured to the beam with a removable “brake” pin to prevent it from rotating.

(i) Find the tension in the cable when everything is at rest.

(ii) Now suppose the brake pin is suddenly removed. When the block has lowered by a vertical distance of 2.50 m, how fast is it moving?

Ignore air drag and any friction in the anchor pin, cable attachment, or pulley axle. Assume that the beam is a solid, uniform rod; the pulley is a solid, uniform disk; and the cord is massless and never slips with respect to the pulley rim.

(i) When all is at rest, it’s static equilibrium: $\Sigma \tau = I\alpha = 0$

Summing the torques about the anchor pin:

$$(F_{T\text{cable}})l'_{T\text{cable}} - (F_{G\text{beam}})l'_{G\text{beam}} - (F_{G\text{pulley}})l'_{G\text{pulley}} - (F_{T\text{block}})l'_{T\text{block}} = 0$$

$$(F_{T\text{cable}}) = \left( (F_{G\text{beam}})l'_{G\text{beam}} + (F_{G\text{pulley}})l'_{G\text{pulley}} + (F_{T\text{block}})l'_{T\text{block}} \right) / l'_{T\text{cable}}$$

From the diagram:

$l'_{T\text{cable}} = (12 - 3.50)\sin 60^\circ = 7.361$ m

$l'_{G\text{beam}} = (12/2)\cos 60^\circ = 3.00$ m

$l'_{G\text{pulley}} = (12)\cos 60^\circ = 6.00$ m

$l'_{T\text{block}} = (6 + 1.50) = 7.50$ m

And:

$F_{G\text{beam}} = 400(9.80) = 3920$ N

$F_{G\text{pulley}} = 30(9.80) = 294$ N

$F_{T\text{block}} = 100(9.80) = 980$ N

So:

$$(F_{T\text{cable}}) = \left( (3920)(3.00) + (294)(6.00) + (980)(7.50) \right) / 7.361$$

$$= 2840 \text{ N} \ (2.84 \times 10^3 \text{ N})$$

(ii) After the brake pin is removed, the block and pulley rim will start to accelerate together. One approach is to analyze the two objects ($\Sigma F$ for the block and $\Sigma \tau$ for the pulley) to find that common acceleration value, $a$, then use kinematics for block (also knowing that $v_i = 0$ and $\Delta y = 2.50$), to calculate its $v_f$.

Or, we can analyze the mechanical energy of the block and pulley system together: $E_{\text{mech.f}} = E_{\text{mech.i}} + W_{\text{ext}}$

Choosing the latter: No work is done except by gravity; and the block loses some $U_G$.

exchanging it for some $K_f$, and some $K_R$ for the pulley:

So:

$K_{f\text{block}} + K_{R\text{pulley}} = U_{G\text{.iblock}}$

Or:

$$\frac{1}{2}m_3v_f^2 + \frac{1}{2}I_{\text{pulley}}\omega_f^2 = m_3gh_i$$

But:

$v_f = r_{\text{pulley}}\omega_f$

And:

$I_{\text{pulley}} = (1/2)m_{\text{pulley}}r_{\text{pulley}}^2$

So:

$$\frac{1}{2}m_3v_f^2 + \frac{1}{4}m_{\text{pulley}}v_f^2 = m_3gh_i$$

Or:

$$v_f^2 \left( \frac{m_3}{2} + \frac{m_{\text{pulley}}}{4} \right) = m_3gh_i$$

So:

$$v_f = \sqrt{\frac{m_3gh_i}{\left( \frac{m_3}{2} + \frac{m_{\text{pulley}}}{4} \right)}}$$

$$= \sqrt{\frac{(100)(9.80)(2.50)}{[100/2 + 30/4]}} = 6.53 \text{ m/s}$$
6. a. A solid, uniform sphere of mass 2.0 kg and radius 1.7 m starts from rest and rolls without slipping down an inclined plane of height 5 m. What is the angular velocity of the sphere at the bottom of the inclined plane?

The fundamental equation:  
\[ E_{\text{mech,f}} = E_{\text{mech,i}} + W_{\text{ext}} \]

In detail:  
\[ K_T + K_R + U_{G,i} + U_{S,i} = K_T + K_R + U_{G,f} + U_{S,f} + W_{\text{ext}} \]

Since all rolling is without slipping (static friction) no work is done here except by gravity, so  \( W_{\text{ext}} = 0 \).

\( v_i = \omega_i = 0 \), and we will let  \( h_i = 0 \). Also, there’s no elastic source in the problem.

With those simplifications:  
\[ (1/2)Mv_i^2 + (1/2)I_\omega_i^2 = Mgh_i \]

Substitute for  \( I_{\text{sphere}} \) and  \( \omega_i \):  
\[ (1/2)M(R\omega_i)^2 + (1/2)(2/5)MR^2 \omega_i^2 = Mgh_i \]

Simplify:  
\[ (1/2)R^2 \omega_i^2 + (1/5)R^2 \omega_i^2 = gh_i \]

Collect terms:  
\[ (7/10)R^2 \omega_i^2 = gh_i \]

Solve for  \( \omega_i \):  
\[ \omega_i = \frac{(10/7)gh_i/R}{2} = \frac{(10/7)(9.80)(0.340)/2}{2} = 4.92 \text{ rad/s} \]

b. A bowling ball rolls without slipping, first along a level track, then up a ramp onto another level section of the track, gaining 0.340 m in altitude. If its translational speed along the lower track level was 3.15 m/s, find its translational speed at the top.

The fundamental equation:  
\[ E_{\text{mech,f}} = E_{\text{mech,i}} + W_{\text{ext}} \]

In detail:  
\[ K_T + K_R + U_{G,i} + U_{S,i} = K_T + K_R + U_{G,f} + U_{S,f} + W_{\text{ext}} \]

Since all rolling is without slipping (static friction) no work is done here except by gravity, so  \( W_{\text{ext}} = 0 \).

Also, there’s no elastic source in the problem, and we will let  \( h_i = 0 \).

With those simplifications:  
\[ (1/2)Mv_i^2 + (1/2)I_\omega_i^2 + Mgh_i = (1/2)Mv_f^2 + (1/2)I_\omega_f^2 \]

Substitute for  \( I_{\text{sphere}} \) and  \( \omega \):  
\[ (1/2)Mv_i^2 + (1/2)(2/5)MR^2 (v_i/R)^2 + Mgh_i = (1/2)Mv_f^2 + (1/2)(2/5)MR^2 (v_f/R)^2 \]

Simplify:  
\[ (1/2)v_i^2 + (1/5)v_i^2 + gh_f = (1/2)v_f^2 + (1/5)v_f^2 \]

Collect terms:  
\[ (7/10)v_f^2 + gh_f = (7/10)v_i^2 \]

Solve for  \( v_f \):  
\[ v_f = \frac{[v_i^2 - (10/7)gh_f]}{2} = \frac{[(3.15)^2 - (10/7)(9.80)(0.340)]}{2} = 2.27 \text{ m/s} \]

c. A basketball of mass  \( m \) and radius  \( r \) rolls without slipping up a hill. The angle between the hill and the horizontal is  \( \theta \). The initial speed of the basketball is  \( v \). The magnitude of the acceleration due to gravity is  \( g \). Expressed in terms of  \( m, r, \theta, v \) and/or  \( g \), what vertical height,  \( h \), does the ball gain before it comes to rest (momentarily)? Assume that the ball is a rigid body, and that friction and air resistance can be ignored.

The fundamental equation:  
\[ E_{\text{mech,f}} = E_{\text{mech,i}} + W_{\text{ext}} \]

In detail:  
\[ K_T + K_R + U_{G,i} + U_{S,i} = K_T + K_R + U_{G,f} + U_{S,f} + W_{\text{ext}} \]

Since all rolling is without slipping (static friction) no work is done here except by gravity, so  \( W_{\text{ext}} = 0 \).

Also, there’s no elastic source in the problem, and we will let  \( h_i = 0 \). And  \( v_i = \omega_i = 0 \).

With those simplifications:  
\[ Mgh_f = (1/2)Mv_i^2 + (1/2)I_\omega_i^2 \]

Substitute for  \( I_{\text{hollow sphere}} \) and  \( \omega \):  
\[ Mgh_f = (1/2)Mv_i^2 + (1/2)(2/3)MR^2 (v_i/R)^2 \]

Simplify:  
\[ gh_f = (1/2)v_i^2 + (1/3)v_i^2 \]

Collect terms:  
\[ gh_f = (5/6)v_i^2 \]

Solve for  \( h_f \):  
\[ h_f = \frac{(5/6)v_i^2}{g} \]

\( h = (5/6)v_i^2/g \)
6. The rear brakes on a bicycle consist of a pair of rubber pads that clamp down and rub ($\mu_K = 0.863$) on the rim of the rear wheel in order to slow it. Suppose the bike is rolling (without slipping) along a straight, level road at 9.25 m/s when the rider applies the rear brakes. If the normal force applied by each brake pad on the wheel rim is 74.1 N, find the speed of bike and rider when the wheels have revolved exactly six times after braking began. Ignore air and rolling resistance, and assume that the bike has two identical wheels; and that the brake pads apply their forces at the wheel’s outer radius.

\[ I_{\text{wheel}} = 0.268 \text{ kg} \cdot \text{m}^2 \quad r_{\text{wheel}} = 0.345 \text{ m} \quad m_{\text{bike}&\text{rider}} = 107 \text{ kg} \]

Use energy to analyze the system (bike and rider) from the moment the braking begins (initial) to the moment when the wheels have turned six full revolutions (final).

The system has $K_T$ (from the speed, $v$, of all parts moving in translational motion) and $K_R$ (from the rotational speed, $\omega$, of the 2 wheels moving in rotational motion). But there’s no height change (so no $U_G$); and no spring or other ideal elastic force (so no $U_S$). Thus: $K_{T_i} + K_{R_i} + W_{ext} = K_{T_f} + K_{R_f}$

The wheels roll without slipping, so the road’s static friction does no work on them. But each of the two brake pads does (negative) work on the rear wheel, via a kinetic friction torque, $|\tau_K| = |F_K| r_{\text{wheel}}$, acting over an angular displacement (6 revolutions $= 12\pi$ rad) of that wheel:

\[ W_{ext} = 2 |\tau_K| (-1) |\Delta \theta_K| = -2 \mu_K F_N r_{\text{wheel}} (12\pi) \]

So:

\[ (1/2)mv_i^2 + 2(1/2)I\omega_i^2 - 2\mu_K F_N r_{\text{wheel}} (12\pi) = (1/2)mv_f^2 + 2(1/2)I\omega_f^2 \]

For rolling w/o slipping, $\omega = v/r_{\text{wheel}}$:

\[ (1/2)mv_i^2 + (I/r_{\text{wheel}}^2)v_i^2 - 2\mu_K F_N r_{\text{wheel}} (12\pi) = (1/2)mv_f^2 + (I/r_{\text{wheel}}^2)v_f^2 \]

Solve for $v_f$:

\[ v_f = \sqrt{\left\{(m/2 + I/r_{\text{wheel}}^2)v_i^2 - 2\mu_K F_N r_{\text{wheel}} (12\pi)\right\}/(m/2 + I/r_{\text{wheel}}^2)} \]

\[ = \sqrt{\left\{((107/2 + 0.268/0.345^2)(9.25)^2 - 2(0.863)(74.1)(0.345)(12\pi))/(107/2 + 0.268/0.345^2)\right\}} \]

\[ = 7.46 \text{ m/s} \]
6. e. In two separate trials, a solid sphere (A) and a hollow spherical shell (B) are each released from rest and allowed to roll for the same distance down the same slope (shown here). Evaluate (T/F/N) each statement. Justify your answers fully with any valid mix of words, drawings and calculations.

(i) Their final translational speeds compare like this: \( v_{fB} = (1.09)v_{fA} \)
False. In each trial: \( U_{Gi} = K_{T,cm.f} + K_{R,cm.f} \)
(No \( K \) initially and no \( U_{Gf} \) finally; and rolling w/o slipping assumes \( W_{ext} = 0 \))
Thus: \( mgh_i = (1/2)mv_{cm.f}^2 + (1/2)(I_{cm}/2R^2)(v_{cm.f}/R)^2 \)
Trial A: \( mgh_i = (1/2)mv_{cm.f}^2 + (1/2)(2/5)mR^2(v_{cm.f}/R)^2 = (7/10)mv_{cm.f}^2 \)
Thus: \( v_{fA}^2 = (10/7)gh_i \)
Trial B: \( mgh_i = (1/2)mv_{cm.f}^2 + (1/2)(2/3)mR^2(v_{cm.f}/R)^2 = (5/6)mv_{cm.f}^2 \)
Thus: \( v_{fB}^2 = (6/5)gh_i \)
Therefore: \( v_{fB}/v_{fA} = \sqrt{(6/5)/(10/7)} = 0.917 \)

(ii) At any given moment during its motion along the slope, the solid sphere has about 28.6% of its total kinetic energy in the form of \( K_R \).
True. Trial A (for any value of \( h_i \)—thus for any point along the slope):
\[ mgh_i = K_{T,cm.f} + K_{R,cm.f} \]
\[ = (1/2)mv_{cm.f}^2 + (1/2)(2/5)mR^2(v_{cm.f}/R)^2 \]
Thus: \( K_{RA}/(K_{TA} + K_{RA}) = (1/5)/[(1/2) + (1/5)] = 0.286 \)

(iii) Their average translational acceleration magnitudes compare like this: \( a_B \approx (0.917)a_A \)
False. Kinematics:
\[ v_f^2 = v_i^2 + 2a\Delta x \]
Rearranging (with \( v_i = 0 \) here):
\[ a = v_f^2/(2\Delta x) \]
From part (i) above:
\[ v_{fA}^2 = (10/7)gh_i \]
and \( v_{fB}^2 = (6/5)gh_i \)
Thus:
\[ a_B/a_A = [v_{fB}^2/(2\Delta x)]/[v_{fA}^2/(2\Delta x)] \]
\[ = v_{fB}^2/v_{fA}^2 \]
\[ = (6/5)/(10/7) \]
\[ = 0.840 \]
6. f. Beginning from rest, a wheel of mass 20.0 kg and outer radius 0.750 m rolls without slipping down a slope onto level ground, losing a total of 4.30 m in altitude. When it arrives on level ground, its total kinetic energy is 75% $K_T$ and 25% $K_R$.

Assume that the wheel’s center of mass is located at its geometric center (and this is the axis around which it turns as it rolls), but do **not** assume that the wheel is a uniform disk. Also, assume normal local $g_{earth}$, and ignore air resistance and rolling friction.

(i) How fast does the wheel’s axle travel over the level ground?

An energy analysis is easiest here....

**The basic equation:**

$$E_{mech,f} = E_{mech,i} + W_{ext}$$

**Itemized:**

$$K_{T,f} + K_{R,f} + U_{G,f} + U_{S,f} = K_{T,i} + K_{R,i} + U_{G,i} + U_{S,i} + W_{ext}$$

**Simplifications:**

$$W_{ext} = 0$$ (Rolling without slipping uses static friction; other resistance is given as negligible.)

$$U_{S,f} = U_{S,i} = 0$$ (No ideal elastic source in this problem.)

$$U_{G,f} = 0$$ (Assign level ground as $h = 0$ for $U_G$.)

$$K_{T,i} = K_{R,i} = 0$$ (Wheel starts from rest.)

**Therefore:**

$$K_{T,f} + K_{R,f} = U_{G,i}$$ (Wheel’s final total $K$ is equal to $U_{G,i}$.)

Thus:

$$K_{T,f} = (3/4)U_{G,i}$$

**Known values:**

$$\frac{1}{2}mv_{c.m,f}^2 = (3/4)mgh_i$$ ($K_T$ computed by the speed of the c.m., $v_{c.m}$)

**Simplifying:**

$$v_{c.m,f}^2 = (3/2)gh_i$$

Solve for $v_{c.m,f}$:

$$v_{c.m,f} = \sqrt{(3/2)gh_i} = \sqrt{(3/2)(9.80)(4.30)} = 7.95 \text{ m/s}$$

(ii) What is the moment of inertia of this wheel (about its center of mass)?

Same logic as above:

$$K_{R,f} = (1/4)U_{G,i}$$

**Known values:**

$$\frac{1}{2}I_{c.m} (v_{c.m}/R)^2 = (1/4)mgh_i$$ (Rolling without slipping: $v_{c.m} = v_T = R\omega$)

**Simplifying:**

$$I_{c.m} (v_{c.m}/R)^2 = (1/2)mgh_i$$

Solve for $I_{c.m}$:

$$I_{c.m} = \frac{mgh_i R^2 / (2v_{c.m}^2)}{(20.0)(9.80)(4.30)(0.750)^2 / [2(7.95)^2]} = 3.75 \text{ kg} \cdot \text{m}^2$$
6. g. A solid sphere and a solid cube have equal masses. You test each, one at a time, on the slope shown here. In each test, you release the object from rest at point A. The sphere rolls without slipping, but the cube slides. In this particular case, it so happens that each object reaches point B with the same translational speed. Find the coefficient of kinetic friction between the cube and the slope.

For each case: $E_{mech,f} = E_{mech,i} + W_{ext}$

The initial point is A; the final point is B.

The distance traveled (A to B) is $d$.

Let $h_i = 0$
Then $h_f = dsin\theta$ ($\theta = 38.6^\circ$)
$U_s = 0$ (no spring involved)
$v_i = 0$
$\omega_i = 0$

The cube: The slope does (negative) work via friction:

$W_{ext} = -F_k(d) = -\mu_k F_N(d) = -\mu_k (mg \cos \theta) d$
Thus: $K_{Tf} + W_{ext} = U_{G,i} + W_{ext}$
Or: $(1/2)mv_f^2 = mgh_i + W_{ext}$
$(1/2)mv_f^2 = mgsin\theta - \mu_k (mg \cos \theta) d$
$v_f^2 = 2gsin\theta - 2\mu_k(g \cos \theta) d$

The sphere: During rolling without slipping, no work is done by the surface: $W_{ext} = 0$

Therefore: $K_{Tf} + K_{Rf} = U_{G,i}$
Or: $(1/2)mv_f^2 + (1/2)I_f \omega_f^2 = mgh_i$
But: $\omega_f = v_f/R$ (R is the sphere’s radius)
And: $I = (2/5)mR^2$
So: $(1/2)mv_f^2 + (1/2)[(2/5)mR^2] (v_f/R)^2 = mgsin\theta$
Simplify: $(1/2)v_f^2 + (1/5)v_f^2 = gdsin\theta$
$v_f^2 = (10/7)gdsin\theta$

But the $v_f$ in each of the two cases above is the same. Therefore:

$(10/7)gdsin\theta = 2gdsin\theta - 2\mu_k(g \cos \theta) d$
$(10/7)sin\theta = 2sin\theta - 2\mu_k (cos \theta)$
$2\mu_k(cos \theta) = 2sin\theta - (10/7)sin\theta$
$\mu_k(cos \theta) = sin\theta - (5/7)sin\theta$
$\mu_k = (2/7)tan \theta = 0.228$
6. h. An ideal spring (stiffness = 910 N/m) with a frictionless “bumper,” as shown, is attached along a level surface to a wall and compressed by a distance of 0.320 from its relaxed equilibrium position. A ball (a solid, uniform sphere, mass = 3.68 kg, radius = 0.465 m) is placed in front of the bumper. When the spring is released, the ball slips (“skids”—i.e. it doesn’t roll) horizontally for a distance of 0.791 m. Then it rolls without slipping thereafter, up the ramp and off the end, which is 0.879 m above the level surface. The coefficient of kinetic friction for the entire surface and ramp is 0.245. Find the rotational speed of the ball as it leaves the ramp. Neglect air resistance.

![Diagram of the spring system and ball](image)

<table>
<thead>
<tr>
<th>Inventory of known and needed facts:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_i = 0 ) ( \omega_i = 0 ) ( h_i = 0 ) ( x_i = -0.320 ) ( I = (2/5)mr^2 )</td>
</tr>
<tr>
<td>( v_f = ? ) ( \omega_f = ? ) <strong>--the question</strong> ( h_f = H = 0.879 ) ( x_f = 0 )</td>
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</tbody>
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<tr>
<th>The fundamental Work-Energy equation:</th>
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<tbody>
<tr>
<td>( E_{mech,f} = E_{mech,i} + W_{ext} )</td>
</tr>
<tr>
<td>( K_{T_f} + K_{R_f} + U_{Gi_f} + U_{Si_f} = K_{T_i} + K_{R_i} + U_{Gi_i} + U_{Si_i} + W_{ext} )</td>
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<tr>
<th>Analysis of the external work being done:</th>
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<tr>
<td>( W_{Fk} = F_k s \cos \theta = \mu_k F_N d \cos \theta = -\mu_k F_N d )</td>
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<tr>
<td>( F_N = mg ) therefore ( W_{ext} = -\mu_k mgd )</td>
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</table>

**Detailed expansion of the Work-Energy equation for this situation:**

\[
\frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 + mgh_i + \frac{1}{2}kx_i^2 - \mu_k mgd = \left( \frac{1}{2} \right)mv_f^2 + \frac{1}{2}I\omega_f^2 + mgh_f + \frac{1}{2}kx_f^2
\]

**Simplify:**

\[
\frac{1}{2}kx_i^2 - \mu_k mgd = \left( \frac{1}{2} \right)mv_f^2 + \frac{1}{2}(2/5)mR^2 \omega_f^2 + mgH
\]

Since the ball is rolling without slipping at the final point, its translational speed is equal to the tangential speed of its edge: \( v = v_f = R\omega \) And therefore: \( v_f^2 = R^2 \omega_f^2 \)

**Substitute for \( v_f^2 \) and \( I \):**

\[
\frac{1}{2}kx_i^2 - \mu_k mgd = \left( \frac{1}{2} \right)2mR^2 \omega_f^2 + \left( \frac{1}{2} \right)(2/5)mR^2 \omega_f^2 + mgH
\]

**Simplify:**

\[
\frac{1}{2}kx_i^2 - \mu_k mgd = \left( \frac{1}{2} \right)2mR^2 \omega_f^2 + \left( \frac{1}{5} \right)mR^2 \omega_f^2 + mgH
\]

**Collect terms:**

\[
\frac{1}{2}kx_i^2 - \mu_k mgd = \frac{7}{10}mR^2 \omega_f^2 + mgH
\]

**Solve for \( \omega \):**

\[
\omega_f^2 = \frac{[(1/2)kx_i^2 - \mu_k mgd - mgH]/[(7/10)mR^2]}{[(7/10)mR^2]}
\]

**Plug in the numbers:**

\[
\omega = \{[(1/2)(910)(.320)^2 - (.245)(3.68)(9.80)(.791) - (3.68)(9.80)(.879)]/[(7/10)(3.68)(.465)^2]\}^{1/2}
\]

\[
= 3.77 \text{ rad/s}
\]