Activity 3: Solution for electric field

Find the electric field in all space due to a ring with total charge Q and radius R

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i\, \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i|^3} \tag{1}$$

For a ring of charge this becomes

$$\vec{\boldsymbol{E}} = \int\limits_{\text{ring}} \frac{1}{4\pi\epsilon_0} \frac{\lambda(\vec{\boldsymbol{r}}')\, |d\vec{\boldsymbol{r}}'|\, \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} \tag{2}$$

where $\vec{\boldsymbol{r}}$ denotes the position in space at which the electric field is measured and $\vec{\boldsymbol{r}}'$ denotes the position of the charge.

In cylindrical coordinates, $|d\vec{\boldsymbol{r}}'| = R\, d\phi'$, where R is the radius of the ring. Thus,

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \int\limits_0^{2\pi} \frac{\lambda(\vec{\boldsymbol{r}}')\, R\, d\phi'\, \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} \tag{3}$$

Assuming constant linear charge density for a ring with charge Q and radius R, $\lambda(\vec{\boldsymbol{r}}') = \frac{Q}{2\pi R}$ Thus,

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int\limits_0^{2\pi} \frac{d\phi'\, \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} \tag{4}$$

Since $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}'$ are not necessarily in the same direction, we cannot simply leave $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ in curvilinear coordinates and integrate directly. One solution to this problem is to go back and forth between cylindrical and cartesian coordinates to represent $\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'$

$$\begin{aligned} \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}' &= (x - x')\hat{\boldsymbol{\imath}} + (y - y')\hat{\boldsymbol{\jmath}} + (z - z')\hat{\boldsymbol{k}} && (5) \\ &= (r\cos\phi - R\cos\phi')\hat{\boldsymbol{\imath}} + (r\sin\phi - R\sin\phi')\hat{\boldsymbol{\jmath}} + (z - z')\hat{\boldsymbol{k}} && (6) \end{aligned}$$

And

$$|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{7}$$

The electric field can now be represented by the elliptic integral

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int\limits_0^{2\pi} \frac{[(r\cos\phi - R\cos\phi')\hat{\boldsymbol{\imath}} + (r\sin\phi - R\sin\phi')\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}]\, d\phi'}{(r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2)^{3/2}} \tag{8}$$

1 The z axis

For points on the z axis, $r = 0$ and the integral simplifies to

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int\limits_0^{2\pi} \frac{[-R\cos\phi'\, \hat{\boldsymbol{\imath}} + -R\sin\phi'\, \hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}]\, d\phi'}{(R^2 + z^2)^{3/2}} \tag{9}$$

Doing the integral results in

$$\vec{\boldsymbol{E}} = \frac{Q}{4\pi\epsilon_0} \frac{z\hat{\boldsymbol{k}}}{(R^2 + z^2)^{3/2}} \tag{10}$$

2 The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{[(r - R\cos\phi')\,\hat{\boldsymbol{\imath}} + -R\sin\phi'\,\hat{\boldsymbol{\jmath}}]\,d\phi'}{(r^2 - 2rR\cos\phi' + R^2)^{3/2}} \tag{11}$$

let $u = r^2 - 2rR\cos\phi' + R^2$, then $du = 2rR\sin\phi' d\phi'$, and for the $\hat{\boldsymbol{\jmath}}$ component the integral becomes

$$\vec{\boldsymbol{E}}_j = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{1}{2r}\int_0^{2\pi}\frac{du\hat{\boldsymbol{\jmath}}}{u^{3/2}} \tag{12}$$

Doing the integral results in

$$\vec{\boldsymbol{E}}_j = 0 \tag{13}$$

Thus the $\hat{\boldsymbol{\jmath}}$ component disappears and results in the elliptic integral with only an $\hat{\boldsymbol{\imath}}$ component

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{(r - R\cos\phi')\,\hat{\boldsymbol{\imath}}\,d\phi'}{(r^2 - 2rR\cos\phi' + R^2)^{3/2}} \tag{14}$$