Activity 4: Solution for magnetic vector potential

Find the magnetic vector potential in all space due to a ring with total charge Q and radius R rotating with a period T

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int\limits_{\text{ring}} \frac{\vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}')\, dl'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \tag{1}$$

where $\vec{\boldsymbol{r}}$ denotes the position in space at which the magnetic vector potential is measured and $\vec{\boldsymbol{r}}'$ denotes the position of the current segment.

For the current

$$\vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}') = \lambda(\vec{\boldsymbol{r}})\vec{\boldsymbol{v}} = \frac{Q}{2\pi}\frac{2\pi R}{T}\hat{\phi} = \frac{QR}{T}\hat{\phi} \tag{2}$$

$$= \frac{QR}{T}(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) \tag{3}$$

In cylindrical coordinates, $dl' = R\,d\phi'$, and, as discussed in previous solutions,

$$|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{4}$$

Thus

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int\limits_0^{2\pi} \frac{QR}{T} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})Rd\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{5}$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int\limits_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{6}$$

1 The z axis

For points on the z axis, $r = 0$ and the integral simplifies to

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int\limits_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})d\phi'}{\sqrt{R^2 + z^2}} \tag{7}$$

Doing the integral results in

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = 0 \tag{8}$$

2 The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int\limits_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})d\phi'}{\sqrt{r^2 - 2rR\cos\phi' + R^2}} \tag{9}$$

This results in a very similar situation as the case for electric field on the x axis, except that now we will address the $\hat{\boldsymbol{\imath}}$ component instead of the $\hat{\boldsymbol{\jmath}}$ component. Using the same process we let $u = x^2 - 2xR\cos\phi' + R^2$, then $du = 2xR\sin\phi' d\phi'$, and for the $\hat{\boldsymbol{\imath}}$ component the integral becomes

$$\vec{\boldsymbol{A}}_x(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{-1}{2x}\int_0^{2\pi}\frac{du\hat{\boldsymbol{\jmath}}}{u^{1/2}} \tag{10}$$

Doing the integral, we find

$$\vec{\boldsymbol{A}}_x(\vec{\boldsymbol{r}}) = 0 \tag{11}$$

Thus the $\hat{\boldsymbol{\imath}}$ component disappears and we are left with an elliptic integral with only a $\hat{\boldsymbol{\jmath}}$ component

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{\cos\phi'\,\hat{\boldsymbol{\jmath}}\,d\phi'}{\sqrt{r^2 - 2rR\cos\phi' + R^2}} \tag{12}$$