Magnetic Vector Potential for Ring

Instructor's Guide

Keywords: Upper-division, E and M, Magnetic Vector Potential, Symmetry, Ring

Brief overview of the activity

In this activity, students work in small groups to write the magnetic potential everywhere in space due to a ring of charge Q and radius R spinning with period T.

This activity brings together student understanding of:

1. Electrostatic potential
2. Spherical and cylindrical coordinates
3. Superposition
4. Integration as "chopping and adding"
5. Linear charge density
6. 3-dimensional geometric reasoning
7. Power series expansion

Student prerequisite skills

This activity is may be used as the fourth in a sequence, following the electric field activity, or may be used on its own. Students will need understandings of:

1. The prerequisites addressed in the `electrostatic field activity`.
2. Spherical and cylindrical coordinates. `Link to spherical and cylindrical coordinates activity`.
3. Integration as chopping and adding. `Link to Integration activity.`
4. `Linear charge density`

Props

- Hula hoop or other thin ring
- Balls to represent point charges
- Voltmeter
- Coordinate system (e.g. with straws or Tinkertoys)
- Poster-sized whiteboards

- markers
- whiteboards around room. `Link to room set-up`.

The activity - Allow 50 minutes.

Overview

What the students will be challenged by and how to facilitate their learning

Highlights of the activity

Working in small groups students are asked to consider a ring with charge Q, and radius R rotating about its axis with period T and create an integral expression for the vector potential caused by this ring everywhere in space. Students also develop the power series expansion for the potential near the center or far from the ring.

Reasons to spend class time on this activity:

1. Surprisingly, it is non-trivial for students to start with Q, R, and T and determine the current density. Although the process does not take long, having students clarify these relationships is important for building understanding. For physicists the picture generated in one's mind can be make this appear so simple that is easy to overlook the important connections that students need to make regarding current density before dealing with other aspects of the problem.
2. This activity allows students to build upon prior understandings and apply them specifically to vector potentials. Even though students had prior experience with this geometry, the instructor found herself discussing with several students the difference between r and r' and which variables could be held constant during integration.

In this problem students must deal with the concept of current density and with the concept of a vector potential. Students will often find it surprising that their intuition about electrostatic potential is not directly applicable to vector potentials.

Instructors may see it as so simple that they wouldn't consider that it would require considerable mental effort from students to find the linear current density from a ring of charge Q and radius R rotating with period T. The reality is that the concept of current density is sufficiently new and unfamiliar that students must spend time grappling with the concept in this context. The understanding gained during this problem "pays off" when students face future problems involving linear, surface and volume current densities.

In addition to determining the magnitude of the current, students will need to consider direction. This may be the first time students have had to consider the vector nature of current beyond simply using the "right hand rule." The understandings gained here will also be needed for the fifth activity where students are required to find the magnetic field.

The concept of magnetic vector potential is new to most students. Since most people try to understand something new by comparing it with something familiar, students will often try to use intuitions about electric potential to understand magnetic vector potential. Because expressions for both potentials contain $\frac{1}{|\vec{r}-\vec{r}'|}$, students may make the assumption that both potentials basically the same thing with a different constant in front. This activity will force students to confront some of the important differences between a scalar potential and a vector potential.

One example of a situation where students must confront this difference is with the magnetic vector potential along the axis of the ring. Whereas the electric potential is positive for all finite values along this axis, the magnetic vector potential is always zero. For many students this result will be counterintuitive and will cause them to think more deeply about the differences between scalar potentials and vector potentials.

Solution for magnetic vector potential in all space due to a ring with total charge Q and radius R rotating with a period T

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int\limits_{\text{ring}} \frac{\vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}')\, dl'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \tag{1}$$

where $\vec{\boldsymbol{r}}$ denotes the position in space at which the magnetic vector potential is measured and $\vec{\boldsymbol{r}}'$ denotes the position of the current segment.

For the current

$$\begin{aligned} \vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}') &= \lambda(\vec{\boldsymbol{r}})\vec{\boldsymbol{v}} = \frac{Q}{2\pi}\frac{2\pi R}{T}\hat{\phi} = \frac{QR}{T}\hat{\phi} & (2) \\ &= \frac{QR}{T}(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) & (3) \end{aligned}$$

In cylindrical coordinates, $dl' = R\, d\phi'$, and, as discussed in previous solutions,

$$|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{4}$$

Thus

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{QR}{T} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})R d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{5}$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{6}$$

1 The z axis

For points on the z axis, $r = 0$ and the integral simplifies to

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) d\phi'}{\sqrt{R^2 + z^2}} \tag{7}$$

Doing the integral results in

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = 0 \tag{8}$$

2 The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{QR^2}{T}\int_0^{2\pi}\frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})d\phi'}{\sqrt{r^2 - 2rR\cos\phi' + R^2}} \tag{9}$$

This results in a very similar situation as the case for electric field on the x axis, except that now we will address the $\hat{\boldsymbol{\imath}}$ component instead of the $\hat{\boldsymbol{\jmath}}$ component. Using the same process we let $u = x^2 - 2xR\cos\phi' + R^2$, then $du = 2xR\sin\phi' d\phi'$, and for the $\hat{\boldsymbol{\imath}}$ component the integral becomes

$$\vec{\boldsymbol{A}}_x(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{-1}{2x}\int_0^{2\pi}\frac{du\hat{\boldsymbol{\jmath}}}{u^{1/2}} \tag{10}$$

Doing the integral, we find

$$\vec{\boldsymbol{A}}_x(\vec{\boldsymbol{r}}) = 0 \tag{11}$$

Thus the $\hat{\boldsymbol{\imath}}$ component disappears and we are left with an elliptic integral with only a $\hat{\boldsymbol{\jmath}}$ component

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{\cos\phi'\,\hat{\boldsymbol{\jmath}}\,d\phi'}{\sqrt{r^2 - 2rR\cos\phi' + R^2}} \tag{12}$$