THE GEOMETRY OF SPECIAL RELATIVITY

Oregon State University

Tevian Dray

I: Circle Geometry

II: Hyperbola Geometry

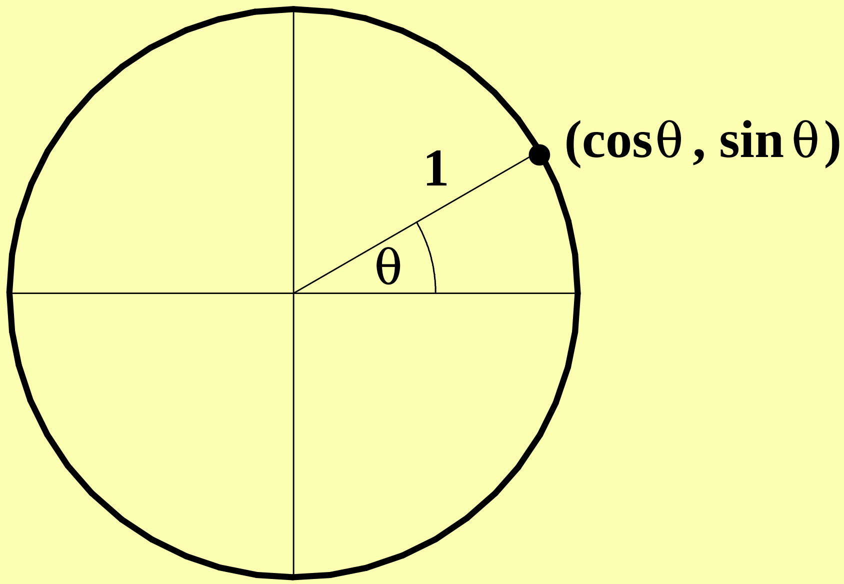
III: Special Relativity

IV: What Next?

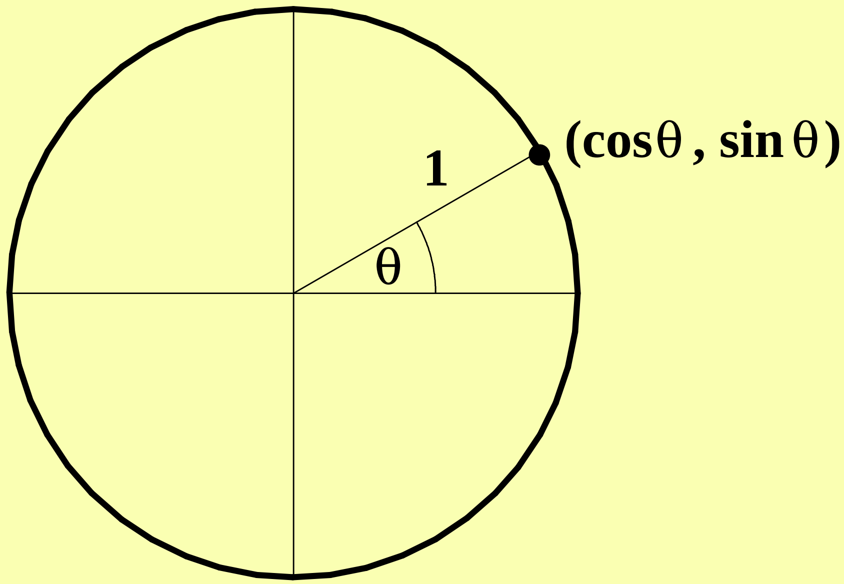
CIRCLE GEOMETRY

Write down something you know about trigonometry

CIRCLE GEOMETRY

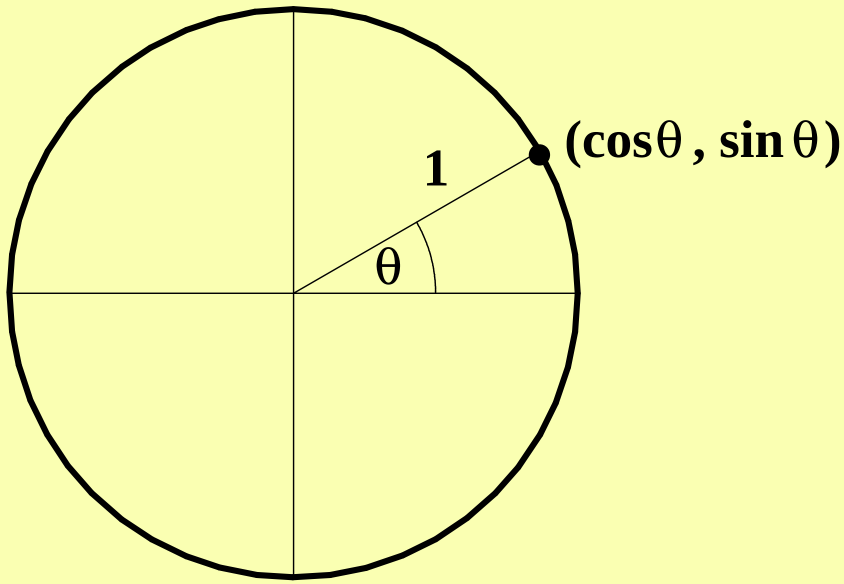


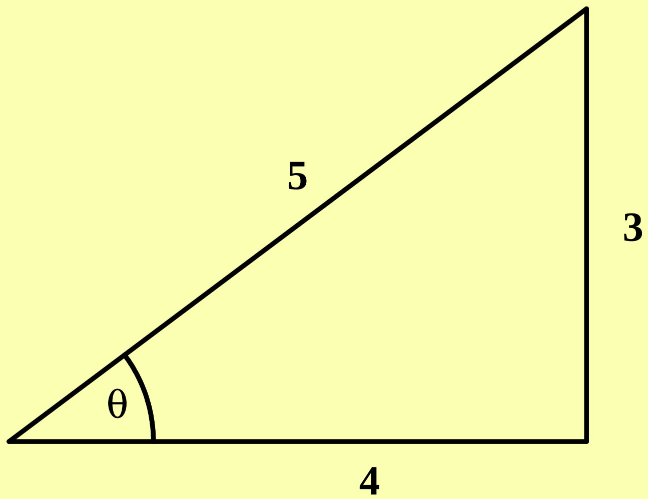
CIRCLE GEOMETRY



$$r\theta = \text{arclength}$$

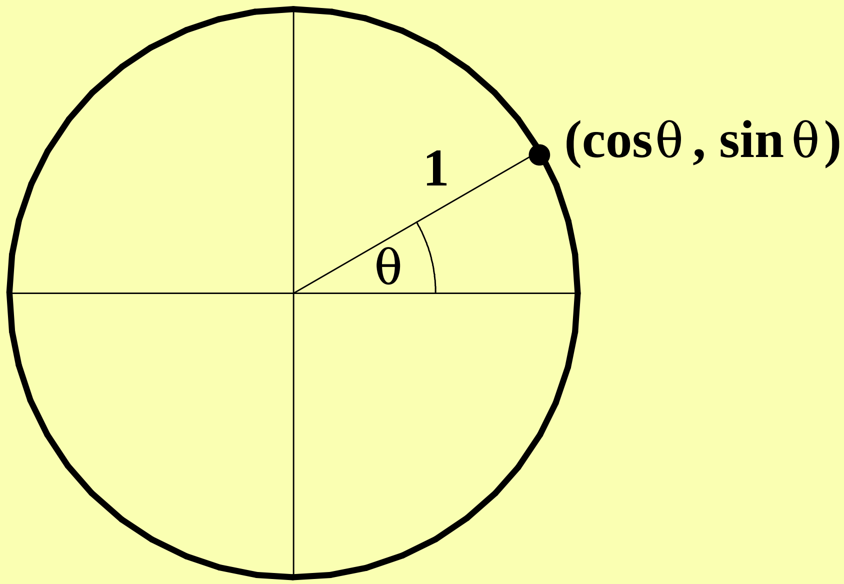
CIRCLE GEOMETRY

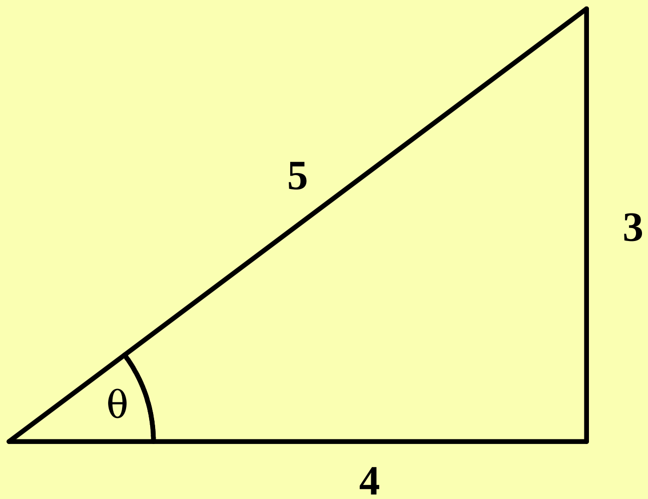




$r\theta = \text{arclength}$

CIRCLE GEOMETRY





$$r\theta = \text{arclength}$$

$$\cos\theta = \frac{4}{5} \implies \tan\theta = \frac{3}{4}$$

WHICH GEOMETRY?

WHICH GEOMETRY?

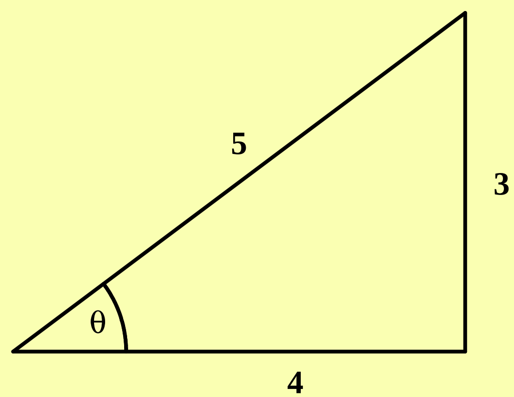
Euclidean

$$ds^2 = dx^2 + dy^2$$

WHICH GEOMETRY?

Euclidean

$$ds^2 = dx^2 + dy^2$$

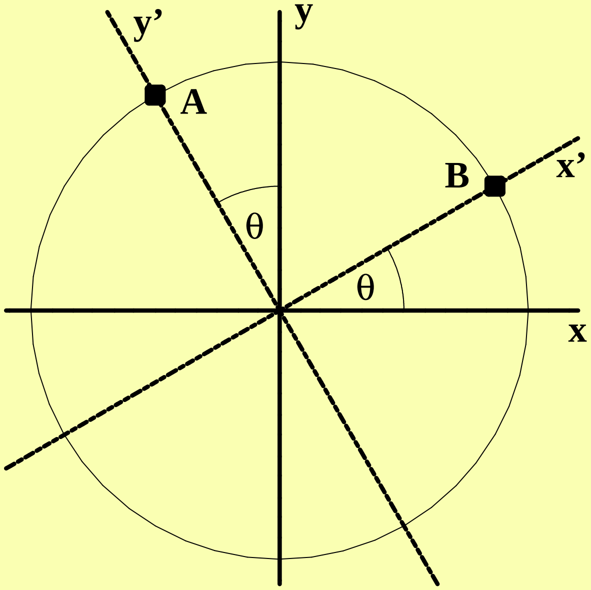


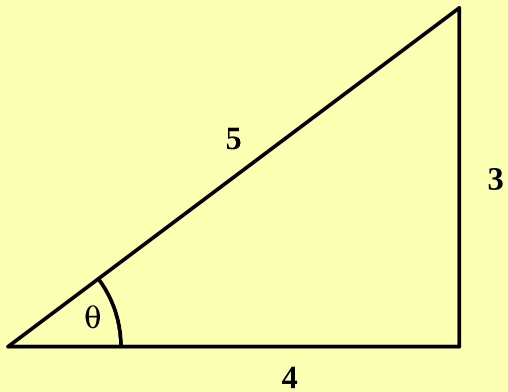
$$\tan\theta = \frac{3}{4}$$

WHICH GEOMETRY?

Euclidean

$$ds^2 = dx^2 + dy^2$$





$$\tan\theta = \frac{3}{4}$$

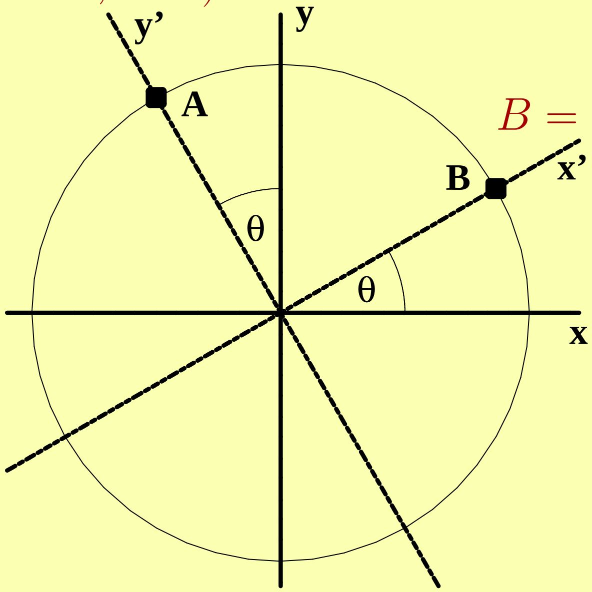
WHICH GEOMETRY?

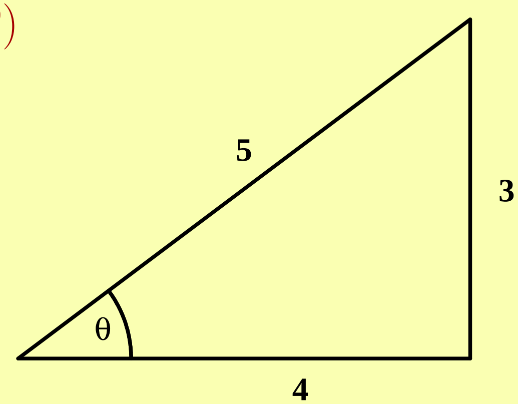
Euclidean

$$ds^2 = dx^2 + dy^2$$

$A = (-\sin\theta, \cos\theta)$

$B = (\cos\theta, \sin\theta)$





$$\tan\theta = \frac{3}{4}$$

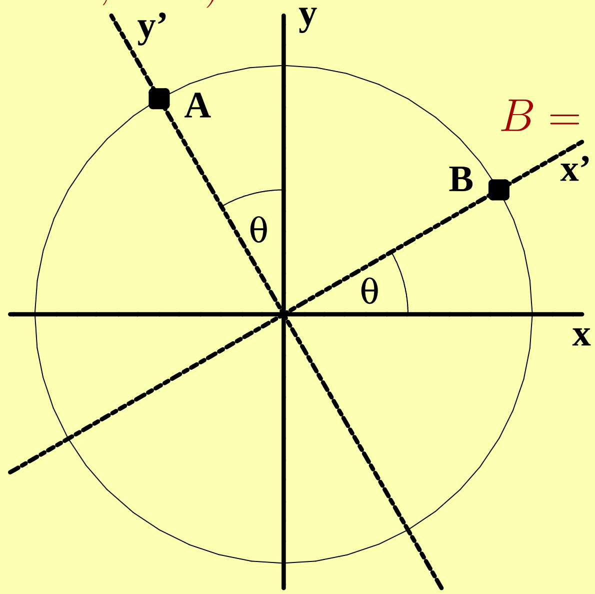
WHICH GEOMETRY?

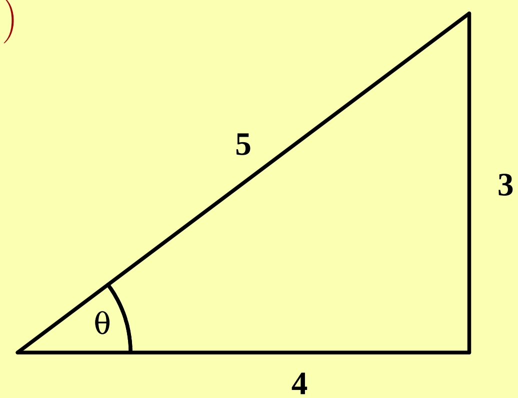
Euclidean

$$ds^2 = dx^2 + dy^2$$

$A = (-\sin\theta, \cos\theta)$

$B = (\cos\theta, \sin\theta)$





$\tan\theta = \frac{3}{4}$

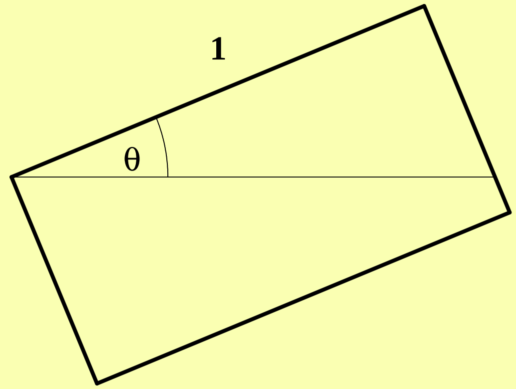
Trigonometry!

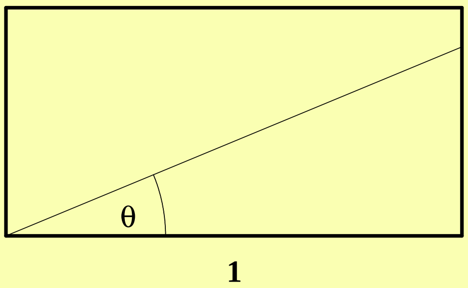
MEASUREMENTS

Return

MEASUREMENTS

Width:

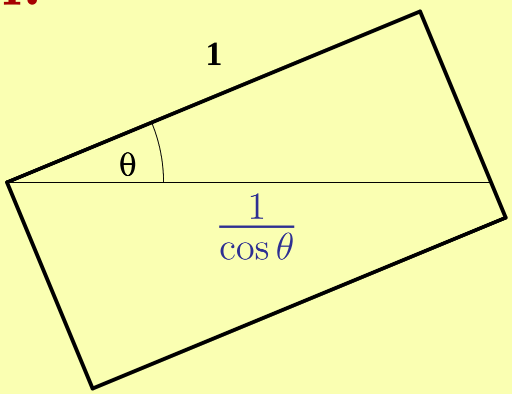


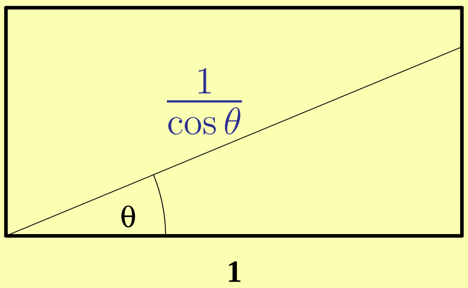


Return

MEASUREMENTS

Width:



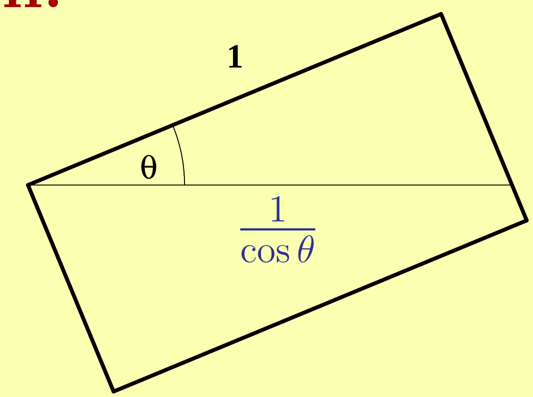


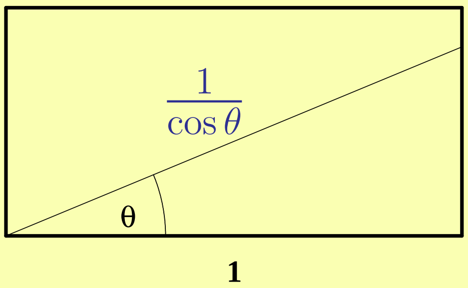
Apparent width > 1

Return

MEASUREMENTS

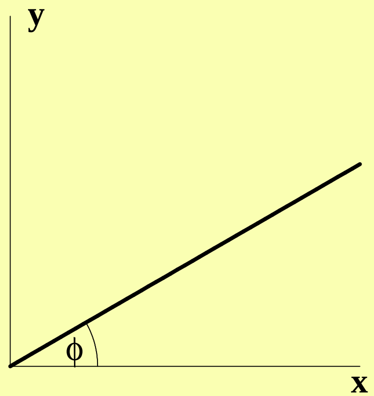
Width:

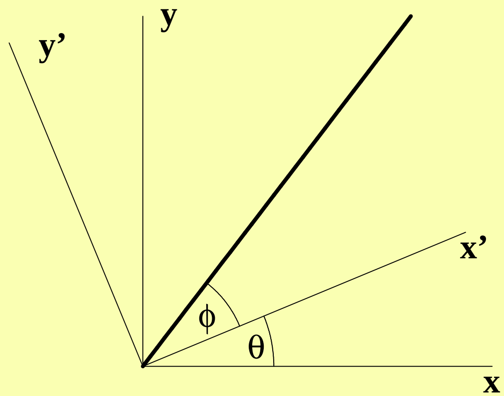




Apparent width > 1

Slope:

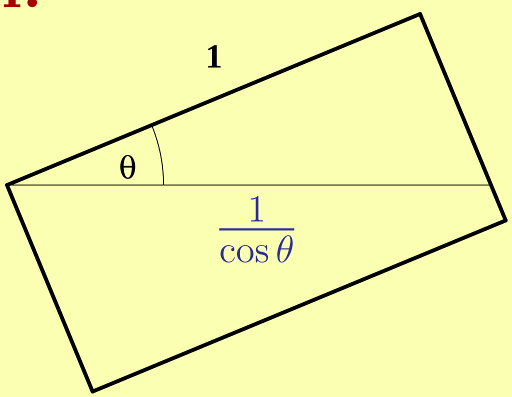


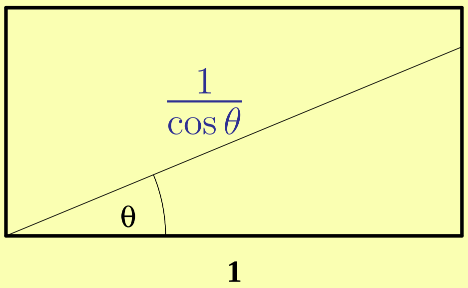


Return

MEASUREMENTS

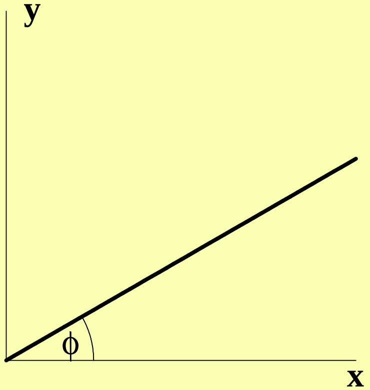
Width:

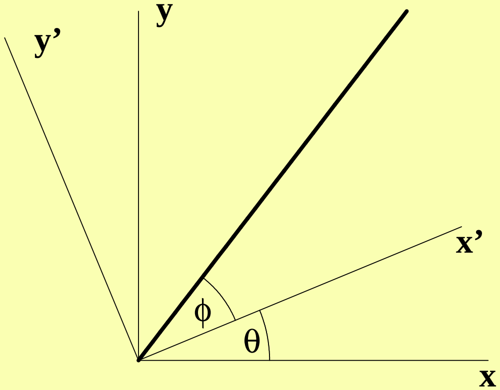




$$Apparent\ width > 1$$

Slope:



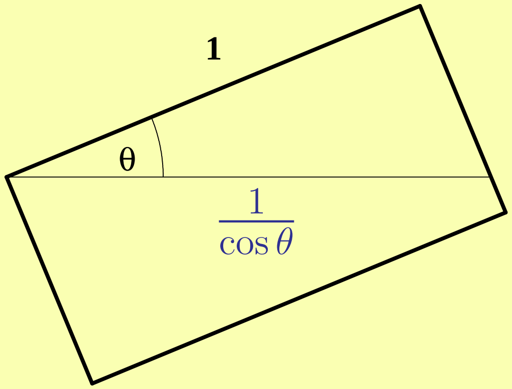


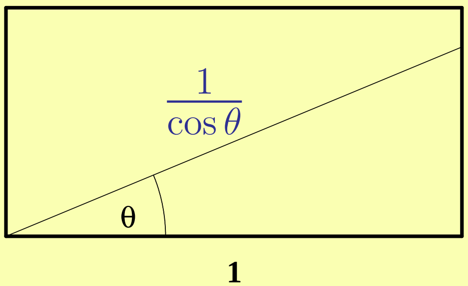
$$m \neq m_1 + m_2$$

Return

MEASUREMENTS

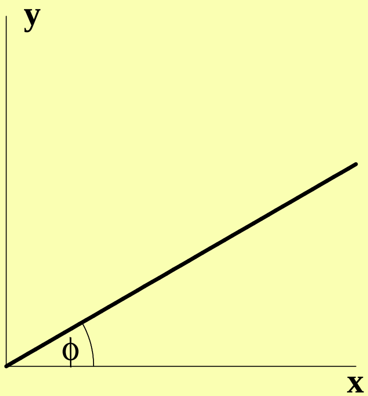
Width:

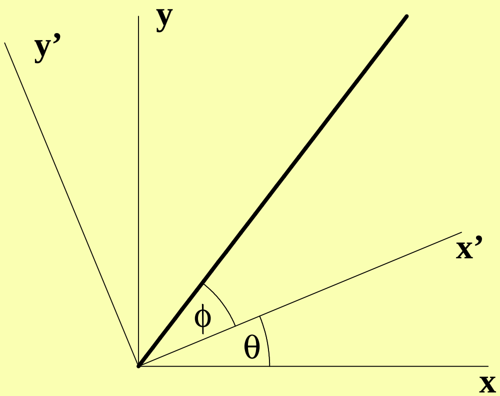




$$Apparent\ width > 1$$

Slope:



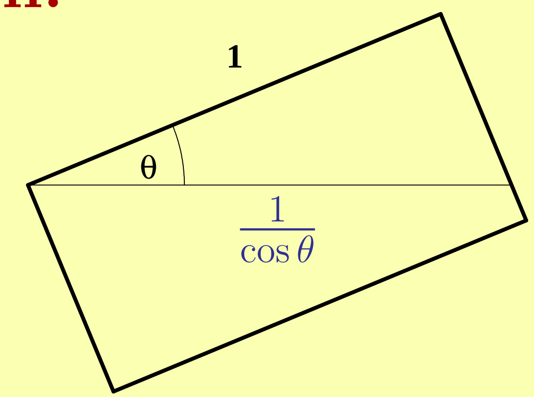


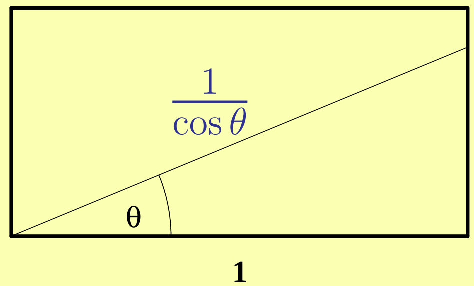
$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

Return

MEASUREMENTS

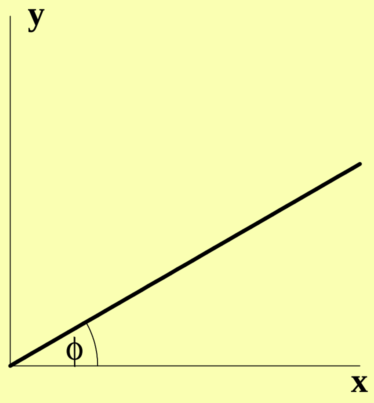
Width:

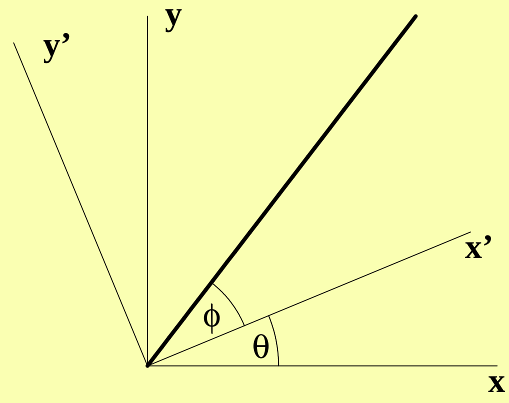




Apparent width > 1

Slope:

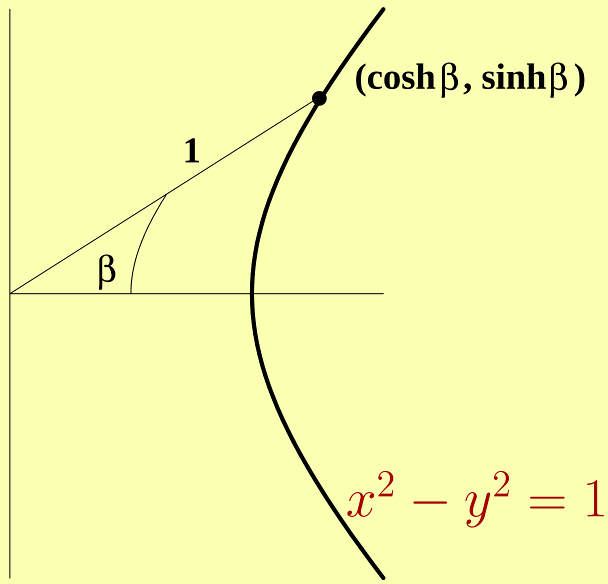




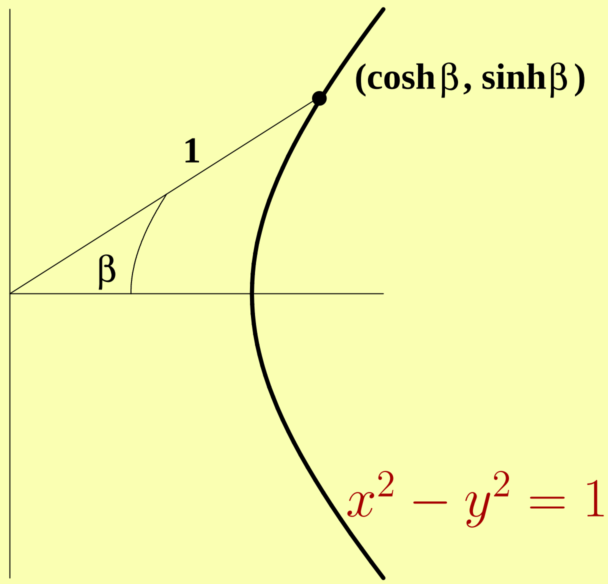
$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{m_1 + m_2}{1 - m_1 m_2}$$

Return

HYPERBOLA GEOMETRY

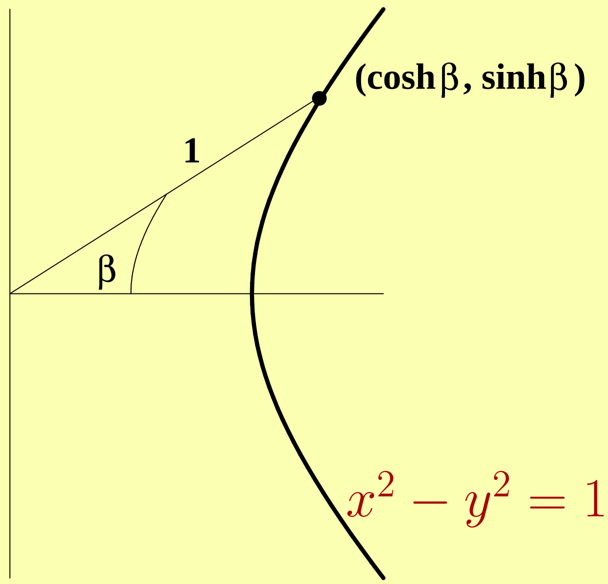


HYPERBOLA GEOMETRY



$$r\beta \;=\; \text{arclength}$$
$$ds^2 \;=\; |dx^2 - dy^2|$$

HYPERBOLA GEOMETRY



$$r\beta \;=\; \text{arclength}$$

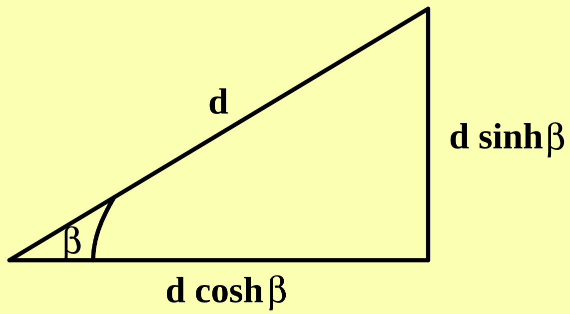
$$ds^2 \;=\; |dx^2 - dy^2|$$

$$\cosh\beta \;=\; \frac{1}{2}\left(e^{\beta} + e^{-\beta}\right)$$

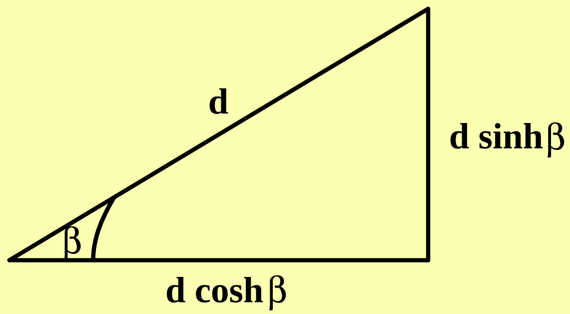
$$\sinh\beta \;=\; \frac{1}{2}\left(e^{\beta} - e^{-\beta}\right)$$

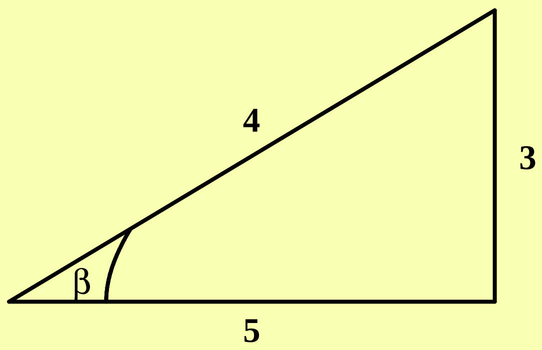
HYPERBOLIC TRIANGLE TRIG

HYPERBOLIC TRIANGLE TRIG



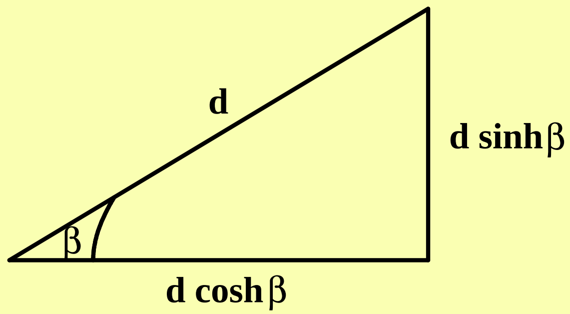
HYPERBOLIC TRIANGLE TRIG

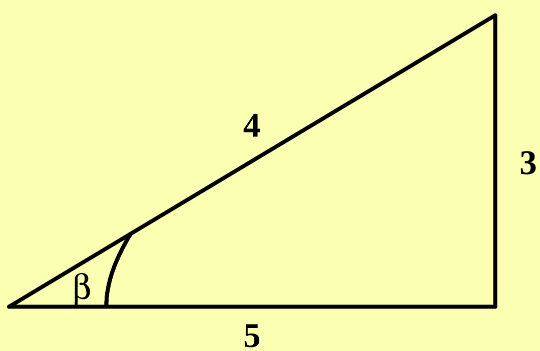




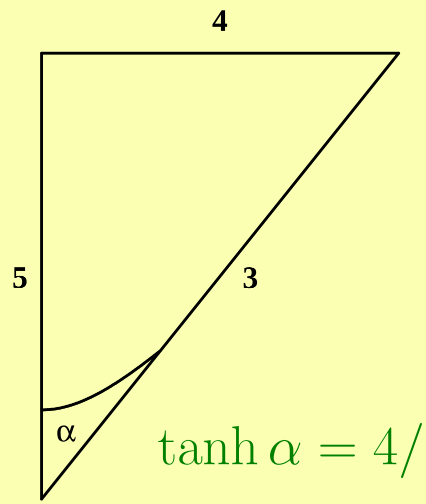
$$\tanh \beta = 3/5$$

HYPERBOLIC TRIANGLE TRIG



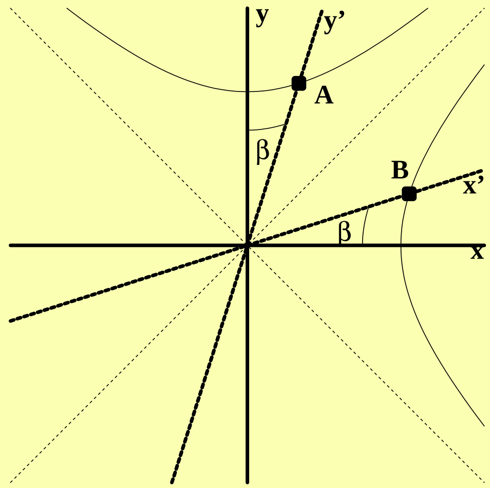


$$\tanh\beta = 3/5$$

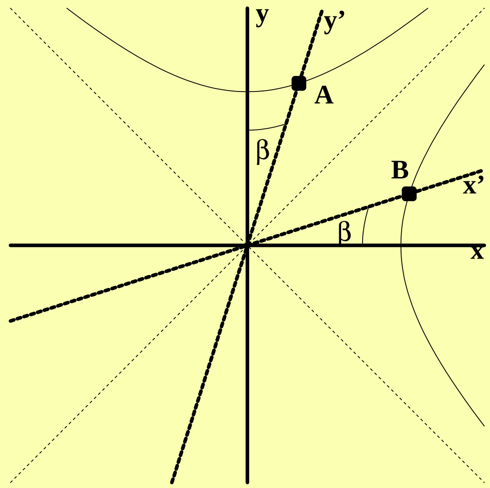


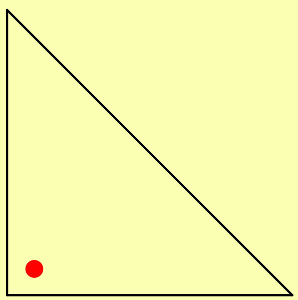
$$\tanh\alpha = 4/5$$

RIGHT TRIANGLES

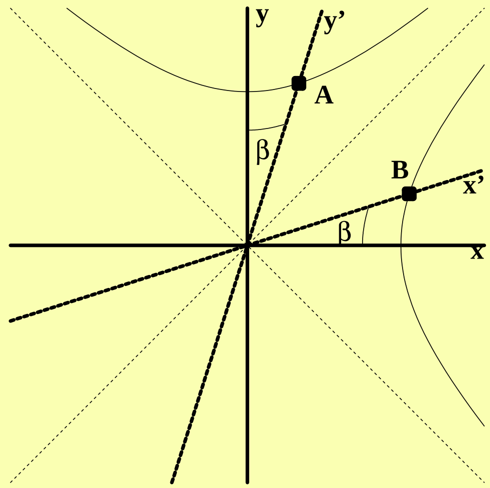


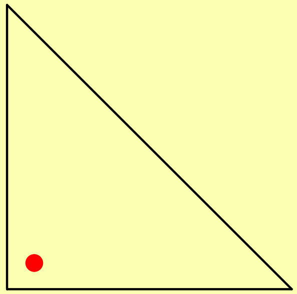
RIGHT TRIANGLES

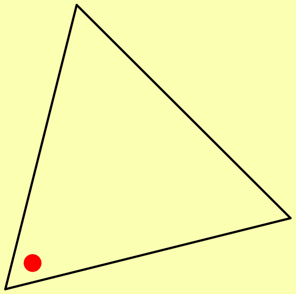




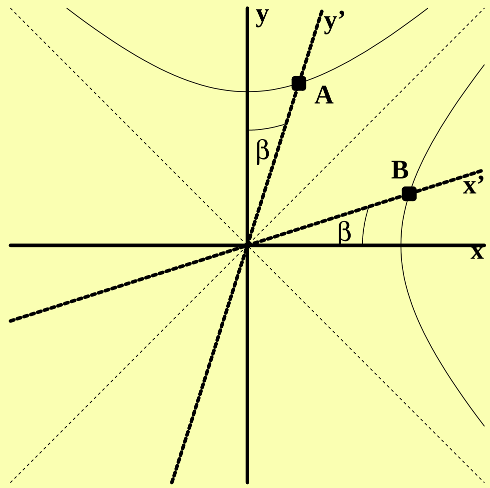
RIGHT TRIANGLES

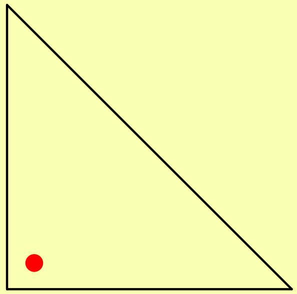


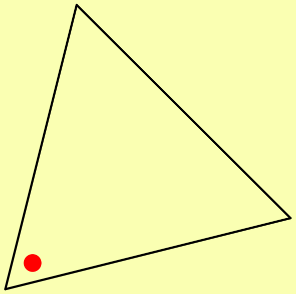


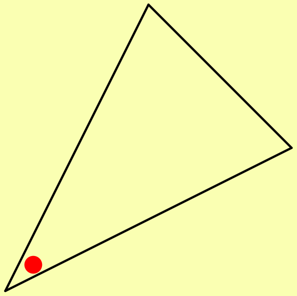


RIGHT TRIANGLES

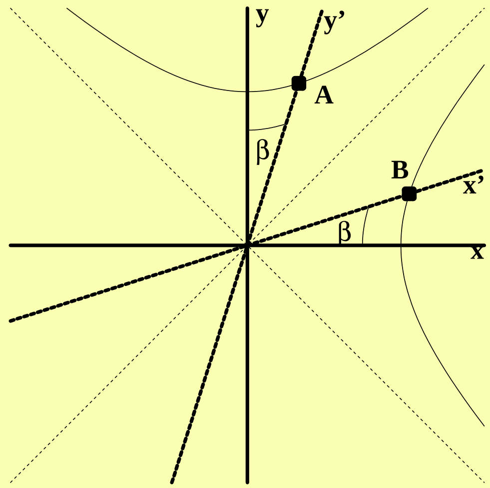


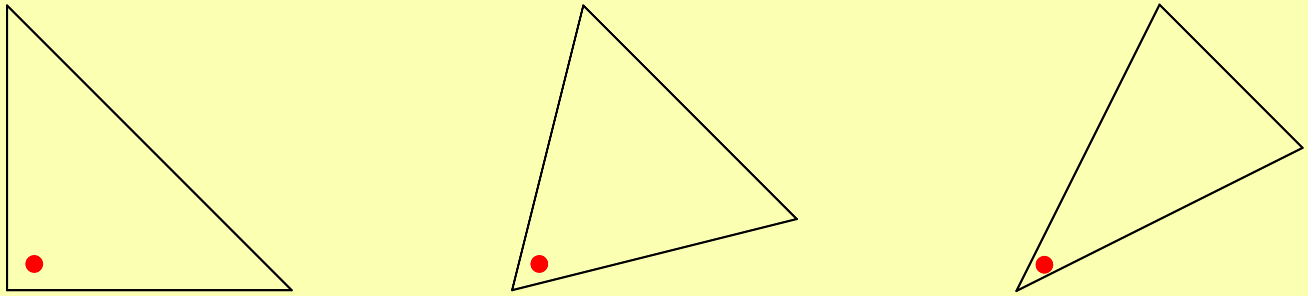






RIGHT TRIANGLES





“right angles” are not angles!

WHICH GEOMETRY?

WHICH GEOMETRY?

signature	
$(+ + \ldots +)$	Euclidean

WHICH GEOMETRY?

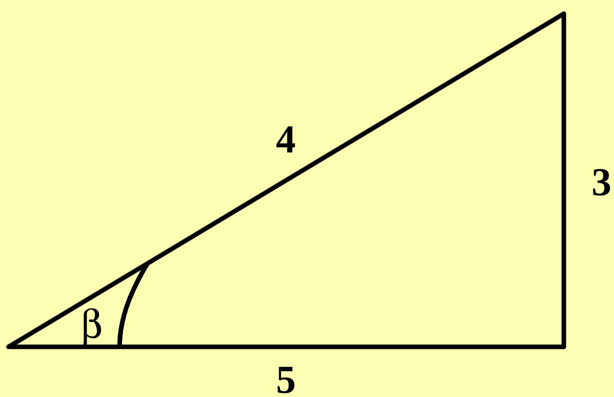
signature	
$(+ + \ldots +)$	Euclidean
$(- + \ldots +)$	Minkowskian

$$ds^2 = -c^2\, dt^2 + dx^2$$

WHICH GEOMETRY?

signature	
$(+ + \ldots +)$	Euclidean
$(- + \ldots +)$	Minkowskian

$$ds^2 = -c^2\, dt^2 + dx^2$$

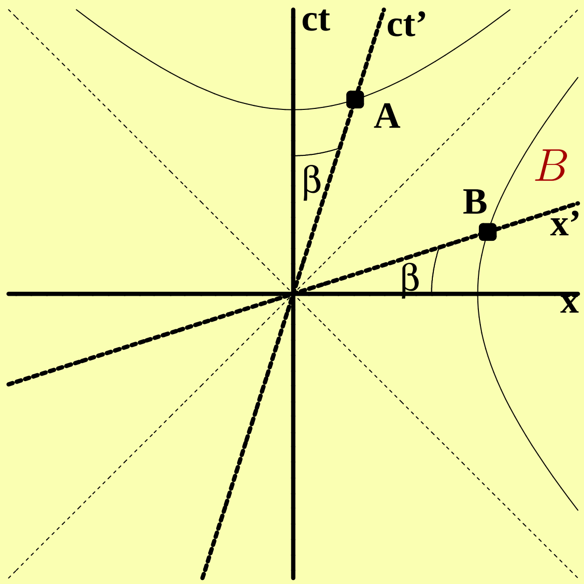


$$\tanh \beta = \frac{3}{5}$$

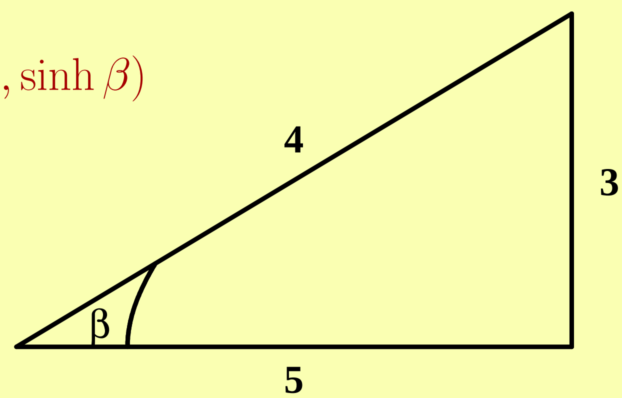
WHICH GEOMETRY?

signature	
$(++\dots+)$	Euclidean
$(-+\dots+)$	Minkowskian

$$ds^2 = -c^2\,dt^2 + dx^2$$



$B = (\cosh\beta, \sinh\beta)$

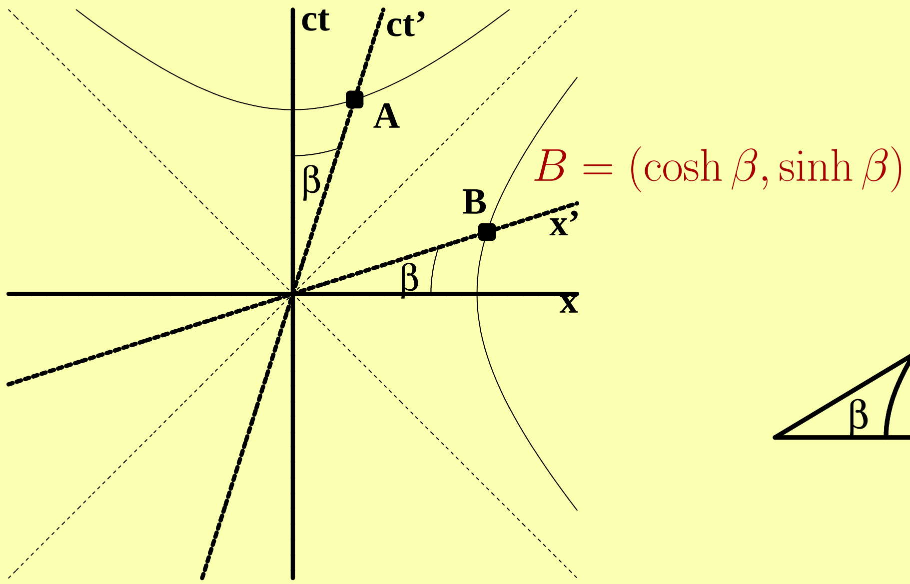


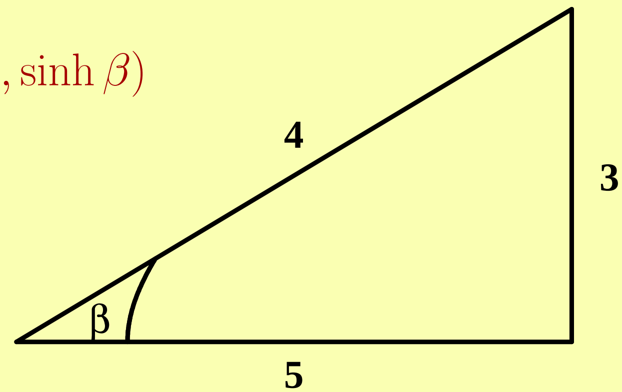
$\tanh\beta = \frac{3}{5}$

WHICH GEOMETRY?

signature	
$(+ + \ldots +)$	Euclidean
$(- + \ldots +)$	Minkowskian

$$ds^2 = -c^2\, dt^2 + dx^2$$





$$\tanh\beta = \frac{3}{5}$$

Special Relativity!

DRAWING SPACETIME DIAGRAMS

DRAWING SPACETIME DIAGRAMS

- Points in spacetime are called *events*.

DRAWING SPACETIME DIAGRAMS

- Points in spacetime are called *events*.
- Lines with slope $m = \pm 1$ represent beams of light.

DRAWING SPACETIME DIAGRAMS

- Points in spacetime are called *events*.
- Lines with slope $m = \pm 1$ represent beams of light.
- Vertical lines represent the worldline of an object at rest.

DRAWING SPACETIME DIAGRAMS

- Points in spacetime are called *events*.
- Lines with slope $m = \pm 1$ represent beams of light.
- Vertical lines represent the worldline of an object at rest.
- Horizontal lines represent snapshots of constant time, that is, events which are simultaneous (in the given reference frame).

DRAWING SPACETIME DIAGRAMS

- Points in spacetime are called *events*.
- Lines with slope $m = \pm 1$ represent beams of light.
- Vertical lines represent the worldline of an object at rest.
- Horizontal lines represent snapshots of constant time, that is, events which are simultaneous (in the given reference frame).
- Lines with slope $|m| > 1$ (called *timelike*) represent the worldlines of observers moving at constant speed.

DRAWING SPACETIME DIAGRAMS

- Points in spacetime are called *events*.
- Lines with slope $m = \pm 1$ represent beams of light.
- Vertical lines represent the worldline of an object at rest.
- Horizontal lines represent snapshots of constant time, that is, events which are simultaneous (in the given reference frame).
- Lines with slope $|m| > 1$ (called *timelike*) represent the worldlines of observers moving at constant speed.
- The speed of such an observer is given by $c \tanh \beta$, where β is the (hyperbolic) angle between the worldline and a *vertical* line.

DRAWING SPACETIME DIAGRAMS

- Points in spacetime are called *events*.
- Lines with slope $m = \pm 1$ represent beams of light.
- Vertical lines represent the worldline of an object at rest.
- Horizontal lines represent snapshots of constant time, that is, events which are simultaneous (in the given reference frame).
- Lines with slope $|m| > 1$ (called *timelike*) represent the worldlines of observers moving at constant speed.
- The speed of such an observer is given by $c \tanh \beta$, where β is the (hyperbolic) angle between the worldline and a *vertical* line.
- The distance between two events along such a line is just the time between them as measured by the moving observer.

DRAWING SPACETIME DIAGRAMS

- Points in spacetime are called *events*.
- Lines with slope $m = \pm 1$ represent beams of light.
- Vertical lines represent the worldline of an object at rest.
- Horizontal lines represent snapshots of constant time, that is, events which are simultaneous (in the given reference frame).
- Lines with slope $|m| > 1$ (called *timelike*) represent the worldlines of observers moving at constant speed.
- The speed of such an observer is given by $c \tanh \beta$, where β is the (hyperbolic) angle between the worldline and a *vertical* line.
- The distance between two events along such a line is just the time between them as measured by the moving observer.
- Lines with slope $|m| < 1$ (called *spacelike*) represent lies of simultaneity as seen by an observer moving at constant speed.

DRAWING SPACETIME DIAGRAMS

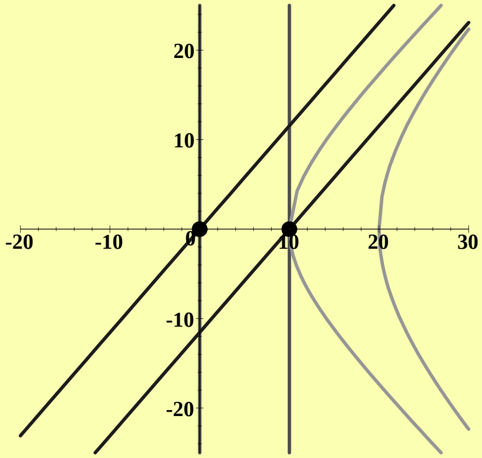
- Points in spacetime are called *events*.
- Lines with slope $m = \pm 1$ represent beams of light.
- Vertical lines represent the worldline of an object at rest.
- Horizontal lines represent snapshots of constant time, that is, events which are simultaneous (in the given reference frame).
- Lines with slope $|m| > 1$ (called *timelike*) represent the worldlines of observers moving at constant speed.
- The speed of such an observer is given by $c \tanh \beta$, where β is the (hyperbolic) angle between the worldline and a *vertical* line.
- The distance between two events along such a line is just the time between them as measured by the moving observer.
- Lines with slope $|m| < 1$ (called *spacelike*) represent lies of simultaneity as seen by an observer moving at constant speed.
- The distance between two events along such a line is just the distance between them as measured by the corresponding observer.

THE POLE AND THE BARN

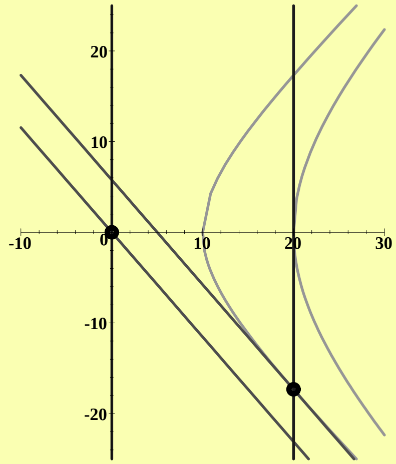
A 20 foot pole is moving towards a 10 foot barn fast enough that the pole appears to be only 10 feet long. As soon as both ends of the pole are in the barn, slam the doors. How can a 20 foot pole fit into a 10 foot barn? Draw a spacetime diagram!

THE POLE AND THE BARN

A 20 foot pole is moving towards a 10 foot barn fast enough that the pole appears to be only 10 feet long. As soon as both ends of the pole are in the barn, slam the doors. How can a 20 foot pole fit into a 10 foot barn? Draw a spacetime diagram!

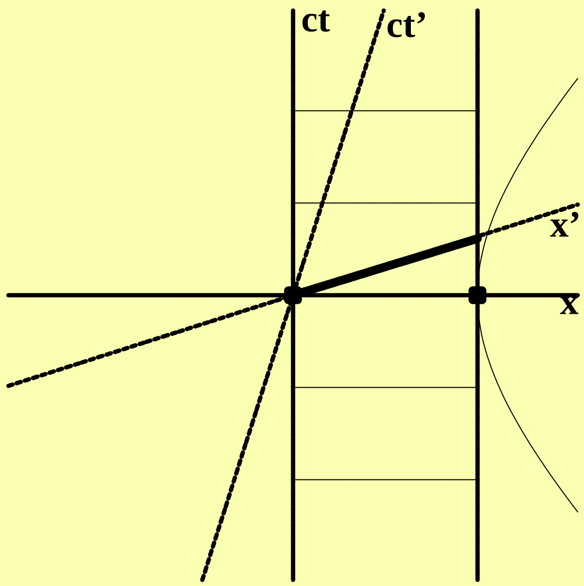


BARN



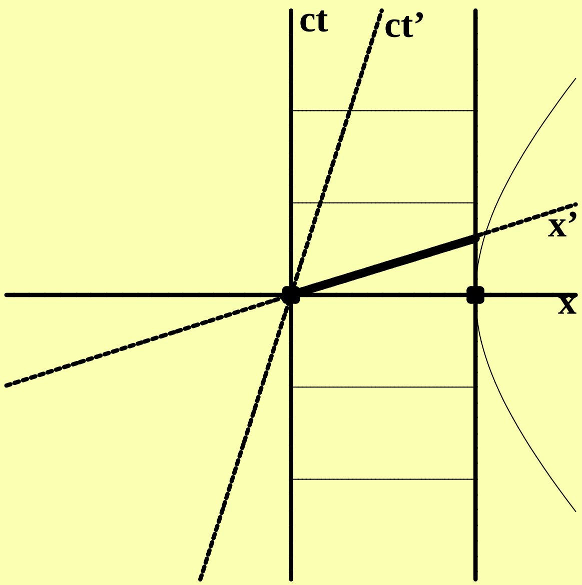
POLE

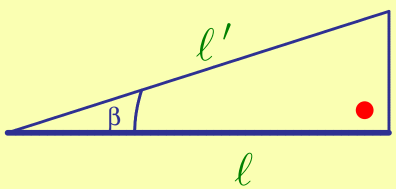
LENGTH CONTRACTION



Compare

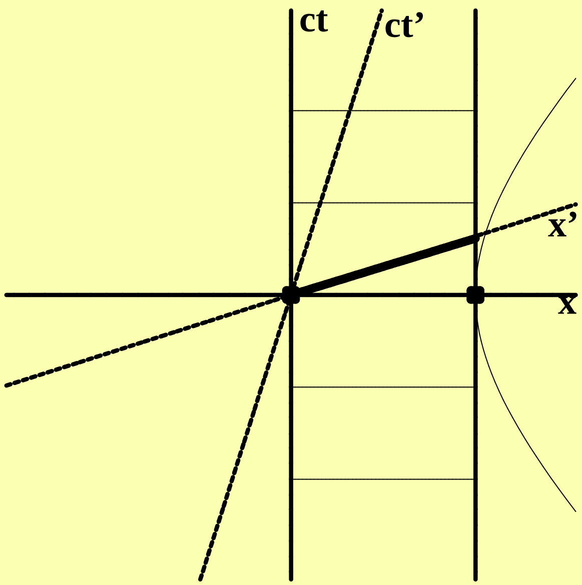
LENGTH CONTRACTION



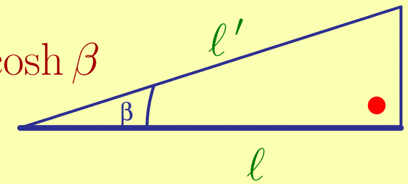


Compare

LENGTH CONTRACTION

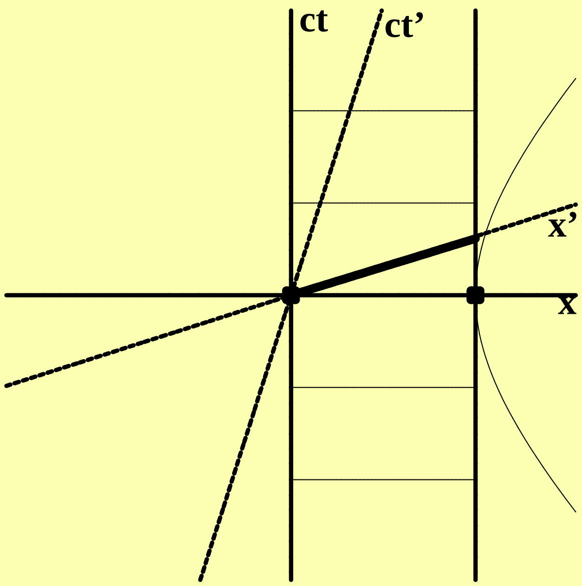


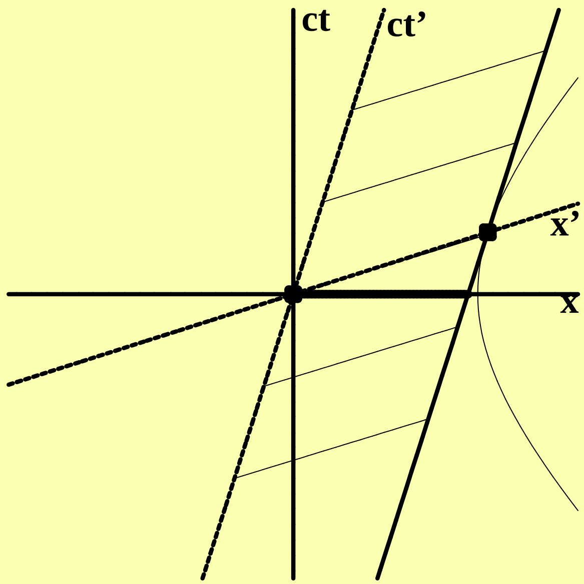
$$\ell = \ell' \cosh \beta$$



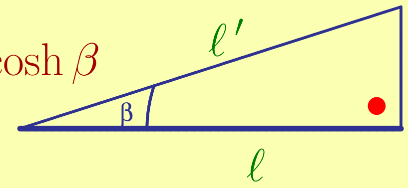
Compare

LENGTH CONTRACTION



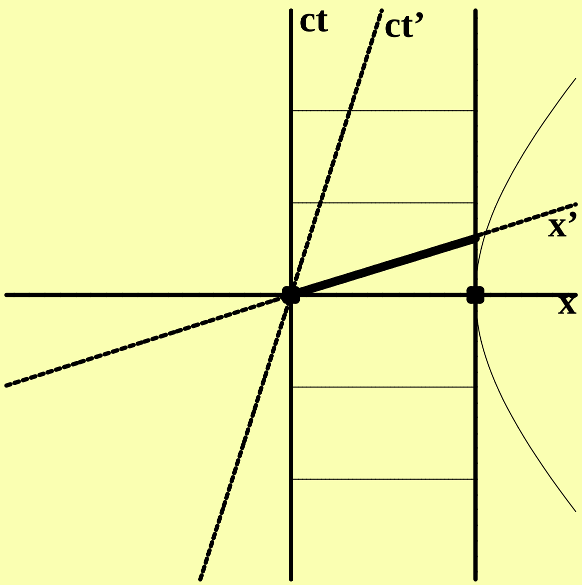


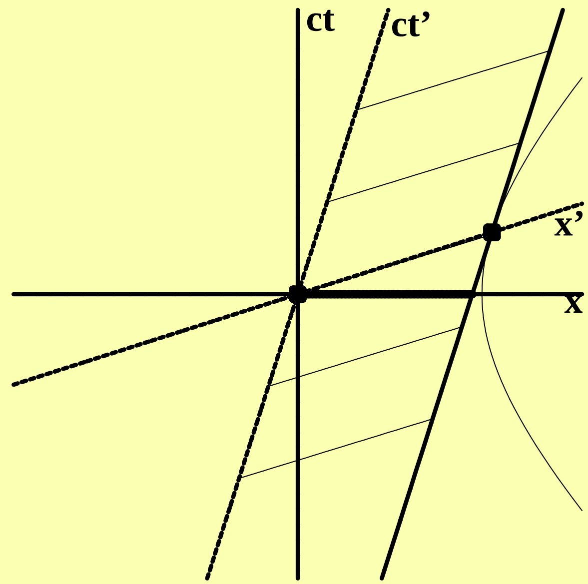
$$\ell = \ell' \cosh \beta$$



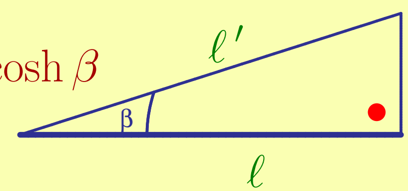
Compare

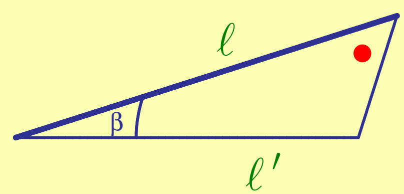
LENGTH CONTRACTION





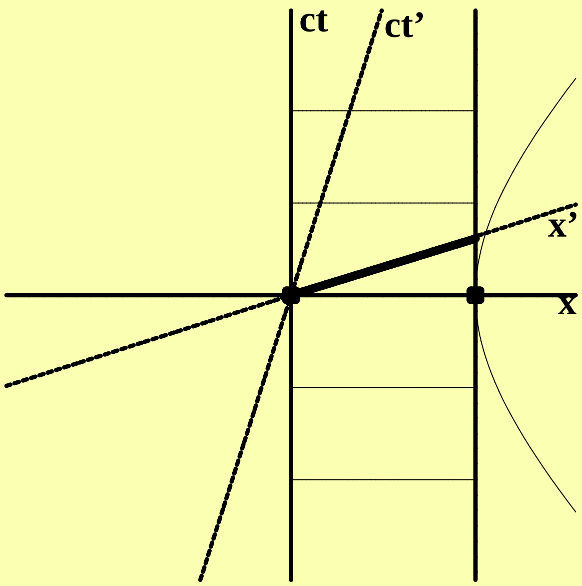
$\ell = \ell' \cosh \beta$

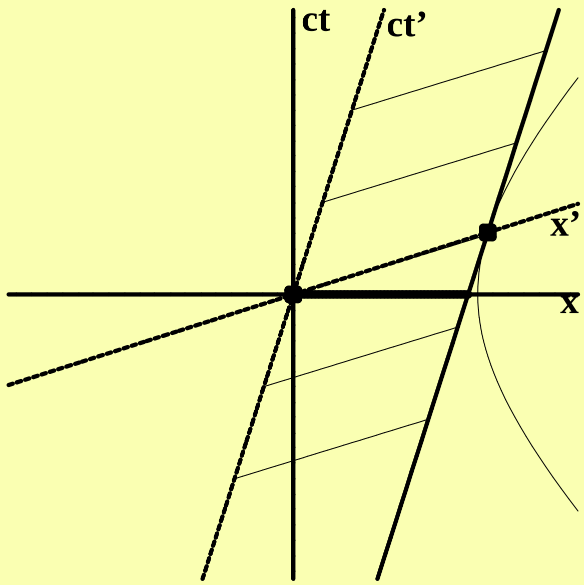




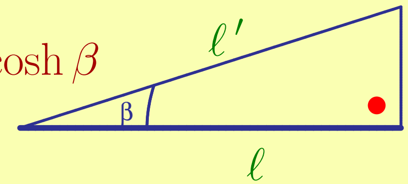
Compare

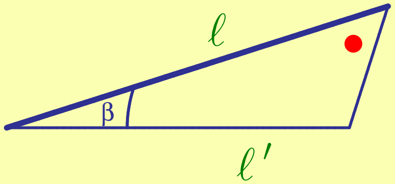
LENGTH CONTRACTION





$\ell = \ell' \cosh \beta$

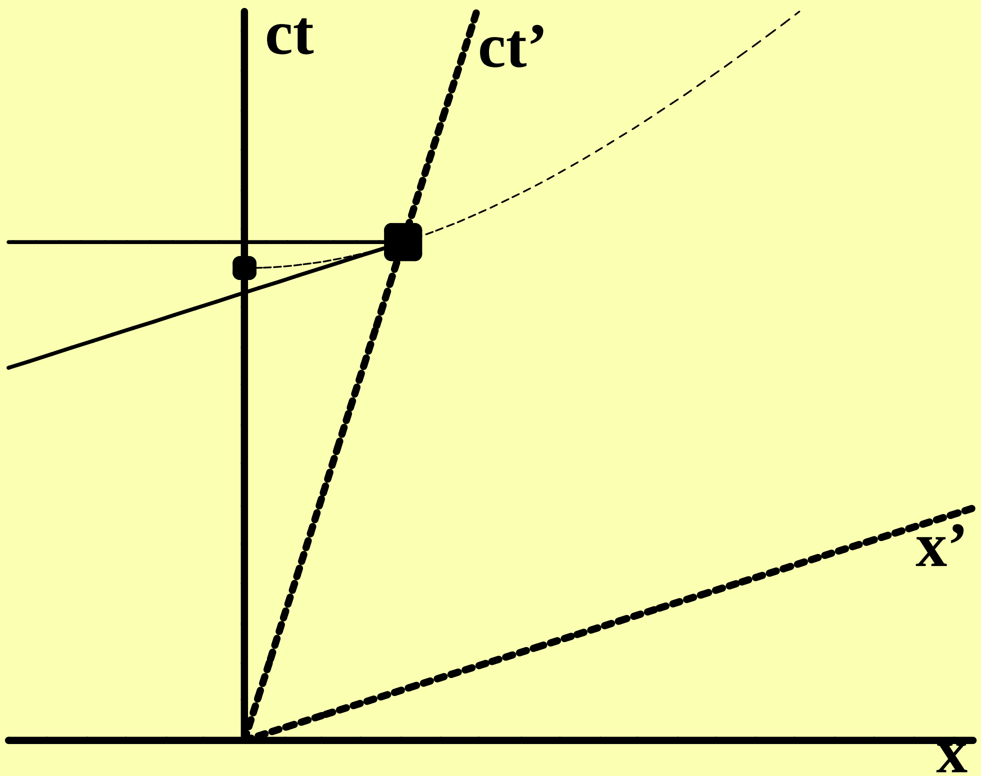




$$\ell' = \frac{\ell}{\cosh \beta}$$

Compare

TIME DILATION

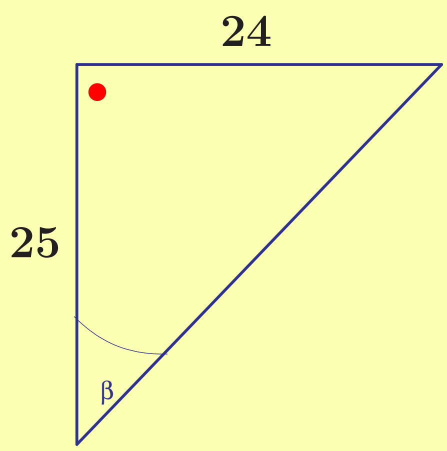


TWIN PARADOX

One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?

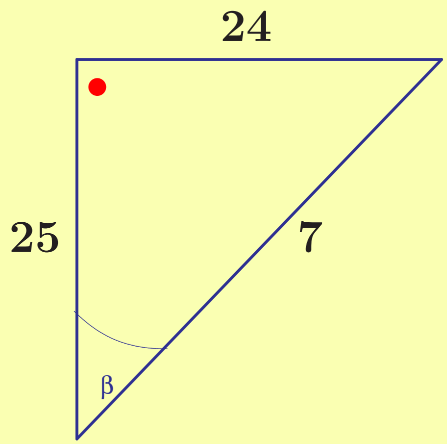
TWIN PARADOX

One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



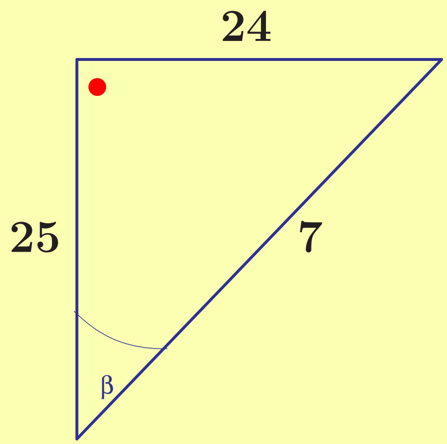
TWIN PARADOX

One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



TWIN PARADOX

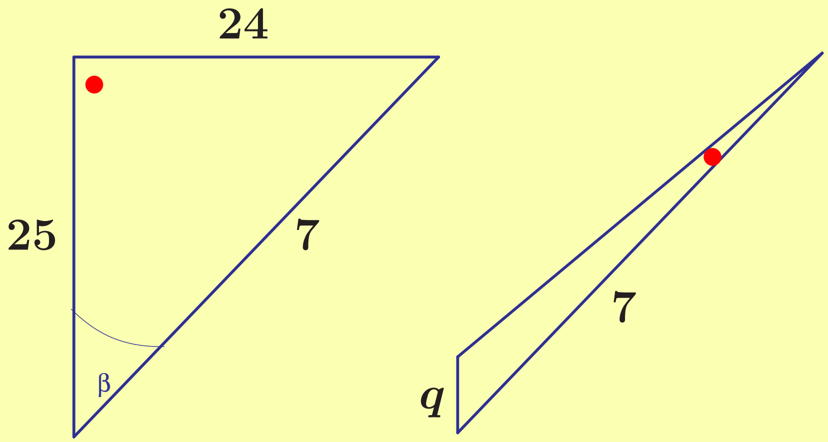
One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



$$\cosh\beta = \frac{25}{7}$$

TWIN PARADOX

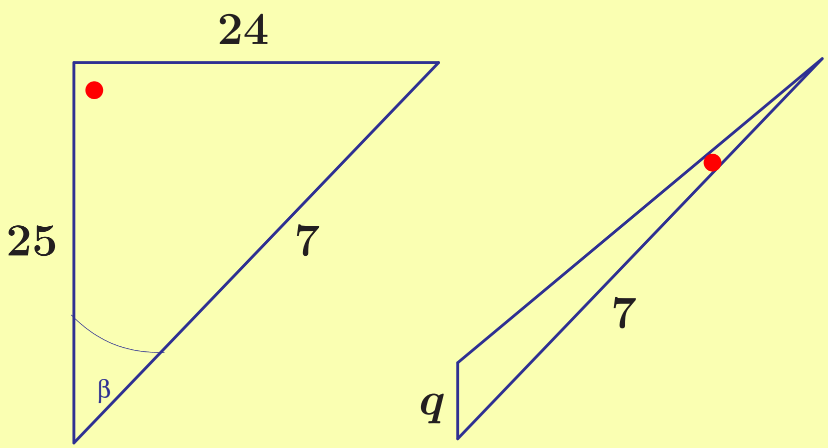
One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



$$\cosh\beta = \frac{25}{7}$$

TWIN PARADOX

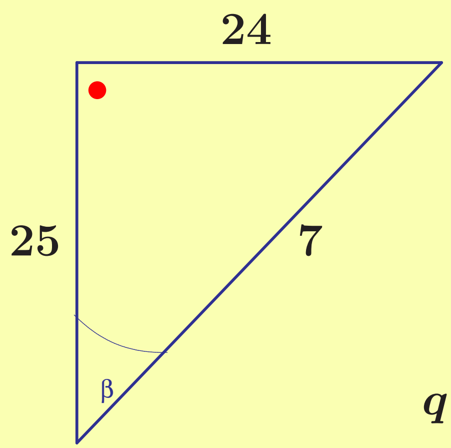
One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



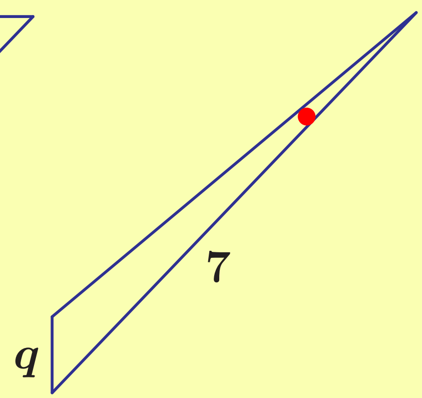
$$\cosh\beta = \frac{25}{7} \qquad q = \frac{7}{\cosh\beta} = \frac{49}{25}$$

TWIN PARADOX

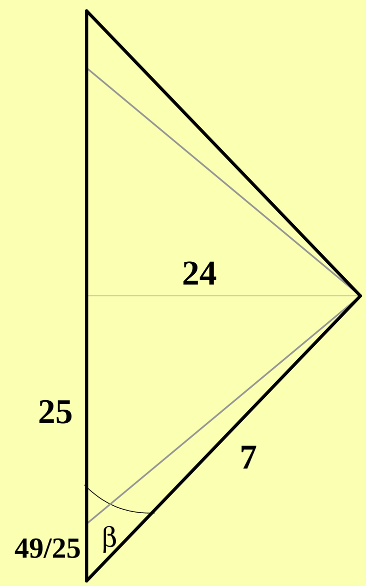
One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



$$\cosh\beta = \frac{25}{7}$$

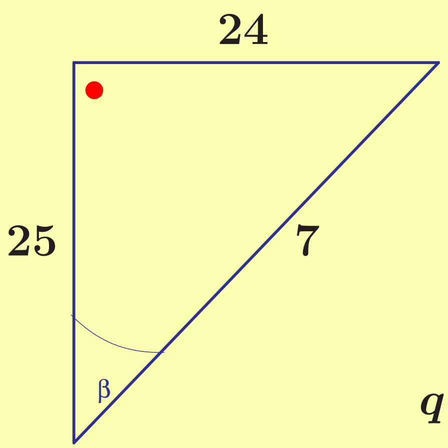


$$q = \frac{7}{\cosh\beta} = \frac{49}{25}$$

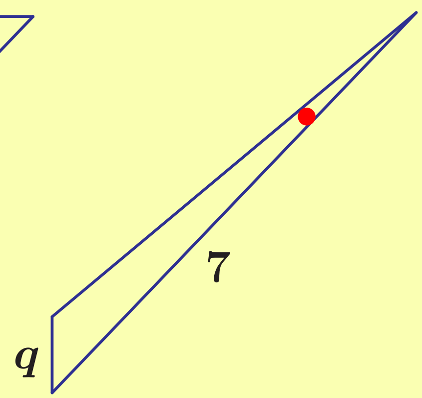


TWIN PARADOX

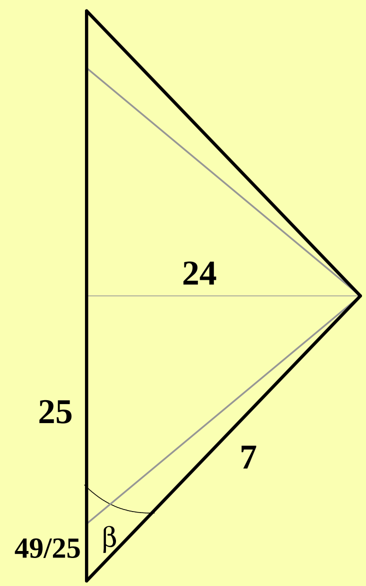
One twin travels 24 light-years to star X at speed $\frac{24}{25}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately turns around, and returns at the same speed. How long does each twin think the trip took?



$$\cosh\beta = \frac{25}{7}$$

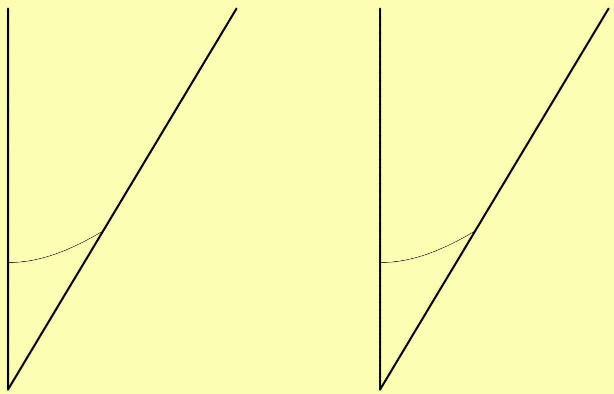


$$q = \frac{7}{\cosh\beta} = \frac{49}{25}$$



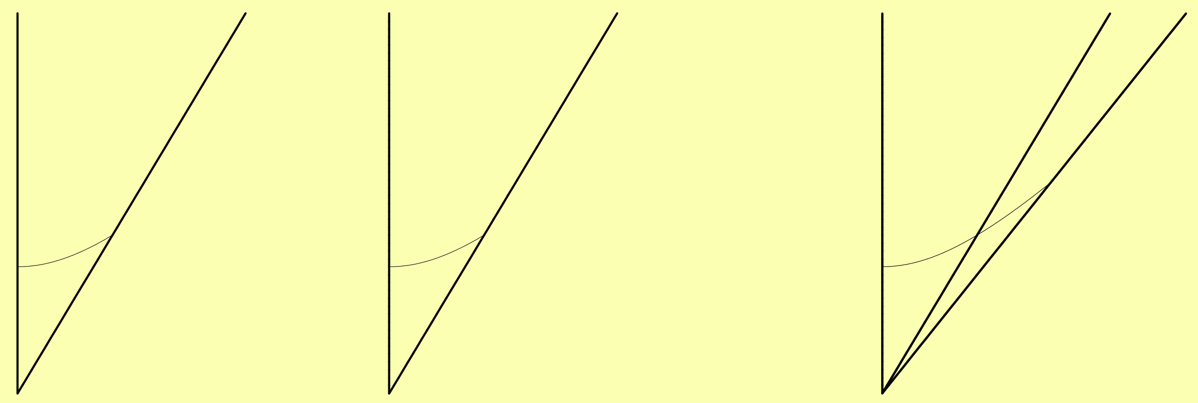
Straight path takes longest!

ADDITION OF VELOCITIES



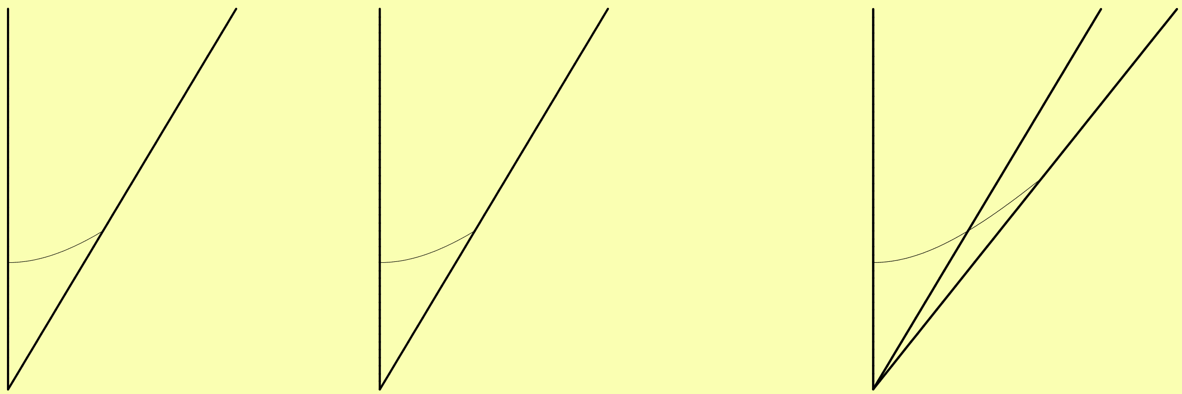
Compare

ADDITION OF VELOCITIES



Compare

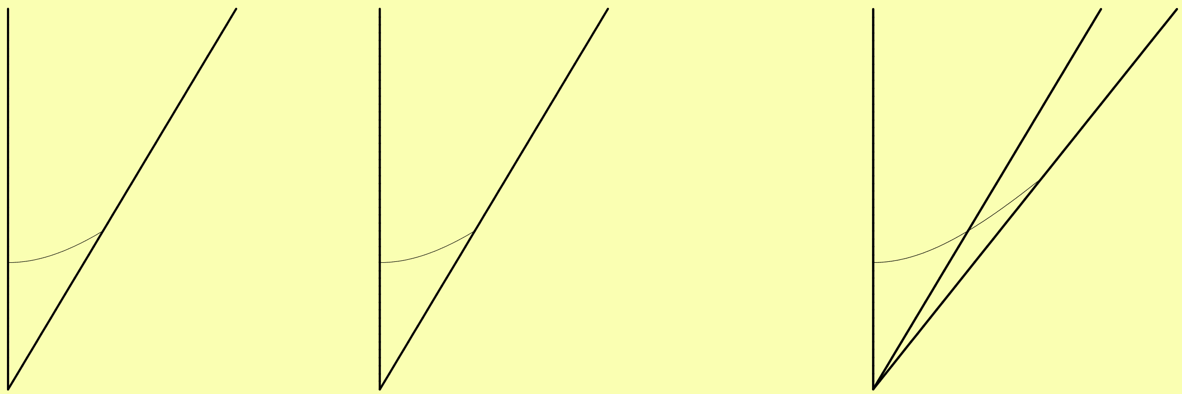
ADDITION OF VELOCITIES



$$\frac{v}{c} = \tanh \beta$$

Compare

ADDITION OF VELOCITIES

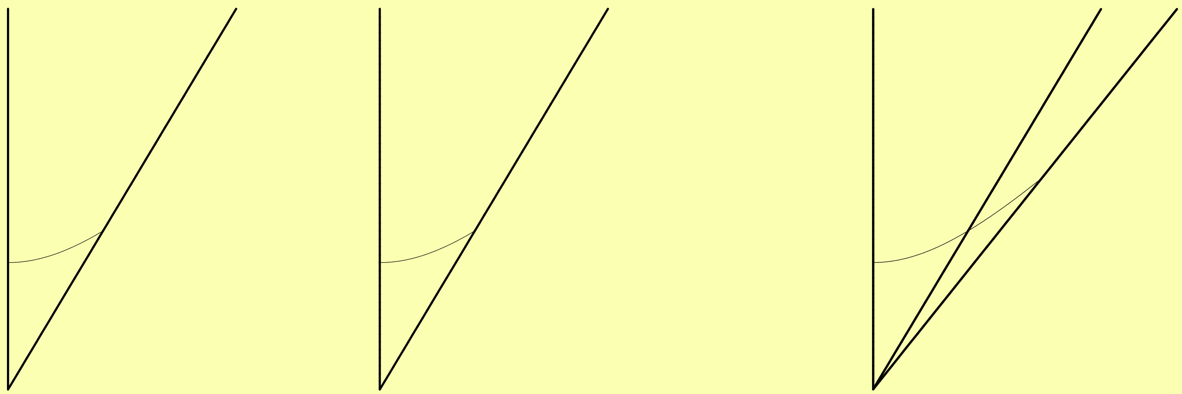


$$\frac{v}{c} = \tanh \beta$$

$$\tanh(\alpha + \beta) = \frac{\tanh \alpha + \tanh \beta}{1 + \tanh \alpha \tanh \beta}$$

Compare

ADDITION OF VELOCITIES

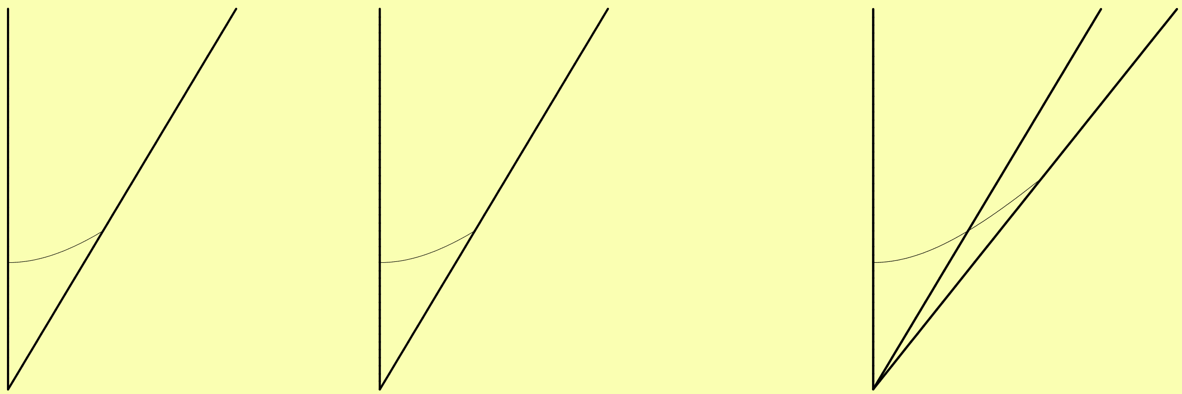


$$\frac{v}{c} = \tanh\beta$$

$$\tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}$$

Compare

ADDITION OF VELOCITIES



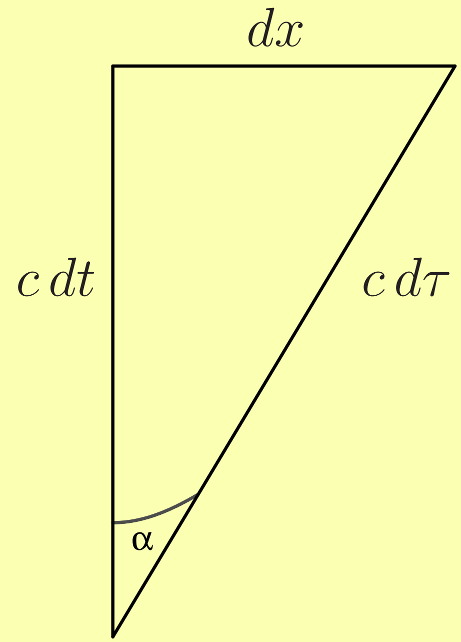
$$\frac{v}{c} = \tanh\beta$$

$$\tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}$$

Einstein addition formula!

Compare

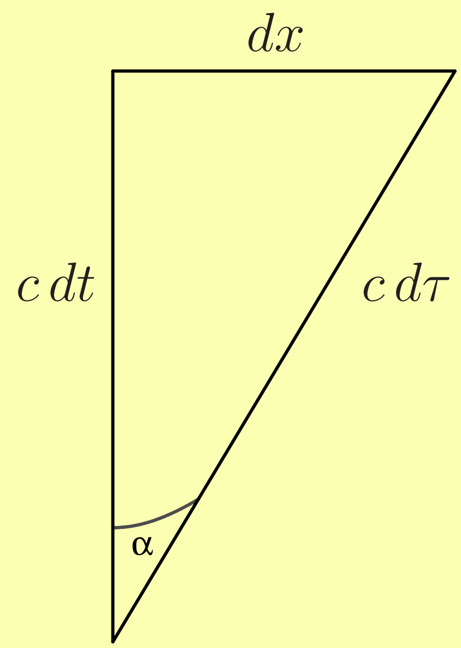
RELATIVISTIC MOMENTUM



RELATIVISTIC MOMENTUM

$$p = m\frac{dx}{d\tau} = mc\sinh\alpha$$

$$E = mc^2\cosh\alpha = mc^2\frac{dt}{d\tau}$$

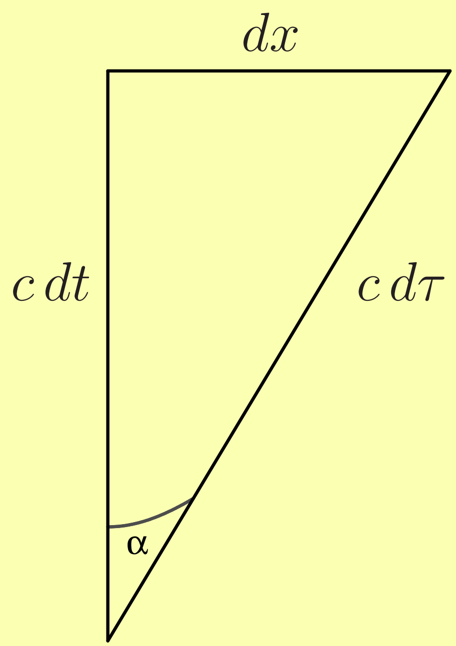


RELATIVISTIC MOMENTUM

$$p = m\frac{dx}{d\tau} = mc\sinh\alpha$$

$$E = mc^2\cosh\alpha = mc^2\frac{dt}{d\tau}$$

$$\begin{pmatrix} \frac{E}{c} \\ p \end{pmatrix} = m\frac{d}{d\tau}\begin{pmatrix} ct \\ x \end{pmatrix}$$

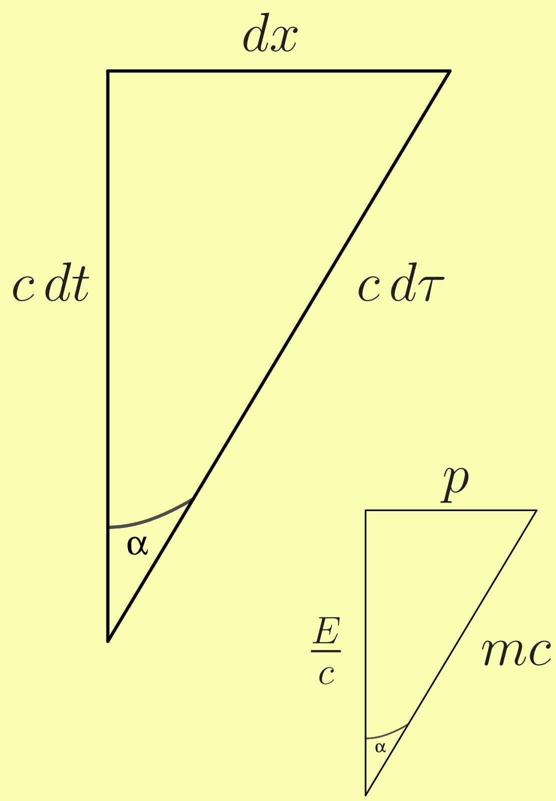


RELATIVISTIC MOMENTUM

$$p = m\frac{dx}{d\tau} = mc\sinh\alpha$$

$$E = mc^2\cosh\alpha = mc^2\frac{dt}{d\tau}$$

$$\begin{pmatrix} \frac{E}{c} \\ p \end{pmatrix} = m\frac{d}{d\tau}\begin{pmatrix} ct \\ x \end{pmatrix}$$

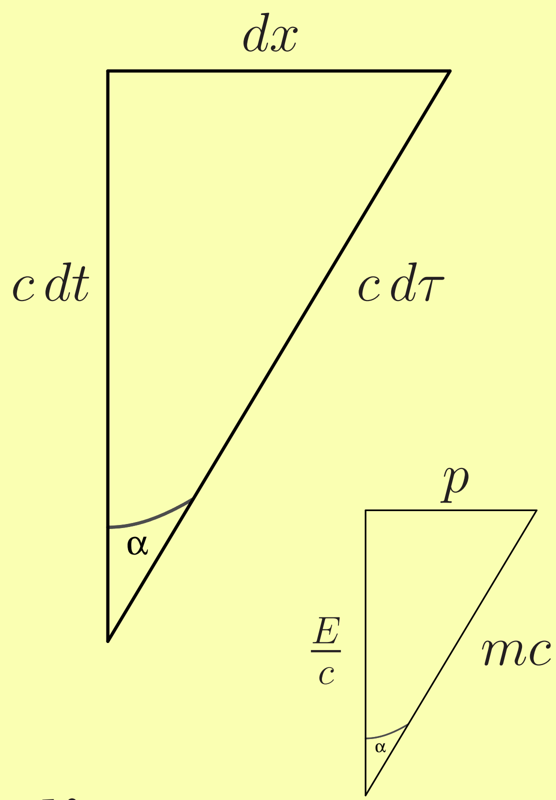


RELATIVISTIC MOMENTUM

$$p = m\frac{dx}{d\tau} = mc\sinh\alpha$$

$$E = mc^2\cosh\alpha = mc^2\frac{dt}{d\tau}$$

$$\begin{pmatrix} \frac{E}{c} \\ p \end{pmatrix} = m\frac{d}{d\tau}\begin{pmatrix} ct \\ x \end{pmatrix}$$



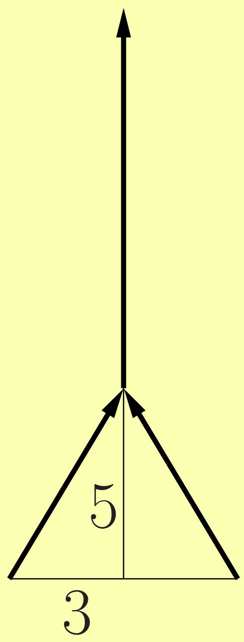
Energy-momentum is conserved!

COLLISIONS

Two identical lumps of clay of (rest) mass m collide head on, with each moving at $\frac{3}{5}c$. What is the mass of the resulting lump of clay?

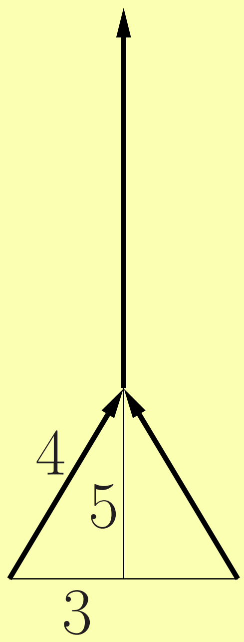
COLLISIONS

Two identical lumps of clay of (rest) mass m collide head on, with each moving at $\frac{3}{5}c$. What is the mass of the resulting lump of clay?



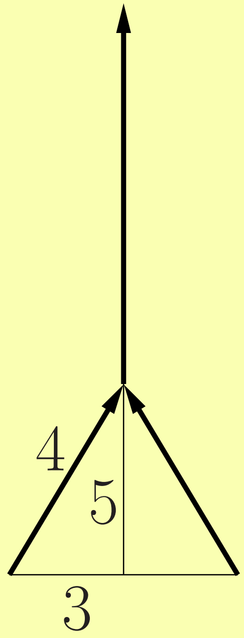
COLLISIONS

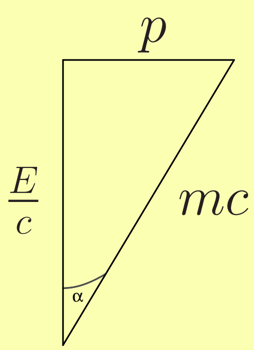
Two identical lumps of clay of (rest) mass m collide head on, with each moving at $\frac{3}{5}c$. What is the mass of the resulting lump of clay?



COLLISIONS

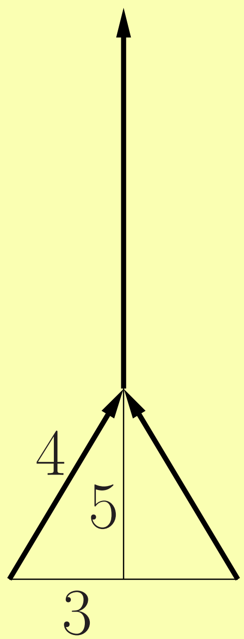
Two identical lumps of clay of (rest) mass m collide head on, with each moving at $\frac{3}{5}c$. What is the mass of the resulting lump of clay?

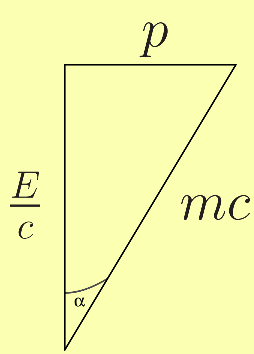




COLLISIONS

Two identical lumps of clay of (rest) mass m collide head on, with each moving at $\frac{3}{5}c$. What is the mass of the resulting lump of clay?

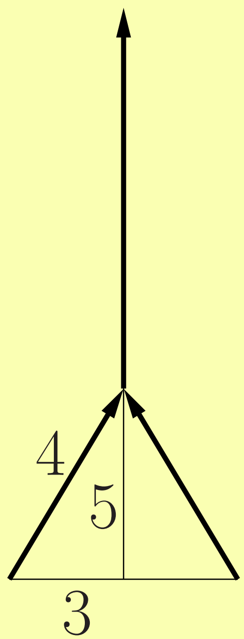


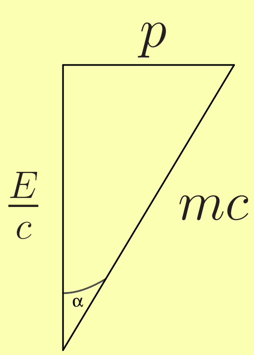


$$E = mc^2 \cosh \alpha = \frac{5}{4} mc^2$$

COLLISIONS

Two identical lumps of clay of (rest) mass m collide head on, with each moving at $\frac{3}{5}c$. What is the mass of the resulting lump of clay?



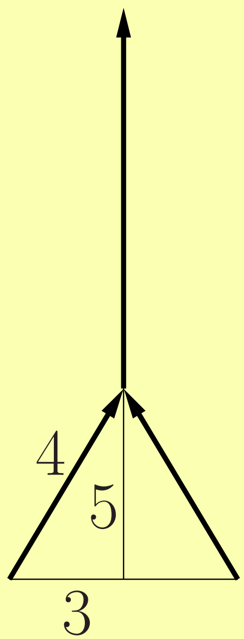


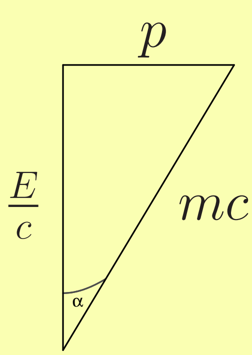
$$E = mc^2 \cosh\alpha = \frac{5}{4}mc^2$$

$$E_{\text{tot}} = 2E = \frac{5}{2}mc^2$$

COLLISIONS

Two identical lumps of clay of (rest) mass m collide head on, with each moving at $\frac{3}{5}c$. What is the mass of the resulting lump of clay?





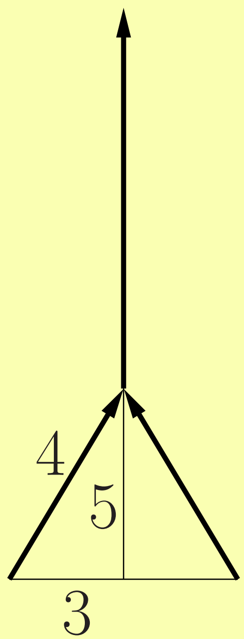
$$E = mc^2 \cosh\alpha = \frac{5}{4}mc^2$$

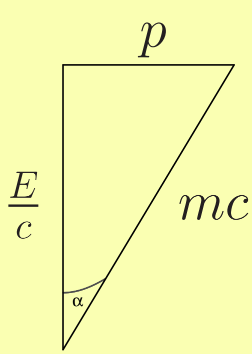
$$E_{\text{tot}} = 2E = \frac{5}{2}mc^2$$

$$E_{\text{tot}} = MC^2$$

COLLISIONS

Two identical lumps of clay of (rest) mass m collide head on, with each moving at $\frac{3}{5}c$. What is the mass of the resulting lump of clay?





$$E = mc^2 \cosh \alpha = \frac{5}{4} mc^2$$

$$E_{\rm tot} = 2E = \frac{5}{2} mc^2$$

$$E_{\rm tot} = MC^2$$

$$M = \frac{5}{2} m > 2m$$

WHICH GEOMETRY?

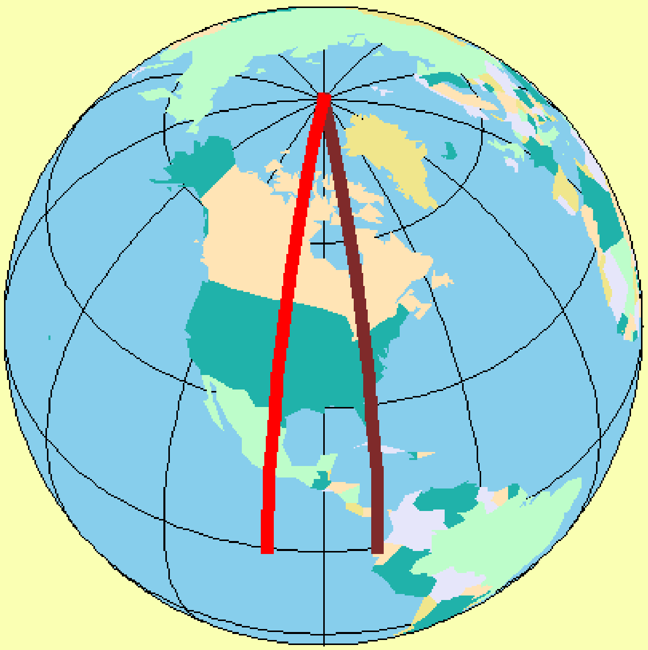
signature	**flat**
$(+ + \ldots +)$	Euclidean
$(- + \ldots +)$	Minkowskian

WHICH GEOMETRY?

signature	**flat**	**curved**
$(+ + \dots +)$	Euclidean	Riemannian
$(- + \dots +)$	Minkowskian	

WHICH GEOMETRY?

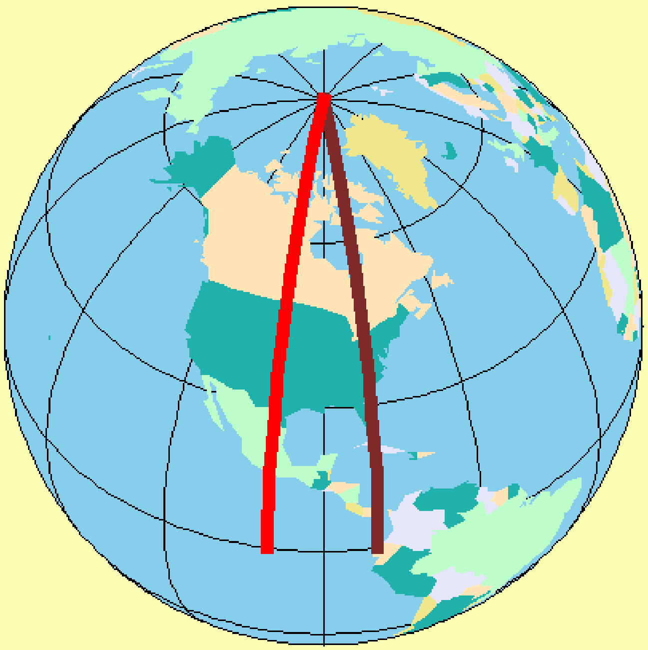
signature	**flat**	**curved**
$(+ + \ldots +)$	Euclidean	Riemannian
$(- + \ldots +)$	Minkowskian	



$$ds^2 = r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

WHICH GEOMETRY?

signature	flat	curved
$(+ + \ldots +)$	Euclidean	Riemannian
$(- + \ldots +)$	Minkowskian	



$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

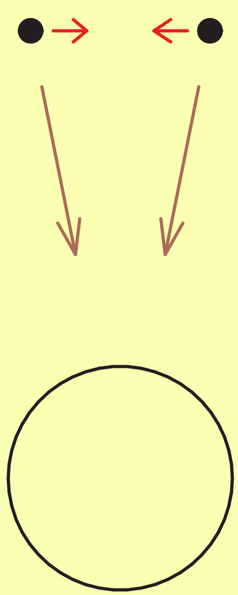
Tidal forces!

WHICH GEOMETRY?

signature	**flat**	**curved**
$(+ + \ldots +)$	Euclidean	Riemannian
$(- + \ldots +)$	Minkowskian	Lorentzian

WHICH GEOMETRY?

signature	**flat**	**curved**
$(+ + \ldots +)$	Euclidean	Riemannian
$(- + \ldots +)$	Minkowskian	Lorentzian



General Relativity!

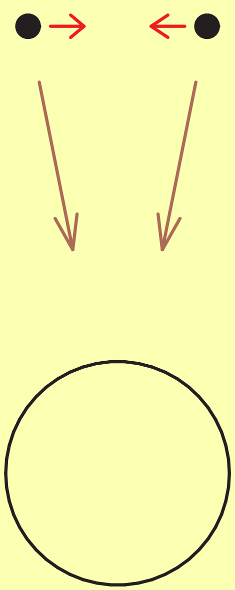
WHICH GEOMETRY?

signature	**flat**	**curved**
$(+ + \ldots +)$	Euclidean	Riemannian
$(- + \ldots +)$	Minkowskian	Lorentzian

$$ds^2 = -dt^2 + a(t)\, dx^2$$

Cosmology!

$(c = 1)$



General Relativity!

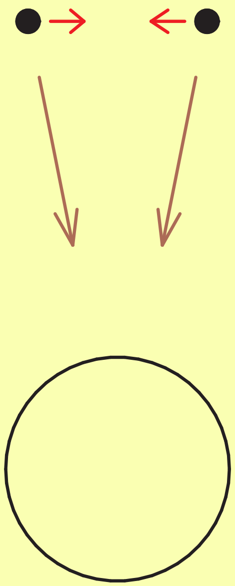
WHICH GEOMETRY?

signature	flat	curved
$(+ + \ldots +)$	Euclidean	Riemannian
$(- + \ldots +)$	Minkowskian	Lorentzian

$$ds^2 = -dt^2 + a(t)\,dx^2$$

Cosmology!

$(c = 1)$



$$ds^2 = -\left(1 - \tfrac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \tfrac{2m}{r}\right)}$$

$$+\, r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

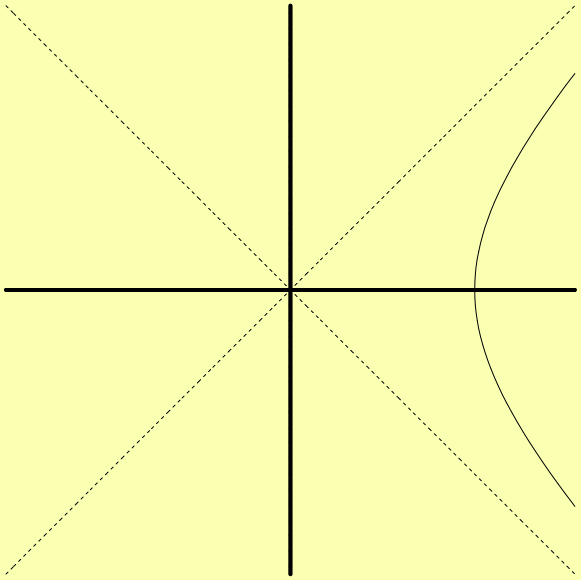
General Relativity!

BLACK HOLES

Einstein: gravity=acceleration

BLACK HOLES

Einstein: gravity=acceleration



THE GEOMETRY OF SPECIAL RELATIVITY

Oregon State University

Tevian Dray

http://www.physics.oregonstate.edu/portfolioswiki

http://www.physics.oregonstate.edu/coursewikis/GSR

http://www.math.oregonstate.edu/~tevian/geometry