Representations of Partial Derivatives

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When making measurements, scientists attempt to change one variable while holding all other independent variables fixed, a process that is mathematically modeled as a partial derivative. This project aims to characterize experts' and students' disciplinary understandings of partial derivatives across STEM subjects. We have developed an expanded framework of the concept image of derivative to include "thick derivatives" and classroom activities to move students toward a more robust understanding of derivatives. We continue to test these materials and explore students' initial and ongoing understandings of partial derivatives. This project will provide classroom-tested curricular materials and associated instructor resources to the education community that will support learning trajectories in multiple STEM disciplines.

Introduction:

A common feature across STEM disciplines is that we are interested in studying change, whether we are studying how changing a design parameter affects a device, how changes in temperature affect a measurement, or how pressure changes when we adiabatically compress a gas. Indeed, the nature of scientific measurement is to control all independent parameters of an experiment except for the one parameter being studied—which itself is being changed. Mathematically, we express the concept of changing one parameter while fixing others by using partial derivatives. However, how we use partial derivatives, and how we talk about partial derivatives varies dramatically across STEM disciplines. We have found that many students—even those with a strong mathematics background—find partial derivatives particularly difficult. This raises the question of how we can best prepare students to use partial derivatives in their fields.

Goals:

Our current grant continues the joint work of two very successful projects: The Paradigms in Physics Project[1,2], begun in 1997, and the Vector Calculus Bridge Project[3], begun in 2001. Written materials produced by these projects include more than 250 group activities and class notes for 20 separate courses.

The major goals of this phase of the project are to:

- Explore how experts use and represent change;
- Move students toward a robust understanding of the quantification of change;
- Develop and test curricular materials for middle-division math and physics courses;
- Establish students' initial and ongoing levels of understanding as they progress through these materials;
- Make these materials freely available online.

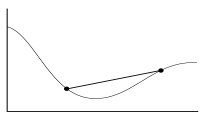
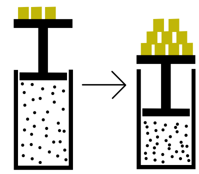
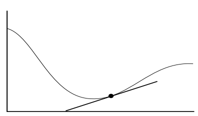
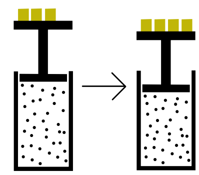
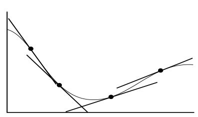
Approach:

A theoretical appreciation of representations is helpful in understanding how students interpret, use, and move between different representations. We draw on the perspective of distributed cognition[4], which provides an account for the role of external entities (including tools, other people, and representations) in cognition. As a part of this project, we are also studying the representations used to work with partial derivatives across the STEM disciplines, and will use these results to analyze and construct learning trajectories for students, as they progress from novice to expert throughout their university career.

Outcomes:

How experts use and represent change: In the process of interviewing professional mathematicians, physicists, and engineers, we have identified shortcomings that arise when applying Zandieh's framework[5] beyond the level of first-year calculus, and in particular outside the field of mathematics. We have found that the concept image for the derivative of physicists and engineers contains substantial elements that are congruent with the three process-object layers identified by Zandieh, but lead to the introduction of new contexts and representations that could also be productive in the instruction of calculus.

Physicists and engineers live and work in a world full of uncertainty, and are accustomed to use the language of equality where there is actually approximation. This language reflects a somewhat “thicker” concept of the derivative than that held by mathematicians. Where a mathematician would speak of the slope of the secant line as an approximation for the derivative, a physicist or engineer might say that the slope of a line drawn between two thoughtfully chosen measurements of a physical observable is the derivative (with some unspecified uncertainty). As we will explain, this “thickness” derives from the impossibility of achieving exact results in experimental or numerical contexts. Attempts to estimate a derivative over too small an interval, for example, could result in a highly erroneous estimate of a derivative due to numerical round-off error or limitations in experimental precision.

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{1.00-0.84}{1.5-1.0}$	
Limit		"instantaneous …"	$\lim_{x\to 0} \cdots$	$\frac{0.89-0.84}{1.1-1.0}$	
Function		"… at any point/time"	$f'(x) = \cdots$	x | y | $\frac{dy}{dx}$ 0.0 | 0.00 | 0.96 0.5 | 0.48 | 0.72 1.0 | 0.84 | 0.32 1.5 | 1.00 | -.18	*tedious repetition*

	Symbolic
	Instrumental Understanding
Function	*rules to "take a derivative"*

Figure 1: Our extended framework for the concept of the derivative.

We have extended the framework of Zandieh (2000) in several ways (see Figure 1): we have elaborated on the physical representation of the derivative; we have added a numerical representation of the derivative; and we have added space in the framework for the set of rules for finding symbolic derivatives. Each of these changes reflects an expansion of the table to incorporate additional answers to the prompt, "find the derivative." By making use of the numerical representation of the derivative, one can answer the prompt numerically. Similarly, if the derivative is situated in a physical context, one can respond with a measurement process. Both of these responses require a conceptual understanding of the derivative in terms of ratio, limit and function, and involve a certain "thickness" in the derivative. In contrast, as pointed out by Zandieh, the instrumental-understanding approach to "find the derivative" using the rules for symbolic derivatives does not require a conceptual understanding of the derivative.

The notion of "thick" derivative has been introduced in order to address the idea of numerical or experimental data that is approximate but "good enough". Our extended framework has appeared in refereed papers for both the mathematics and physics communities and been presented at several conferences and colloquia. In addition, Michelle Zandieh's[5] framework for describing student concept images of the derivative has been extended to partial derivatives[6,7].

We have also examined how experts use partial derivatives in solving thermodynamics problems and have identified several epistemic games experts play while manipulating partial derivatives.[8]

Moving students toward a robust quantification of change: We are continuing to design and classroom test our learning trajectories and classroom materials in physics and mathematics courses.

In particular, a significant innovation has been the development of the Partial Derivatives Machine (PDM)[9,10]. The PDM was designed with the goal of introducing partial derivatives in a physical context that is familiar to students through a mechanical analogue of a thermodynamic system.

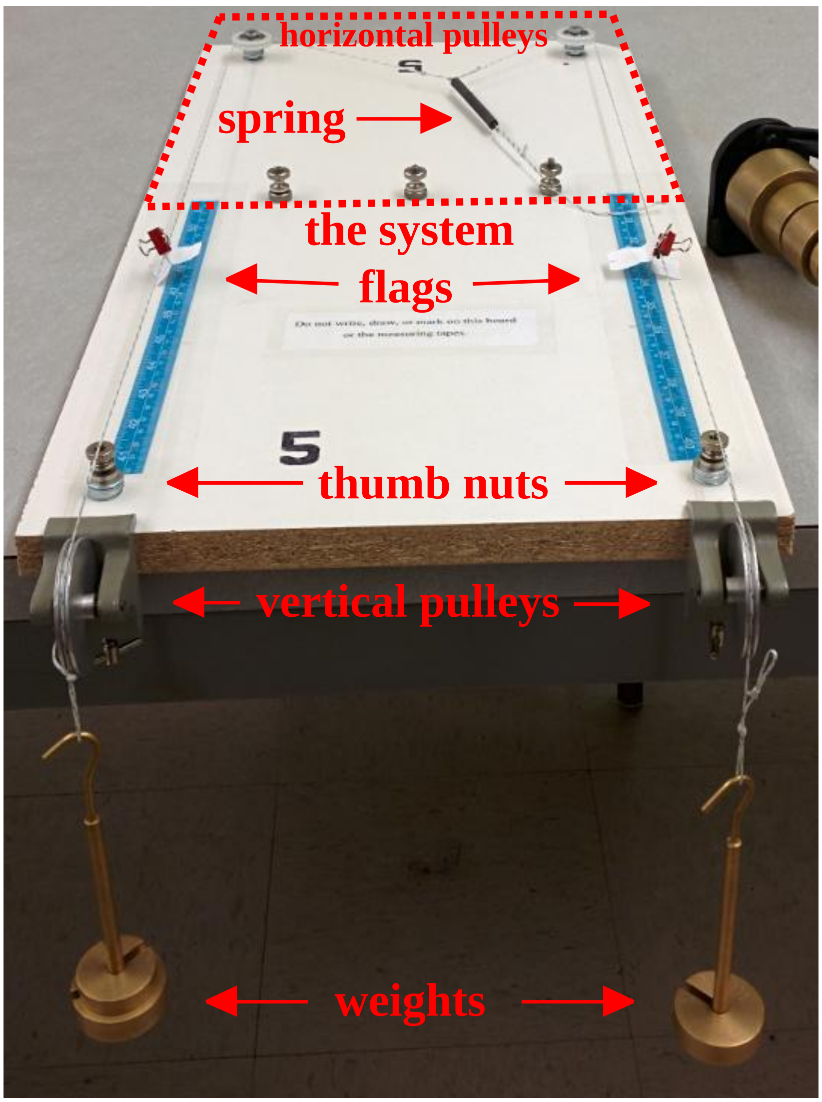


Figure 2: Picture of a Partial Derivatives Machine with parts labeled.

The PDM (Figure 2) consists of a spring system that can be stretched via strings under tension from weights. In order to more easily measure the stretching of the system, a measuring tape is placed on the board parallel to each string and flags are added to the strings. Therefore, four variables may be measured: the weights on each of the two overhanging strings (F_1 and F_2) and the two positions of the flags on the strings (x_1 and x_2).

Several activities with the PDM have been developed and deployed in classes to help students develop more functional and robust understandings of partial derivatives. For example, the thermodynamics course is the first time that our students encounter scenarios in which the quantities held fixed when taking a partial derivative are ambiguous. In mathematics courses, students are taught that when taking partial derivatives, all the *independent* variables are held fixed, or at least all variables are able to be systematically varied and held constant. Nevertheless, through experience in the classroom we have found that most students come into our course with a firm belief that when taking a partial derivative *everything else is held fixed*. An activity that addresses this issue is asking the students to use the PDM to measure $\frac{\partial x_1}{\partial F_{x_1}}$. The students have to consider two possible options: $\left(\frac{\partial x_1}{\partial F_{x_1}}\right)_{x_2}$ and $\left(\frac{\partial x_1}{\partial F_{x_1}}\right)_{F_2}$. The values of these two derivatives are, in principle, different and different procedures are required to measure them. The instructor asks students to consider if this activity is consistent with or contradicts the idea that one takes a partial derivative while holding "everything else" constant.

We have developed sets of activities in physics and mathematics courses to help students develop deep conceptual understandings of partial derivatives[11]. While the PDM is useful in thermodynamics courses, we find physical models of 2D functions as "surfaces" to be more useful in multivariable calculus. We have begun an active collaboration with researchers from the "Raising Calculus to the Surface" project (NSF 1246094) to develop additional derivatives activities.

Establishing students initial and ongoing levels of understanding: We are working on characterizing how students' ways of thinking about partial derivatives develop within multivariable calculus and middle-division physics. Data sources for this research include a combination of individual, semi-structured interviews, classroom video, and assessments of students in both courses. Characterization of how their thinking about partial derivatives develops will allow us to revise the initial learning trajectories and curricular materials in the vein of a design experiment.

Broader Impacts:

This project will directly impact mathematics and physics education at the middle-division undergraduate level by providing classroom-tested curricular materials and associated instructor resources to the education community through existing, proven online resources (an activities wiki and textbook). Mathematics materials will support learning trajectories in multiple STEM disciplines, not just mathematics and physics.

The addition of the new materials will make the existing resources easier to adopt by providing more complete coverage, in line with most common course structures.

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Biographical Information

Elizabeth Gire is an Assistant Professor of Physics at the University of Memphis. Her research is in the area of physics education, with emphasis on problem-solving and the development of epistemological beliefs and metacognitive skills of undergraduate physics majors.

David Roundy does research in computational condensed matter physics, and his curriculum development has focused on improving the teaching of computational and thermal physics. He was the PI on an NSF-funded grant to develop a computational laboratory with a primary goal being to improve student learning of upper division physics content.

Tevian Dray is a geometer with a longstanding interest in the interface between mathematics and physics, as well as in mathematics education. He has published more than 50 journal articles in mathematical physics, including collaborations with several world-class theoretical physicists. He has been the director of the Vector Calculus Bridge Project since 2001 and a co-PI on the Paradigms Project since its inception in 1996.

Corinne Manogue has directed the Paradigms project from the beginning. She has thirteen years' experience not only developing and teaching multiple courses in the Paradigms program, but also working with the entire Paradigms team to create a coherent curriculum characterized by active learning. Her special interest, investigating the role that active-engagement experiences play in helping students transition from lower-division mathematics to upper-division physics, led to her involvement as co-PI in the Vector Calculus Bridge Project.