AN EXTENDED THEORETICAL FRAMEWORK FOR THE CONCEPT OF THE DERIVATIVE

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Abstract

This paper extends the theoretical framework for exploring student understanding of the concept of the derivative, which was developed by Zandieh (2000). We expand upon the concept of a physical representation for the derivative by extending Zandieh's map of the territory to provide higher resolution in regions that are of interest to those operating in a physical context. We also introduce the idea of "thick" derivatives, which are ratios of *small* but not *infinitesimal* changes, which are practically equivalent to the true derivative.

Key words: derivative, theoretical framework, physical, experiment

In this theoretical report we extend the theoretical framework for exploring student understanding of the concept of the derivative which was developed by Zandieh (2000). We expand upon the concept of a physical representation for the derivative. As with Zandieh's original framework, this work is not meant to explain how or why students learn as they do, nor to propose a learning trajectory. Rather, this work extends Zandieh's map of the territory, to provide higher resolution in regions that are of interest to those working with derivatives in a physical context. In addition to focusing on the physical context, we discuss challenges that have arisen in applying Zandieh's framework to an understanding of the derivative beyond the level of first-year calculus.

This work is motivated by preliminary results of a project to study understanding of the derivative across STEM fields. In the process of interviewing physicists and engineers, we have identified shortcomings that arise when applying Zandieh's framework beyond the level of first-year calculus, and in particular outside the field of mathematics. We have found that the concept image for the derivative of physicists and engineers contains substantial elements that are congruent with the three process-object layers identified by Zandieh, but lead to the introduction of new contexts and representations that could also be productive in the instruction of calculus.

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Physicists and engineers live and work in a world full of uncertainty, and are accustomed to use the language of equality where there is actually approximation. This language reflects a somewhat "thicker" concept of the derivative than that held by mathematicians. Where a mathematician would speak of the slope of the secant line as an approximation for the derivative, a physicist or engineer might say that the slope of a line drawn between two carefully chosen measurements of a physical observable *is* the derivative (with some unspecified uncertainty). As we will explain, this "thickness" derives from the impossibility of achieving exact results in physical or numerical contexts. Attempts to estimate a derivative over too small an interval, for example, could result in a highly erroneous estimate of a derivative due to numerical round-off error or limitations in experimental precision.

Theoretical background

Concept Image

In this work, we extend the theoretical framework of Zandieh (2000), which itself draws on the idea of concept image (Vinner, 1983). Vinner (1983) describes the concept image as the set of properties associated with a concept together with mental pictures of the concept. Thompson (2013) argues that the development of coherent meanings is at the heart of the mathematics that we want teachers to teach and what we want students to learn. He argued that meanings reside in the minds of the person producing them and the person interpreting them.

Zandieh's framework for the concept of the derivative

Zandieh (2000) introduced a framework for the concept of the derivative, aimed at mapping student concept images at the level of first-year calculus. This framework maps out the *correct* concepts as understood by the mathematical community, and thus does not incorporate incorrect understandings. We reproduce in Fig. 1 below Zandieh's outline of her framework. This table consists of columns corresponding to *representations* or *contexts*, and rows corresponding to *process-object layers*. The *process-object* framework is taken from Sfard (1991), who conceives of mathematics as proceeding through processes acting on objects, with those processes then becoming reified into objects.

Process-object layer	Graphical Slope	Verbal Rate	Physical Velocity	Symbolic Difference Quotient	Other
Ratio					
Limit					
Function					

Figure 1: Zandieh's outline of the framework for the concept of the derivative.

Representations

Each of the representations in Zandieh's table can be used to convey the concepts behind the three process-object layers. She also likens these columns to "contexts" in the sense that each of these provides a context within which we can think about the derivative. In the paragraphs below, we give a brief summary of each position in Fig. 1

Graphical. The graphical representation of the derivative is slope. At the ratio layer, this is the slope of a secant line between two points on the curve describing a function. When taking the limit, we arrive at the slope of the tangent line at a point. Finally, considering the derivative as a function requires us to recognize that the slope is different for different values of the independent variable.

Verbal. The verbal representation for the derivative discussed by Zandieh is the "rate of change." At the ratio layer, this is expressed as an "average rate of change." When taking the limit, this becomes the "instantaneous rate of change." Understanding this verbal description as a function requires us to visualize the instantaneous rate of change for the inputs over the domain of the function.

Physical. The physical representation, or *paradigmatic physical* representation is velocity: average velocity, instantaneous velocity, and the velocity as a function of time. These physical concepts provide a language that we can use to understand the derivative: a large derivative means "faster" and a varying derivative means there is acceleration going on.

Symbolic. The symbolic representation of the derivative is the formal definition of the derivative in terms of the limit of a difference quotient. In this case, the distinction between the limit layer and the function layer can be subtle. They differ in the recognition that the variable describing the point at which the limit is taken can be treated as the argument of a function. Zandieh expresses this with a notational distinction between x_0 and x.

Other. Finally, we point out that Zandieh explicitly placed in her framework space for additional contexts. In particular, when discussing the physical context, she mentioned that there is a wide set of physical contexts for understanding the derivative. In this paper, we will discuss some of the subtleties we have encountered in investigating understanding of the derivative within the context of a mechanical system (AUTHOR).

Extensions to Zandieh's framework

Likwambe and Christiansen (2008) extend Zandieh's framework in three ways. Firstly, they recognize the importance in a concept image that we be able to make connections between different representations, and extend the use of the table to include arrows indicating that a student has made a connection between two representations or ideas. Secondly, they add a "non-layer" row, which indicates a recognition or use of that representation of the derivative without indication of an understanding of any of the three process-objects layers. Finally, Likwambe and Christiansen (2008) added a separate category for what they refer to as *instrumental understanding*, a term taken from Skemp (1978). Instrumental understanding (as opposed to *relational understanding* refers to the knowledge of and ability to follow a procedure. Both Skemp (1978) and Lithner (2003) point out that instrumental understanding is commonly emphasized in both homework assignments and exams. Zandieh explicitly omits instrumental understanding from her framework, but Likwambe and Christiansen (2008) add an additional box for instrumental understanding, in order to include "the only learning exhibited by most of the interviewees."

Extending Zandieh's framework for the derivative

In our research on expert understanding of the derivative across disciplines, we have encountered several issues that led us to an extension of Zandieh's framework for the derivative, with a particular focus on physical contexts. We propose a deeper understanding of the "physical" representation, and add an additional "numerical" representation, which fills out the *Rule of Four*: graphical, verbal, symbolic and numerical (Hughes-Hallett et al., 1998). In addition, we follow Likwambe and Christiansen (2008) in adding an *instrumental understanding* category that lives outside the three process-object layers.

Figure 2 shows our framework for the concept of the derivative. This figure is modeled after Fig. 1, the framework of Zandieh, and is best understood in terms of the differences between these two frameworks. We have added one additional column labeled *numerical* (and removed the *Other* column to make space). We have added the *instrumental understanding* of Likwambe and Christiansen (2008) (which is to say, the rules of differentiation) as an entirely separate table, partially to reflect its weak connection to any other aspect of the concept of the derivative.

Finally, we have added into each entry of the table (which Zandieh left blank) an iconic description of the concept meant by that entry. These entries are intended to aide in understanding the table by compactly describing the conception of the derivative indicated by that combination of row and column.

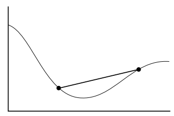
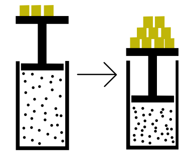
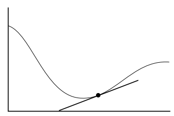
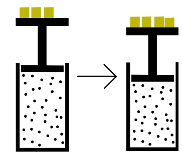
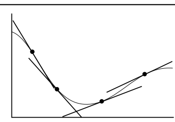
Changes in the framework

In this section, we discuss individually the extensions we have made to Zandieh's framework.

Physical. We begin by noting that the physical examples given by Zandieh (2000) each involve a time derivative: velocity, acceleration, and the time rate of change of temperature. We suggest that although these quantities do reside in a physical context, perhaps at least *some* uses of these phrases properly belong in the realm of verbal representation. We propose here a more "physical" (as opposed to verbal) concept of the physical representation of the derivative.

We define the physical representation for the derivative to be a process to *measure* that derivative (see, for instance Roundy, Kustusch, & Manogue, 2014; Styer, 1999). Of course, the *concept* does not require us to actually perform a measurement, just to imagine one. However, we note that it is the *process of measurement* itself that is the physical representation. Actually obtaining a numerical measurement would (also) require the use of the numerical representation, and describing the measurement may involve a verbal or graphical representation (Roundy et al., 2014; Styer, 1999), but the measurement process itself is the physical representation of the concept of the derivative.

As an example, consider the derivative dV/dp of the volume of a piston full of air with respect to the pressure on the piston, as controlled by a set of weights on the piston (illustrated in Fig. 2). At the ratio layer, one can say that you need to measure the volume twice, with two different pressures, and the derivative is the change in volume divided by the change in pressure. The limit

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2 - y_1}{x_2 - x_1}$ numerically	
Limit		"instantaneous ..."	$\lim\limits_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"... at any point/time"	$f'(x) = \cdots$	... depends on x	tedious repetition

	Symbolic
	Instrumental Understanding
Function	*rules to "take a derivative"*

Figure 2: Our extended framework for the concept of the derivative.

layer imposes on this process the idea that the two pressures need to be quite similar in order for this ratio to "be" the derivative in the thick sense used by physicists and engineers. However, it is not desirable to choose *too* small a value for Δp, because this would result in an imprecise measurement, since the change in volume would be too small to be precisely measured, resulting in increased error in the value of the measured derivative. Finally, the function layer requires us to recognize that this ratio will depend on the pressure itself, and that to fully explore the derivative, we must perform repeated experiments—or more likely a single experiment in which we gradually add weight to the piston and repeatedly measure its volume.

The physical representation of a derivative can often (but not always) be felt or percieved directly, which leads scientists to give derivatives names such as compressibility, velocity, thermal conductivity, etc. Qualitatively, the derivative dV/dp describes the compressibility of the air: how easy it is to compress. We anticipate that as the piston is compressed at higher pressures, it will require more and more pressure to compress it further. Because the volume cannot be negative, we can conclude on physical grounds that the derivative must eventually approach zero as the pressure increases.

Numerical. The *numerical* representation is the one member of the Rule of Four (Hughes-Hallett et al., 1998) that was not present in the framework of Zandieh (2000). We recognize a numerical representation of the derivative that is closely allied to but distinct from the physical representation. This representation parallels the formal symbolic concept of the derivative, but differs in ways that are of practical importance in the use of the derivative in the sciences and in numerical analysis.

The numerical concept of the derivative begins with a ratio of change:

$$\frac{y_2 - y_1}{x_2 - x_1},$$

where it is understood that the values in this equation are numerical values. When we take the limit numerically, we do not formally write $\lim_{\Delta x \to 0}$, and we do not apply a formal procedure. Rather we select a value of Δx that is *small*, where small is understood in terms of the desired precision. As in the case of physical measurements, practically speaking it is possible to make the change Δx *too* small, in this case due to truncation error in a computer or calculator. In this regard, when operating numerically we think of derivatives as having some "thickness," in contrast to the formal definition which requires an infinitesimal limit. Finally, the derivative as function is understood as a sequence of numerical ratios of differences, just as a function can be understood numerically as an array of numbers or set of ordered pairs.

Conclusions

We have extended the framework of Zandieh (2000) in several ways: we have elaborated on the physical representation of the derivative; we have added a numerical representation of the derivative; and we have added space in the framework for the set of rules for finding symbolic derivatives. Each of these changes reflects an expansion of the table to incorporate additional answers to the prompt, "find the derivative." By making use of the numerical representation of the derivative, one can answer the prompt numerically. Similarly, if the derivative is situated in a physical context, one can respond with a measurement process. Both of these responses require a conceptual understanding of the derivative in terms of ratio, limit and function, and involve a certain "thickness" in the derivative. In contrast, as pointed out by Zandieh, the instrumental-understanding approach to "find the derivative" using the rules for symbolic derivatives does *not* require a conceptual understanding of the derivative.

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