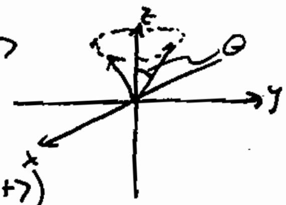
Bloch Sphere & Atomic Clocks

Review of spin-1/2: $m = \pm\frac{1}{2} \rightarrow |\pm\rangle$

what does this "look like"? e.g. $|+\rangle$



$S = \frac{1}{2} \rightarrow$ fixes $|\hat{\vec{S}}|$ ($\hat{S}^2|+\rangle = \hbar^2 S(S+1)|+\rangle$)

$m = +\frac{1}{2} \rightarrow$ fixes $\hat{S}_z$ ($\hat{S}_z|+\rangle = \hbar m|+\rangle$)

$\rightarrow \Theta$ fixed

but ϕ completely uncertain (Heisenberg uncertainty for S_z & ϕ, like p & x)

$\rightarrow$ so no well-defined spin vector in QM.

Also $\rightarrow$ problem w/ rotation

turns out that $|\pm\rangle$ under rotation about z axis by angle ϕ

is $e^{-im\phi}|\pm\rangle$ $(m = \pm 1/2)$

2π rotation $\rightarrow e^{-im\phi} = e^{\mp i(1/2)2\pi} = e^{\mp i\pi} = -1$

so $|\pm\rangle$ not invariant under 2π rotation!

(spin-1/2 not a physical angular momentum like orbital)

Bloch Sphere

- visual representation of spin-1/2 as 3-D unit-length vector (point on unit sphere)
- doesn't mean spin points this way

arb. state: $|\psi\rangle = c_+|+\rangle + c_-|-\rangle$

define 3-vector: $\vec{\Psi} := u\hat{x} + v\hat{y} + w\hat{z}$ $\rightarrow$ Bloch vector

where
$$u := 2\,\mathrm{Re}[c_+c_-^*] = c_+c_-^* + c.c.$$
$$v := -2\,\mathrm{Im}[c_+c_-^*] = i(c_+c_-^* - c.c.)$$
$$w := |c_+|^2 - |c_-|^2$$

Remarks:

- $|\vec{\Psi}| = 1 \Rightarrow \vec{\Psi}$ on unit sphere (Bloch sphere)
  - all possible Bloch vectors for $|\psi\rangle$

Proof: (activity)
$$|\vec{\Psi}|^2 = u^2+v^2+w^2 = \underbrace{(c_+c_-^* + c_-c_+^*)^2 - (c_+c_-^* - c_-c_+^*)^2}_{=4|c_+|^2|c_-|^2} + \underbrace{(|c_+|^2 - |c_-|^2)^2}_{+|c_+|^4 + |c_-|^4 - 2|c_+|^2|c_-|^2}$$
$$= |c_+|^4 + 2|c_+|^2|c_-|^2 + |c_-|^4 = (|c_+|^2 + |c_-|^2)^2 = 1$$

Remarks (cont'd):

– if $|\psi\rangle = |+\rangle \Rightarrow c_+ = 1,\ c_- = 0$
$\Rightarrow u = v = 0,\ w = +1$

– if $|\psi\rangle = |-\rangle \Rightarrow c_+ = 0,\ c_- = 1$
$\Rightarrow u = v = 0,\ w = -1$

generally: w represents population of $|+\rangle$ vs. $|-\rangle$
$(+z)$ $(-z)$

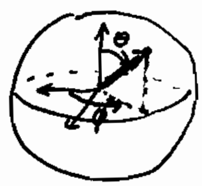
– other points on sphere → superpositions

e.g. $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \Rightarrow u = 1, v = 0, w = 0$

$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle) \Rightarrow u = 0, v = 1, w = 0$

– polar form: $|\psi\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$

$\Rightarrow$



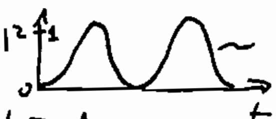
θ → related to population
ϕ → indicates relative phase

– free evolution: (B_z-field) $H = \frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \rightarrow |\psi(t)\rangle = c_+(0)e^{-i\omega_0 t/2}|+\rangle + c_-(0)e^{i\omega_0 t/2}|-\rangle$

relative phase $\phi = \omega_0 t$
$\Rightarrow$ rotation of Bloch vector about z-axis at con. rate ω_0

– Rabi oscillation: (B_x-field) $H = \frac{\hbar\Omega}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ e.g. if $|\psi(0)\rangle = |-\rangle$

population oscillates



~ frequency Ω

→ rotation of Bloch vector about x-axis at Ω freq.

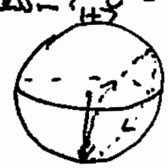
– general $(B_z + B_x)$ → combined rotation (any transformation (evolution) is rotation of sphere)

$\dot{\vec{\Psi}} = \vec{\Omega} \times \vec{\Psi}$ (like precession: $\vec{\tau} = \dot{\vec{L}} = \vec{\Omega} \times \vec{L}$)

"Feynman – Vernon – Hellwarth representation"

$\vec{\Omega} := \Omega\hat{x} + \omega_0\hat{z}$

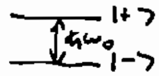
note: $|\vec{\Omega}| = \sqrt{\Omega^2 + \omega_0^2}$ ("Generalized Rabi freq")



← incomplete Rabi oscillation

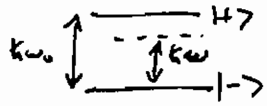
Interaction w/ ac field

const B_z :



const B_x → causes oscillations $|+\rangle \leftrightarrow |-\rangle$
strongest when $\hbar\omega_0 = 0$

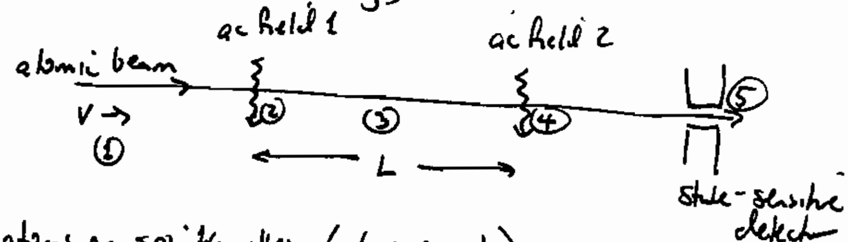
Similar for <u>oscillating</u> B_x → $B_x = B_0 \cos\omega t$
- Rabi oscillations
- strongest on <u>resonance</u> : $\omega = \omega_0$



basically substitute $\Delta := \omega - \omega_0$
for ω_0 in static case.
(approximation, but good one)

Ramsey's Method of Separated, Oscillating Fields

(Nobel prize 1989 → Ramsey)



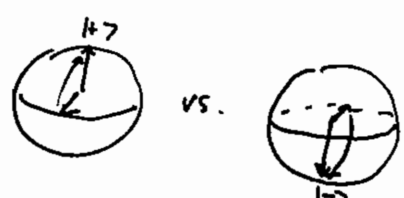
(1) atoms ~ spin 1/2 ptcles (close enough)
start in $|-\rangle$

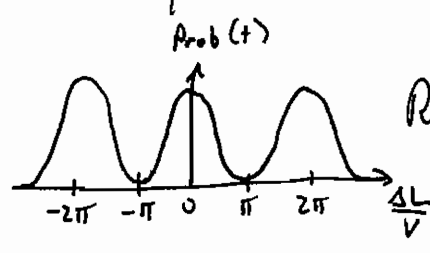
(2) (ac field)
Rabi oscillation by $\frac{1}{4}$ period ("$\frac{\pi}{2}$-pulse")

(3) free evolution at freq $\Delta := \omega - \omega_0$
for duration $\frac{L}{v}$ $\left(\phi = \frac{\Delta L}{v}\right)$

(4) ac fields 1 & 2 are <u>identical</u>
- another $\frac{\pi}{2}$ pulse

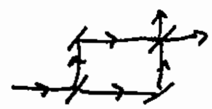
(5) read out final state, is it $|+\rangle$ or $|-\rangle$?
(Stern-Gerlach, laser, etc.)
final state depends on <u>phase</u>





Ramsey fringes.

- Ramsey fringes → like interferometer for light



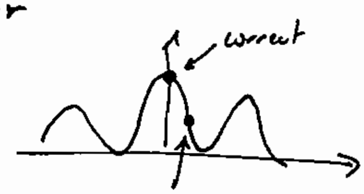
$|+\rangle, |-\rangle \Leftrightarrow$ arms of interferometer
$\pi/2$ pulses $\Leftrightarrow$ beamsplitters. (mixes $|\pm\rangle$)
Ramsey method is interferometer in time

Atomic Clocks

Definition of time: ground state of ^{133}Cs → split by ~~small~~ weak interaction of electron w/ nuclear spin
→ hyperfine splitting

second is defined such that hyperfine splitting of ^{133}Cs is $h \cdot 9.192\,631\,770$ GHz (exactly)

Atomic clock is basically Ramsey interferometer w/ microwave fields near 9.2 GHz.



Best atomic clocks → use hydrogen maser as "flywheel" (good short term stability)
occasionally correct via Ramsey fringes

Example: NIST-7 (Boulder, CO, 1993-9)
$L = 1.53$ m, $v = 230$ m/s.
uncertainty $\frac{\delta\omega}{\omega} \sim 5\times10^{-15}$ (large correction due to altitude, GR)

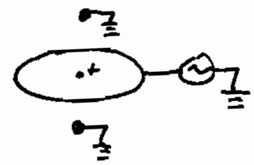
Agilent 5071A (rackmount)
$\frac{\delta\omega}{\omega} = 5\times10^{-13}$ Naval ensemble of these

- Longer interaction time ⇒ better accuracy ($\frac{\Delta L}{v}$ ~ const $L\uparrow \Leftrightarrow \Delta\downarrow$)
make L longer (NIST-7)
or v - smaller

NIST-F1 : atomic "fountain" (laser-cooled atoms to 10's of μK)

$\frac{\delta\omega}{\omega} \sim 5\times10^{-16}$ as of 2005, still in service.

Future Atomic Clocks



1. single, trapped ion (e.g. Hg^+, Al^+, Yb^+)
   - advantage: no collisions between atoms (biggest limit for Cs clocks)
   - disadvantage: 1 ion, smaller signal than $\sim 10^6$ atoms in fountain.
   - net effect of trapping field is zero (ac field).
   - major advantage: ω at optical frequencies ($\sim 10^{14}$ - 10^{15} Hz), not microwave (10^{10} Hz) $\quad \frac{\delta\omega}{\omega}$ bigger
2. neutral atoms in optical lattices (e.g. Sr)
   - optical lattice: light interference pattern

   - advantage: optical frequency, no collisions, many atoms.
   - no error due to optical lattice ("Stark shift") if λ chosen properly ("magic wavelength")
   - currently best candidate

Applications

- navigation (e.g. GPS)
- other precision measurements ($\dot{\alpha}$, magnetometry, ...)
- secure communication