Group Activity 14: Change of Variables

I Essentials

(a) Main ideas

- There are many ways to solve this problem!
- Using Jacobians (and inverse Jacobians)

(b) Prerequisites

- Surface integrals
- Jacobians
- Green's/Stokes' Theorem

(c) Warmup

Perhaps a discussion of single and double integral techniques for solving this problem.

(d) Props

- whiteboards and pens

(e) Wrapup

This is a good conclusion to the course, as it reviews many integration techniques. We emphasize that (2-dimensional) change-of-variable problems are a special case of surface integrals.

Here are some of the methods one could use to do these integrals:

- change of variables (at least 2 ways)
- Area Corollary to Green's Theorem (at least 2 ways)
- ordinary single integral (at least 2 ways)
- ordinary double integral (at least 2 ways)
- surface integral

II Details

(a) In the Classroom

- Some students will want to simply use Jacobian formulas; encourage such students to try to solve this problem both by computing $\frac{\partial(x,y)}{\partial(u,v)}$ and by computing $\frac{\partial(u,v)}{\partial(x,y)}$.
- Other students will want to work directly with $d\vec{\boldsymbol{r}}_1$ and $d\vec{\boldsymbol{r}}_2$. This works fine if one first solves for x and y in terms of u and v.
- Students who compute $d\vec{\boldsymbol{r}}_1$ and $d\vec{\boldsymbol{r}}_2$ directly can easily get confused, since they may try to eliminate x or y, rather than u or v. [1] Emphasize that one must choose parameters, both on the region, and on each curve, and that u and v are chosen to make the limits easy.

(b) Subsidiary ideas

- Review of Green's Theorem
- Review of single integral techniques
- Review of double integral techniques

(c) Homework (none yet)

(d) Essay questions (none yet)

(e) Enrichment

- Discuss the 3-dimensional case, perhaps relating it to volume integrals.

[1] Along the curve $v =$ constant, one has $dy = v\,dx$, so that $d\vec{\boldsymbol{r}}_1 = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} = (\hat{\boldsymbol{\imath}} + v\,\hat{\boldsymbol{\jmath}})\,dx$, which some students will want to write in terms of x alone. But one needs to express this in terms of du! This can be done using $du = x\,dy + y\,dx = x(v\,dx) + y\,dx = 2y\,dx$, so that $d\vec{\boldsymbol{r}}_1 = (\hat{\boldsymbol{\imath}} + v\,\hat{\boldsymbol{\jmath}})\,\frac{du}{2y}$. A similar argument leads to $d\vec{\boldsymbol{r}}_2 = (-\frac{1}{v}\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\,\frac{x\,dv}{2}$ for $u =$ constant, so that $d\vec{\boldsymbol{A}} = d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2 = \hat{\boldsymbol{k}}\,\frac{x}{2y}\,du\,dv = \hat{\boldsymbol{k}}\,\frac{du\,dv}{2v}$. This calculation can be done without solving for x and y, provided one recognizes v in the penultimate expression.