SPIN and QUANTUM MEASUREMENT (PH 425)

INSTRUCTOR GUIDE

Material covered

This course on Spin and Quantum Measurement was developed as part of the Paradigms in Physics project at Oregon State University, which entailed a reform of the junior level physics curriculum. The Spin and Quantum Measurement course is an introduction to quantum mechanics through the analysis of sequential Stern-Gerlach spin measurements. The approach and material are based upon previous presentations of spin systems by Feynman, Sakurai, Cohen-Tannoudji, and Townsend. The postulates of quantum mechanics are illustrated through their manifestation in the simple spin-1/2 Stern-Gerlach experiments. The organization of the postulates follows the presentation of Cohen-Tannoudji. The table below lists the postulates briefly and their manifestations in the spin-1/2 system as presented in the course.

	Postulates of Quantum Mechanics	Spin 1/2 manifestation
1)	State defined by ket	$\lvert+\rangle$, $\lvert-\rangle$
2)	Operators, observables	S_i, $\mathbf{S}$, H
3)	Measure eigenvalues	$\pm\hbar/2$
4)	Probability	$\lvert\langle+\vert\psi\rangle\rvert^2$
5)	State reduction	$\lvert\psi\rangle \rightarrow \lvert+\rangle$
6)	Schrödinger equation evolution	Larmor precession

The specific examples covered are: sequential Stern-Gerlach measurements of spin-1/2 and spin-1 systems, spin precession in a magnetic field, spin resonance in an oscillating magnetic field, neutrino oscillations, and the EPR experiment. The tools of Dirac notation and matrix notation are used throughout the course. General two- and three-state quantum mechanical systems are also covered as simple extensions of the spin systems.

The Stern-Gerlach experiments are discussed in class and are performed by the students using a software program (SPINS) that simulates the experiments on spin-1/2 and spin-1 systems (also SU(3) for those ambitious enough!). The program permits the students to study any configuration of sequential Stern-Gerlach measurements, interferometers, spin precession in a magnetic field, and which-path (Welcher Weg) detection to destroy interference. The program provides the student with unknown quantum states that must be determined through experiment (i.e., simulation).

Goals

Primary Content/Knowledge Goals

1) Understand and analyze sequential Stern-Gerlach measurements on spin systems.
2) Analyze generic quantum problems using matrix mechanics.
3) Use time evolution to understand spin precession.
4) Analyze generic time dependent quantum problems using matrix mechanics.

Analytical/Metacognitive Goals

1) To give the students an immersion into the **quantum** spookiness of quantum mechanics by focusing on simple measurements that have no classical explanation.
2) To give the students experience with the **mechanics** of quantum mechanics in the form of Dirac bra-ket notation and matrix representations.

Since these goals are so at odds with classical mechanics, the simplicity of the spin-1/2 system allows the students to focus on these new features instead of the complexity of the problem at hand.

Skills Objectives

1) Use bra-ket notation to calculate probabilities in Stern-Gerlach experiments.
2) Use matrix notation to calculate probabilities in Stern-Gerlach experiments.
3) Use matrix notation to analyze generic two-level quantum systems.
4) Find eigenvalues and eigenvectors of matrices representing operators.
5) Use the projection postulate to analyze sequential Stern-Gerlach experiments.
6) From Stern-Gerlach measurements, deduce initial state vectors.

7) Use Schrödinger equation to analyze time dependent spin problems.
8) Use Schrödinger equation to analyze general time dependent QM problems.

Material not covered

We have deliberately left out some material, either for lack of time or for pedagogical reasons. We do not cover systems with spin greater than 1. Spin 3/2, 2, etc require more complex calculations, but provide no additional concepts. Rotation operators are left out because they are mathematically abstract and can also lead to too much geometric thinking. With more time, they could be introduced as in Townsend (T p.28), and then used to show that the time evolution in spin precession is the same as a simple angular rotation (T p.97). We also don't introduce raising and lowering operators, due to lack of time and lack of applications later in the course. But they are used in the Central Forces Paradigm when angular momentum is discussed.

There are other topics that could be covered with more time. Density matrices provide another calculational tool and can be used to describe systems that are not pure states (C pp. 437-442). Density matrices are a key aspect of the Energy and Entropy Paradigm. Perturbation theory can often be simplified to the two levels of interest and then the two-level results can be used (C pp. 405-415).

Important emphases

Calculate states from measurements: Homework problems based on the SPINS software ask students to determine quantum states from measured probabilities in computer simulations. This is the opposite of the common problem of finding probabilities from given states, and is more similar to what physicists have to do in the lab.

Projection postulate and projection operators: We emphasize the projection postulate (p. 46 of text) and make use of the projection operator to find the state after a measurement. Most texts simply state that the wave function is reduced and don't use the formulaic statement of the projection postulate.

Spin-1/2 vs. spin-1 distinctions: The use of the unknown states in the SPINS program leads students to note that the spin-1/2 state $\left|+\right\rangle_n$ is the most general state for a spin-1/2 system (i.e., any state in the vector space can be written as $\left|+\right\rangle_n = \cos\frac{\theta}{2}\left|+\right\rangle + \sin\frac{\theta}{2}e^{i\phi}\left|-\right\rangle$ with appropriately chosen θ and ϕ), whereas the spin-1 states $\left|1\right\rangle_n$, $\left|0\right\rangle_n$, or $\left|-1\right\rangle_n$ are not the most general states in the spin-1 system (i.e., there are states in the spin-1 system that cannot be written as a projection eigenstate of 1, 0, or –1 along an axis). In the spin-1/2 case, this means that it is always possible to align the S_n detector in a way that the probability of a measurement is 100%. On the other hand, the third unknown state in the spin-1 case is $\left|\psi_3\right\rangle = \frac{1}{\sqrt{3}}\left|1\right\rangle - \frac{i}{\sqrt{3}}\left|0\right\rangle - \frac{1}{\sqrt{3}}\left|-1\right\rangle$, and it is not possible to align the S_n detector to get a 100% measurement. One way to understand this distinction is to note that the S_n detector has two degrees of freedom (θ and ϕ). A general state in the spin-1/2 case can be written as $\left|\psi\right\rangle = a\left|+\right\rangle + b\left|-\right\rangle$, where *a* and *b* are complex and so represent four numbers. However, the state must be normalized and an overall phase has no physical significance, so the state can be rewritten as $\left|\psi\right\rangle = |a|\left|+\right\rangle + \sqrt{1-|a|^2}\,e^{i\phi}\left|-\right\rangle$, showing that there are really only two degrees of freedom, matching the two degrees of the eigenstate $\left|+\right\rangle_n = \cos\frac{\theta}{2}\left|+\right\rangle + \sin\frac{\theta}{2}e^{i\phi}\left|-\right\rangle$. However, the general state in the spin-1 case is $\left|\psi\right\rangle = a\left|1\right\rangle + b\left|0\right\rangle + c\left|-1\right\rangle$, with six numbers to specify it. The conditions on normalization and overall phase reduce this to $\left|\psi\right\rangle = |a|\left|1\right\rangle + |b|e^{i\alpha}\left|0\right\rangle + \sqrt{1-|a|^2-|b|^2}\,e^{i\beta}\left|-1\right\rangle$, with four degrees of freedom, which is more than the eigenstates of S_n.

Mixtures vs superpositions: We discuss the important distinction between a coherent superposition of states and a mixture of states (which has a random phase) (p. 19 of text). We don't have any examples or problems related to this, but it could be a useful addition. [Ref. 1]

Bell inequalities: We cover the Einstein-Podolsky-Rosen (EPR) paradox and Bell inequalities as an application of what can be done with the new tools students have learned. [Refs. 2, 3]

Neutrino oscillations: We do the modern example of neutrino oscillations as an example of a time dependent application, rather than the standard example of the ammonia maser. [Refs. 4, 5, 6]

Course Activities

Lecture

The course is roughly divided into 3 parts corresponding to the 1st three chapters of the text. The content of the three parts is summarized below. The breakdown of the material in hourly blocks as presented in the course is detailed in the appendix.

Part 1:

The six quantum mechanical postulates are introduced immediately, but their explanations are reserved until needed in the context of the spin examples. The Stern–Gerlach (SG) experiment is explained and then four experiments using sequential measurements are presented. The concept of a ket as a simple way to label the output of a SG magnet is introduced and the 1st postulate (state defined by ket) is explained. The mathematics of kets is reserved for later. The four SG experiments introduce the new quantum ideas of state preparation, state analysis, probabilistic or random measurement results, incompatible measurements, interference, and probability amplitudes. (Lab 1) After qualitative discussions of the 4 experiments, the mathematics of bras and kets is explained. The 4th postulate (probability) and the inner product are explained. These allow the students to analyze the first 2 experiments quantitatively. The results of experiment 2 lead to the idea of superposition, which is contrasted with the idea of a mixture of states (as opposed to a pure quantum state). Matrix notation (for bras and kets) is introduced and used to analyze the experiments again. Both bra-ket and matrix calculations are done as example of the two techniques. The extension of these ideas to general quantum systems is introduced. (Lab 2)

Part 2:

The idea of using operators to represent physical observables is introduced along with their representation as matrices. Postulates 2 (operators, observables) and 3 (measure eigenvalues) are explained at this point. The importance of eigenvalues and eigenvectors is explained and the students are shown how to "diagonalize" a matrix. The students work with the spin component operators S_x etc. in a variety of homework problems and labs. The projection operators are introduced as another example of a quantum mechanical operator and their use in the reduction postulate (postulate 5) is explained. This permits quantitative analysis of experiments 3 and 4. Measurement expectation values and standard deviations are discussed and related to commuting observables and the uncertainty principle. $\boldsymbol{S}^2$ is introduced as another important operator. The extension to spin 1 is discussed. (Lab 3)

Part 3:

The Schrödinger equation (postulate 6) is introduced and the importance of energy eigenstates as stationary states is discussed. A recipe is given for solving time dependent problems. This is applied to spin precession and the Larmor frequency is derived. (Lab 4) Rabi's formula is derived for the spin-1/2 system and then used in other problems: neutrino oscillations and magnetic resonance.

The course ends with a discussion of the EPR paradox, Bell's inequalities, and Schrödinger's cat paradox.

Computer Laboratory

The SPINS computer lab is conducted on the 2 hour days (Tues, Thurs), but only 1 of the 2 hours is used for the lab. The other hour is for lecture. This should make it easy to adapt to more normal schedules. The SPINS program is an updated Java version of a program originally written with Pascal on a Mac. [Ref. 7] The Java version can be run as a standalone application or as an applet from within a network browser. The spinhelp file has tips on using the program, and the javahelp file has tips on running the java code. Further details on each lab are available in the instructor versions of the lab handouts.

Lab 1 (day 2): Introduction to successive Stern-Gerlach spin-1/2 measurements. Randomness of measurements is demonstrated and students use statistical analysis to deduce probabilities from measurements. The results demonstrate the orthonormality of the S_z basis kets and are used to deduce the S_x and S_y basis kets in terms of the S_z kets.

Lab 2 (day 4): Deduce quantum state vector from measurements of spin projections. Unknown states are easy at first and then become progressively harder. Students are given a recipe for solving for unknown states. This procedure is chosen to be in concert with the specific unknown states in the program. Unknown #1 ($|+\rangle$) is obvious from the data and is trivial to solve for with the recipe. Unknown #2 ($|-\rangle_y$) is also obvious from the data, but is not so trivial to solve for with the recipe. The recipe forces students to take the longest route to the solution so they get practice with all the trigonometry and complex number arithmetic. Solutions to unknowns can be checked by using a spin analyzer aligned along a new direction. New unknown states can be made with the User State option, a spin analyzer aligned along a new direction, or by spin precession with the magnet, which they don't know about yet, but you can tell them what to do (so it's a black box). This latter option provides information for later problems with spin precession (Lab 4). Use the projection postulate to calculate expected results in an interferometer. Perform which-path experiments to see perturbative effect of measurement. Spin unknown states can be changed in Java code if desired (definitions of unknown states start at line 215 in Experiment.java file).

Lab 3 (day 7): Repeat Labs 1 (part 4) and 2 (parts 2 and 3) for the spin-1 system.

Lab 4 (day 9): Spin precession in magnetic field. No worksheets handed out. Students asked to design experiment to figure out how magnet affects spins. They must take data, analyze, develop hypothesis, test hypothesis, and determine scale for magnetic field parameter displayed on screen. They can get a head start if they have done the magnet part of Lab 2.

Lab wrap up (day 12): Simply make sure everyone knows all the workings of the program, especially the unknown states and how to confirm them with an aligned detector.

Homework

HW 1: Bra-ket exercises, spin projection probabilities

HW 2: Matrix exercises, change of bases, commutators, SPINS Lab unknowns, generic QM 3-level system

HW 3: Spin precession, generic QM 2-level time-dependent system, spin 1 interferometer (could be in HW2), generic QM 3-level time-dependent system

At the beginning of the term a set of key problems (2.22, 2.23, 3.7, and 3.13) is handed out that is linked to the four primary content goals of the course. These same problems are part of the homework during the course. These four types of problems are used again in the final exam.

Final Exam

In our course we give a two-hour final exam on the Monday evening after the three-week course. The exam has four questions similar to the four key problems handed out at the beginning of the course. These problems test the students on the four primary course goals:

1) Understand and analyze sequential Stern-Gerlach measurements on spin systems.
2) Analyze generic quantum problems using matrix mechanics.
3) Use time evolution to understand spin precession.
4) Analyze generic time dependent quantum problems using matrix mechanics.

Necessary Preparation for this Paradigm

Physics Concepts

1) Understanding of behavior of a magnetic dipole in a magnetic field (General Physics with Calculus PH 213).
2) Resonance ideas (Oscillations Paradigm PH 421).

Mathematical Techniques

1) Familiarity of vector concepts of orthogonality, normality, and scalar products.
2) Ability to find the determinant of a matrix.
3) Ability to solve for eigenvalues and eigenvectors of 2 x 2 and 3 x 3 matrices.

These ideas are covered in the mathematical preface at the beginning of the term (see appendices B and C).

Current Problems

Problems encountered

Compared to other Paradigms courses, this one has more lecturing and could use more group activities. One thing we have tried is to have students work on some of the derivations in the text (*e.g.*, p. 17 and p. 21).

Often students will take data in labs and then run, without taking time to solve or begin solving problems. The labs do require more time than we give them in the class period, but it is important that they get started with the calculations in the class.

Other ideas tried

An analogy can be drawn between a spin-1/2 system and polarized light (S p. 6, T p. 51). In the first year, we spent one session doing an optics experiment to illustrate this analogy. We did it early in the course and found that it didn't add much value and could confuse some students.

One year we tried starting the course in the computer lab with the SPINS program, but that turned out to be too early to be useful.

Student problems and misunderstanding

Some students rely too strongly on a classical or geometric view of spin.

Many students have trouble with the complex arithmetic. For example, many do not know that they can use $a+ib$ or $re^{i\theta}$ as they find convenient. We stress the amplitude-phase approach since it leads to simple trig and is better practice for other complex problems.

The 2 x 2 matrix algebra of the spin-1/2 system is often so easy that students know the answers for the simple problems, but don't really know how to work out the algebra when a slightly harder problem is posed. To help those students we have given a recipe for the solutions of the SPINS lab unknown states.

Some students have trouble recognizing that basis vectors are always unit vectors when written in their own basis; i.e., if I say there is a basis $|1\rangle, |2\rangle$, they often don't associate this with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Problem 2.23 is an example of where this arises.

In the labs, some students will record actual numbers of counts rather than probabilities.

Students are often taught the proper matrix diagonalization technique in math classes and have to be reminded that we only need to find the eigenvalues and eigenvectors, and we stop short of finding the transformation that diagonalizes the matrix and actually producing the diagonal matrix (since that would be in a different basis than we are working in).

Links to Other Paradigms/Capstones

1) Bra-ket notation used in Central Forces (PH 426) and Energy and Entropy (PH 423).
2) Eigenvalue and eigenvector ideas used in Central Forces (PH 426) and Rigid Bodies (PH 428)
3) SPINS software used in Energy and Entropy (PH 423).

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Appendix

PH 425: Brief notes on class sessions

Lecture ideas, topics, key points, activities, connections to other ideas, broken down by hours in our course as taught at OSU. We have 15 class sessions and 21 total class hours since we have six 2-hour sessions on T/Th.

Hour	Text pp.	Material
1	1-5	Stern-Gerlach expt, magnetic moment, force in inhomogeneous field, SG results –> two beams, Postulate 1
2	5-11	Sequential SG measurements, Expt. 1, Expt. 2, randomness, Expt. 3, incompatible observables, Expt. 4, interference, state vector, ket
3		Do SPINS Lab 1
4	11-19	Bras and kets, basis vectors, inner product, bra+ket=bracket, square to find probability, Postulate 4, Analyze Expts. 1 & 2, use data from Expt. 2 to find S_x eigenkets in terms of S_z states,
5	19-28	Analyze Expts. 3 &4, discuss important distinction between coherent superposition of states and mixture of states (which we won't use in course), introduce matrix notation as bookkeeping for amplitudes of states, introduce representation notation, analyze Expt. 2 with second analyzer along *y*-axis as practice in using matrix notation and as way to find S_y eigenkets, relate bra-ket notation to generic QM two-state system.
6		Do SPINS Lab 2
7	34-41	Operators, Postulate 2, Postulate 3, eigenvalue equation, eigenvalues, eigenvectors, find S_z matrix from eigenvalue equations, generic matrix elements, matrix diagonalization procedure (even though never complete diagonalization!), Hermitian operators, action on bras and kets.

8	41-54	Other operators?, ket-bra as operator, Projection operators, Postulate 5, measurement perturbation, how to do measurements, probabilities, expectation values, do simple examples, standard deviation, do same simple examples
9	54-63	Commutator, commuting observables, simultaneous eigenstates, uncertainty relation, $\mathbf{S}^2$ operator, spin vector model, spin-1 example
10		Do SPINS Lab 3
11	68-71	Schrödinger equation and time evolution, Postulate 6, Hamiltonian, energy eigenstates, energy basis importance, time evolution simply written in terms of energy basis, stationary states, relative phase, Bohr frequency
12	72-76	Application of time evolution to spin 1/2, magnetic dipole in magnetic field, spin Hamiltonian, H already diagonal, stationary states, precessing states
13		Do SPINS Lab 4
14	76-78	Larmor precession, Larmor frequency, relation to classical torque, time dependent probabilities
15	78-84	Apply additional field along x-axis to flip spins aligned along B field (z-axis), calculate new Hamiltonian, now not diagonal, diagonalize H, calculate probability of spin flip, Rabi's formula
16	84-87	Apply time dependent results to other 2-level systems, Neutrino oscillations, mass eigenstates vs weak eigenstates, mixing angle, relativistic neutrino energy, calculate probability of electron neutrino to muon neutrino oscillation
17		Finish computer labs and make sure everyone understood all aspects

18	87-93	Magnetic resonance, how do we make spin flip without applying huge additional field?, explain first with classical picture in rotating frame, apply rotating field, get time dependent Hamiltonian, solve by using rotating state vector to take time dependence out of equations, then get time-independent equations for new vector, use old Rabi solution, get Rabi flopping equation with resonance characteristic, discuss relation to NMR
19,20	97-105	Class discussion on EPR Paradox, Bell's inequalities, and Schrödinger's cat, class handouts
21		wrap up and review