Electrostatic Potential for Ring of Charge

Instructor Guide

Keywords: Upper-division, E and M, Electrostatic Potential, Symmetry, Ring

Brief overview of the activity

In this activity, students work in small groups to write the electrostatic potential everywhere in space due to a charged ring.

This activity brings together student understanding of:

1. Electrostatic potential
2. Spherical and cylindrical coordinates
3. Superposition
4. Integration as ”chopping and adding”
5. Linear charge density
6. 3-dimensional geometric reasoning
7. Power series expansion

Student prerequisite skills

This activity is may be used as the second in a sequence, following the electrostatic potential - discrete charges activity, or may be used on its own. Students will need understandings of:

1. The prerequisites addressed in the electrostatic potential - discrete charges activity.
2. Spherical and cylindrical coordinates. Link to spherical and cylindrical coordinates activity.
3. Integration as chopping and adding. Link to Integration activity.
4. Linear charge density

Props

- Hula hoop or other thin ring
- Balls to represent point charges
- Voltmeter
- Coordinate system (e.g. with straws or Tinkertoys)
- Poster-sized whiteboards
- markers
- whiteboards around room. Link to room set-up.

The activity - Allow 50 minutes.

Overview

Part I

Students should be assigned to work in groups of three and given the following instructions using the visual of a hula hoop or other large ring: “This is a ring with total charge $Q$ and radius $R$. Find the electrical potential due to this ring in all space.” Students do their work collectively with markers on a poster-sized sheet of whiteboard at their tables. Link to worked solution resulting in an elliptic integral.

Part II

Students determine the power series expansion to represent the electrostatic potential due to the charged ring along a particular axis. Link to worked solutions for power series expansions. Note: students should not be told about part II until they have completed part I.
What the students will be challenged by and how to facilitate their learning

Part I - Finding the potential everywhere in space: Creating an elliptic integral

1. The first concept students need to understand is linear charge density. Given that the ring has a charge $Q$ students will need a few minutes to realize that the charge density $\lambda = \frac{Q}{2\pi r}$. In general students come up with this on their own without help.

2. Students will grapple with how the linear density relates to the ‘chopping and adding’ aspect of integration. Students frequently leave math classes understanding integration as ‘the area under a curve’. This activity pushes students to transform their understanding of integration to focus on ‘chopping and adding.’ Students may reach a correct representation on their own in a few minutes or the instructor may assist by using a hula hoop as a prop to help students in describing the ‘chopped’ bits of hoop.

3. Students must use an appropriate coordinate system to take advantage of the symmetry of the problem. Students attempting to do the problem in rectangular coordinates can be given a few minutes to struggle and see the problems that arise and then should be guided to using curvilinear coordinates. Most students will choose to do this problem in cylindrical coordinates, but an interesting problem for groups who finish early is to redo the problem in spherical coordinates.

4. Putting the whole thing together requires three dimensional geometric understanding. One of the big advantages to doing this problem in class as opposed to homework is that the instructor can interact with student making 3-dimensional arguments. Either a hoop or a ring drawn on the table can be used to ask students about the potential at points in space that are outside the plane of the ring.

5. This activity also gives students the opportunity to use curvilinear coordinates and then realize that they cannot successfully integrate without transforming them into rectangular coordinates. Understanding that $\bar{r} - \bar{r}'$ cannot be integrated by simply using $\bar{r}$ in curvilinear coordinates is an important realization. Some instructors may even miss
this point if they have not carefully considered it prior to this activity. Unlike linear coordinates where \( x - x' \) always refers to vectors in the same direction, this is not the case for curvilinear coordinates where \( r \) and \( r' \) can be oriented in different directions at any angle. Solving this problem entirely in rectangular coordinates from the beginning is overly cumbersome, but the curvilinear coordinates which very successfully simplify the problem can lead one to incorrectly think that using \( \vec{r} - \vec{r}' \) in curvilinear coordinates can be successfully integrated. To see how this fits into the whole process, see the link to worked solution resulting in an elliptic integral.

6. The final component is that students need to recognize an elliptic integral and what to do when they run into one. Most commonly students have never seen such ‘unsolvable’ integrals in their calculus classes. In our case we had students do the power series expansion before the integral (see below).

**Part II - Finding the potential along an axis: Power series expansion**

With the charged ring in the \( x, y \) plane, students will make the power series expansion for either near or far from the plane on the \( z \) axis or near or far from the \( z \) axis in the \( x, y \) plane. Once all students have made significant progress toward finding the integral from part I, and some students have successfully determined it, then the instructor can quickly have a whole class discussion followed by telling students to now create a power series expansion. The instructor may choose to have the whole class do one particular case or have different groups do different cases. Link to worked solutions for power series expansions.

If you are doing this activity without having had students first create power series expansions for the electrostatic potential due to two charges, students will probably find this portion of the activity very challenging. If they have already done the Electrostatic Potential - Discrete Charges activity, or similar activity, students will probably be successful with the \( y \) axis case without a lot of assistance because it is very similar to the \( y \) axis case for the two \(+Q\) point charges. However, the \( y \) axis presents a new challenges because the “something small” is two terms. It will probably not be obvious for students to let \( \epsilon = \frac{2R}{r} \cos \phi' + \frac{R^2}{r^2} \) (see Eq.17 in the solutions) and sugges-
tions should be given to avoid having them stuck for a long period of time. Once this has been done, students may also have trouble combining terms of the same order. For example the $\epsilon^2$ term results in a third and forth order term in the expansion and students may not realize that to get a valid third order expansion they need to calculate the $\epsilon^3$ term.

**Debriefing, Whole-Class Discussion, Wrap-up and Follow-up**

- Discuss which variables are variable and which variables are held constant - Students frequently think of anything represented by a letter as a ‘variable’ and do not realize that for each particular situation certain variables remain constant during integration. For example students do frequently do not see that the $R$ representing the radius of the ring is held constant during integrating over all space while the $r$ representing the distance to the origin is varying. Understanding this is something trained physicists do naturally while students frequently don’t even consider it. This is an important discussion that helps students understand this particular ring problem and also lays the groundwork for better understanding of integration in a variety of contexts. **Link to helping students understand what is variable are what is held constant.**

- Maple representation of elliptic integral - After finding the elliptic integral and doing the power series expansion, students can see what electric potential ‘looks like’ over all space by using Maple. **Link to Maple worksheet**

**Suggested Homework**

- Use Maple to solve a nasty integral
- Integrate over a volume with a charge distribution