Activity 5: Solution for magnetic field

Find the magnetic field in all space due to a ring with total charge Q and radius R rotating with a period T

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int\limits_{\text{ring}} \frac{\vec{I}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')\, dl'}{|\vec{r} - \vec{r}\,'|^3} \tag{1}$$

where $\vec{r}$ denotes the position in space at which the magnetic field is measured and $\vec{r}\,'$ denotes the position of the current segment. As described in previous solutions,

$$dl' = R\, d\phi' \tag{2}$$

$$\vec{I}(\vec{r}\,') = \frac{Q}{T}(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) \tag{3}$$

$$\vec{r} - \vec{r}\,' = (r\cos\phi - R\cos\phi')\hat{\imath} + (r\sin\phi - R\sin\phi')\hat{\jmath} + (z - z')\hat{k} \tag{4}$$

$$|\vec{r} - \vec{r}\,'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{5}$$

Thus $\vec{B}(\vec{r}) =$

$$\frac{\mu_0}{4\pi}\frac{QR}{T}\int\limits_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) \times [(r\cos\phi - R\cos\phi')\hat{\imath} + (r\sin\phi - R\sin\phi')\hat{\jmath} + z\hat{k}]d\phi'}{(r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2)^{3/2}} \tag{6}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int\limits_0^{2\pi} \frac{(z\cos\phi'\hat{\imath} + z\sin\phi'\hat{\jmath} + [R - r\cos(\phi - \phi')]\hat{\mathbf{k}})d\phi'}{(r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2)^{3/2}} \tag{7}$$

1 The z axis

For points on the z axis, $r = 0$ and ϕ can be arbitrarily taken as zero. Thus, the integral simplifies to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int\limits_0^{2\pi} \frac{[z\cos\phi'\hat{\imath} + z\sin\phi'\hat{\jmath} + R\hat{\mathbf{k}}]d\phi'}{(R^2 + z^2)^{3/2}} \tag{8}$$

Doing the integral results in

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\frac{2\pi R\,\hat{\mathbf{k}}}{(R^2 + z^2)^{3/2}} \tag{9}$$

2 The x axis

For points on the x axis, $z = 0$ and $\phi = 0$. Because $z = 0$ the $\hat{\imath}$ and $\hat{\jmath}$ components disappear and the integral simplifies to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int\limits_0^{2\pi} \frac{(R - r\cos\phi')\hat{\mathbf{k}}d\phi'}{(r^2 - 2rR\cos\phi' + R^2)^{3/2}} \tag{10}$$