Group Activity 12: Stokes' Theorem

I Essentials

(a) Main ideas

- Practice visualizing surfaces
- Stokes' Theorem

(b) Prerequisites

- Ability to do line and surface integrals
- Definition of curl
- Statement of Stokes' Theorem

(c) Warmup

None, but be prepared to talk about appropriate surfaces for Stokes' Theorem (perhaps using a "butterfly net" as a prop).

(d) Props

- whiteboards and pens
- a butterfly net; homemade is fine, such as a plastic bag on a wire rim
- formula sheet for div and curl in spherical and cylindrical coordinates (Each group may need its own copy.)

(e) Wrapup

- Discuss the various surfaces one could use for the second question.
- Discuss the various ways one could compute the curl.
- This could be a good time to emphasize the similarity between the basic theorems.

II Details

(a) In the Classroom

- Students like this lab; it should flow smoothly and quickly.
- Make sure students choose surfaces which can catch butterflies!
- The curl is easy but slightly messy in rectangular coordinates, starting from the formula $\hat{\boldsymbol{\phi}} = \dfrac{-y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}}{\sqrt{x^2+y^2}}$.
- It is easier to factor $\vec{\boldsymbol{F}}$ as $(r^2)(r\,\hat{\boldsymbol{\phi}})$ than as $(r^3)(\hat{\boldsymbol{\phi}})$.
- The (curl and the) resulting surface integrals are much easier in cylindrical (or possibly spherical) coordinates.
- Some students want to write "$\vec{\boldsymbol{F}} \times \vec{\nabla}$" rather than $\vec{\nabla} \times \vec{\boldsymbol{F}}$.
- A possibly related problem is that students will often write $\vec{\nabla} \times \vec{\boldsymbol{F}}$ even when the vector field is called something else, such as $\vec{\boldsymbol{G}}$.
- Students using a disk or cylinder may well want to use cylindrical basis vectors here; this should be encouraged.
- Some students will draw a cone whose tip is at the origin; this is wrong.
- Students using a hemisphere will probably reinterpret r as the spherical radial coordinate; this is fine, although the instructor needs to be prepared to help students understand why they get a different answer for curl; see below.

(b) Subsidiary ideas

- Different ways of calculating curl.
- Calculating the curl in curvilinear coordinates.

(c) Homework (none yet)

(d) Essay questions (none yet)

(e) Enrichment

- Many students who try the paraboloid will discover that they don't in fact need to substitute the equation of the paraboloid! That is, leaving both dr and dz intact results in the dz term canceling anyway. Such students have in fact done a nearly arbitrary surface! (If it's not the graph of a function a further argument is needed.)
- Mention the product rule for curl, namely

$$\vec{\nabla} \times (f\vec{\boldsymbol{G}}) = (\vec{\nabla} f) \times \vec{\boldsymbol{G}} + f(\vec{\nabla} \times \vec{\boldsymbol{G}})$$

  Discuss the fact that *all* product rules take the form [1]

  > *The derivative of a product is the derivative of the first quantity times the second plus the first quantity times the derivative of the second.*

  The only complication here is figuring out which derivative to take, and what multiplication to use! A similar product rule holds for the divergence.
- The vector field is deliberately given in *polar* coordinates; the extension off the plane (or for that matter off the circle) doesn't matter! Most students will assume there is no z-dependence without thinking about it; this is fine, and does not need to be discussed. But students using spherical coordinates will most likely interpret r as the spherical radial coordinate, thus obtaining a different vector field than the above (which would be $r^3 \sin^3\theta\,\hat{\boldsymbol{\phi}}$). It is important to realize that this is fine! *Any* vector field which has the correct limit to the circle (and is differentiable) will work!
- The "wire" singularity for the vector field $\frac{\hat{\boldsymbol{\phi}}}{r}$ from an earlier activity can in fact be handled by interpreting r as the spherical radial coordinate, and using Stokes' Theorem on a hemisphere. This is of course no longer the magnetic field of a wire carrying a steady current, and the curl of this vector field isn't zero.
- Ask students how to apply Stokes' Theorem to an open cylinder, with neither top nor bottom.

[1] The product rules for derivatives of $\vec{\boldsymbol{F}} \times \vec{\boldsymbol{G}}$ do not obviously have this form, but can be rewritten (in terms of differential forms or covariant differentiation) so that they do.