Making Sense of Quantum Operators, Eigenstates and Quantum Measurements

Elizabeth Gire[*] and Corinne Manogue[†]

[*]University of Memphis, Memphis, TN 38152
[†]Oregon State University, Corvallis, OR 97331

Abstract. Operators play a central role in the formalism of quantum mechanics. In particular, operators corresponding to observables encode important information about the results of quantum measurements. We interviewed upper-level undergraduate physics majors about their understanding of the role of operators in quantum measurements. Previous studies have shown that many students think of measurements on quantum systems as being deterministic and that measurements mathematically correspond to operators acting on the initial quantum state. This study is consistent with and expands on those results. We report on how two students make sense of a quantum measurement problem involving sequential measurements and the role that the eigenvalue equation plays in this sense-making.

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INTRODUCTION

Previous research in students' understanding of quantum mechanics found that some students interpret the mathematics of an operator of an observable acting on wavefunctions as being a representation of making a quantum measurement of that observable[1-4]. Although this interpretation is ultimately incorrect, it is easy to imagine how students might come to this misunderstanding. Eigenvalue equations are used in quantum mechanics to determine the eigenvalues (possible values of measurements) and eigenstates (possible states after a measurement is made). Additionally, it is common for students to over generalize eigenvalue equations and think that they hold for all states and are not limited to eigenstates[1]. In mathematics, operators transform vectors; in physics, measurements change the state of a quantum system; but these are two different uses of the idea "to change." In order to calculate the state of particle after a measurement has been made, one can use a projection operator made up of a superposition of outerproducts of eigenstates that correspond to the eigenvalue that is measured. However, this projection operator is not the same thing as the Hermitian operator corresponding to the observable quantity.

In this study, we describe two approaches to thinking about quantum measurements. We give rich descriptions of problem-solving interviews with two upper-level physics students. We explore how these students make sense of quantum measurements and how the eigenvalue equation plays a role in this sense-making.

METHODS

Semi-structured clinical interviews were conducted with fourteen juniors at Oregon State University in the spring of 2011. Students were asked to reflect on their experiences in their quantum courses, describe how they would explain to a friend or roommate what an operator is and how it is used in quantum mechanics, think aloud while solving a problem related to sequential measurements on identically prepared hydrogen atoms, and consider seven statements about operators and quantum measurement and discuss whether they agree with these statements. The students were instructed to talk aloud during problem solving and while considering the Agree/Disagree statements. Students wrote on tabletop whiteboards and were video and audio recorded. Of these interviews, two students were purposefully selected for detailed analysis to illustrate different approaches to this issue of measurement. Ann was selected as an example of a student who uses a correct approach, while Billy was selected as an interesting example of a student who is trying to make sense of using an incorrect approach.

All of the students interviewed had recently taken the quantum Paradigms[5,6]. These courses take a "spins first" approach to quantum mechanics and introduce the formalism of quantum mechanics in the context of a simulation of Stern-Gerlach experiments. The interviews occurred four weeks after the completion of the third, winter term, Central Forces Paradigm.

The students were asked to consider the following Sequential Hydrogen Measurement problem - Imagine you have a collection of hydrogen atoms identically prepared in the state:

$$|\Psi\rangle = \frac{1}{3}|3,2,1\rangle - \frac{i}{3}|3,2,0\rangle + \frac{\sqrt{7}}{3}|2,1,1\rangle$$

and perform the following sequence of experiments: measure L_z, measure L^2, and measure L_z.

What state do you expect the particles to be in immediately after the last measurement? How would your answer change for the sequence of experiments: measure L^2, measure L_z, and measure L^2?

The Agree/Disagree Statements include: (Operator as Measurement) "When an operator $\hat{A}$ corresponding to an observable A acts on a wavefunction Ψ, it corresponds to a measurement of that observable"; (Operator Is Not a Measurement) "$\hat{A}$ acting on Ψ is not a statement about the measurement of A"; (Psi prime) "When an operator $\hat{A}$ corresponding to an observable A acts on a wavefunction Ψ, it corresponds to a measurement of that observable"

NARRATIVES

Ann

Ann starts by describing operators as something that transforms functions or matrices. "an operator is something that, just like a function is sort of a box that you know you take numbers in and spit numbers out, an operator is something that you take functions or matrices in and spit functions or matrices back out… in essence it's, it operates on things."

When asked to give examples of computations where she had to use an operator, she mentions the energy eigenvalue equations, using rotation matrices, and coordinate transformations.

She starts the Sequential Hydrogen Measurement problem by describing a general procedure she would go through to address the problem. "So, um what happens to each of them as they move through the measuring device is that the state of the particle gets projected onto whatever state it is you measure. And so, say that you're measuring L_z, and you measure it in a particular state. Then what you're doing is taking this initial state and using these projection operators and having those act on the ket so that what you get out in the end is basically a rescaled, the components of the initial one that corresponds to whatever it is that you study. And so you're going to repeat that process with L^2 and L_z again. And so what you get out in the end, you have to add all of those different operators that you do on the state in order to get the final one out. So then, what state you expect to be in after the last measurement? I would expect them to be in the state of whatever z, whatever z-component of the angular momentum that I measured."

She seems to anchor her discussion on her understanding that the particle's state after a measurement is the result of a projection of its initial state onto the states (eigenstates) that correspond to the value that is measured. In this discussion, she specifically mentions projection operators as providing the relevant transformation of the state.

The interviewer then prompts her to do the computation. After she picks up her pen, she says she's going to project the initial state and then she asks, "Do I know what...the results of the measurement, like L_z are, for instance?" The interviewer asks what possible results she might expect, and she turns her attention to the quantum numbers in the kets. After she identifies that the m quantum number could be 0 or 1, she talks about creating a projection operator by taking an outer product of kets, but her kets are empty. She then says that in this case, it's easier to do a scalar product between the eigenstates and the initial state and then rescale, but she still needs to know what value of L_z was measured. The interviewer picks m=0 and Ann performs the computation. She has some minor difficulty rescaling the resulting state. Then the interviewer asks her about the case when m=1, the degenerate case. Here, she uses a similar strategy of selecting the eigenstates that correspond to m=1 (without using a projection operator), and she is careful to maintain the relative probabilities of the two terms that remain.

She repeats the same procedure with the second L^2 and the last L_z measurement. For the L^2 measurement, she again asks for information about which of the possible values of l she would measure, and she does not correctly remember the eigenvalues of L^2.

When asked if the final state of the particle at the end of the sequence would be the same if the order of the measurements were reversed, she explains that even if the order were the same, the end state could be different if different values of L_z were measured. "Um, not necessarily because what I could have done, for instance is measured different L_z at the beginning, right, like, so I'm assuming I do these measurements and they have the possibility of getting different results, right. So if I had done the say the L_z measurement and gotten a different result then I would have picked out a different component of this initial wavefunction and so I might not have ended up with the same thing." Again, here she anchors her discussion on the fact that the measurement could result in different values for the angular momentum.

Ann disagrees with the Operator as Measurement Statement. "I guess not exactly, because if you want to measure, say, the energy of a state you have to act on an eigenfunction of that state in order to get your energy back out. (Here she is equating "measurement" with getting a particular number out of a calculation.) What you can do is get an expectation value, for instance, by having the bra of the, so do something like $\langle\Psi|\hat{H}|\Psi\rangle$. So there you can get back out an expectation value for your measurement. But just having this act doesn't necessarily get you an energy. This act gets you an energy times each component that you then have to deal with." Here, Ann takes issue with the fact that Ψ is not an eigenstate. She understands that when the operator acts on a wavefunction, each eigenfunction in the expansion of the wavefunction gets multiplied by an eigenvalue. Here, she talks about a similar computation - calculating an expectation value - and distinguishes between the expectation value computation with having the operator act on the wavefunction.

Similarly, Ann agrees with the Operator Not A Measurement statement. "The only thing you can get information out of, really, is the operator acting on each of the different components, as long as you have it written in the same basis so that you actually pull something out of this operator acting on Ψ." Her reasoning here is anchored on her understanding that the operator acts differently on each component of the eigenfunction expansion and doesn't pick out one of the possible values that could be measured.

Billy

From the beginning, Billy describes operators as acting on states and equates the operator with the measurement apparatus. "Um, I would say an operator is what acts - so mathematically, an operator is what acts on some state, on some eigenvector… So, kinda like the Stern-Gerlach experiment, where you have a spin up and then *you send it through some operator* (emphasis added), either some mixed state or not, then you see what comes out of that operator, or out of that, basically out of that projection." He mentions projections several times in his opening statements, and it's clear that he understands measurements to be closely associated with projections, but he does not make any clear distinction between projection operators and operators which represent observables. Projection operators are among the first operators he mentions as examples he's used in quantum mechanics, and using a projection operator is the first example he gives of a computation in quantum mechanics that involves operators. The first thing he chooses to write on the board is a correct mathematical expression for the projection that happens when the wave function collapses (Figure 1).

When Billy begins thinking about the Sequential Hydrogen Measurement problem, the first thing he wants to do is to see what the L_z operator does to the initial state. "So, I would do, I'd first see how L_z acts on Ψ. And then you'd get some new state, essentially, and I think this is $-i\hbar\partial/d\phi$. And, then, I guess the way I'd first do it, because I'm not exactly sure how it looks in just ket notation, is I'd do the long route, when you actually have to do the derivatives of the continuous form. From there you get some state." He confirms that the Ψ he is talking about is the initial state of the particle. He writes an equation on his board that indicates L_z acting on Ψ yields a new state, Ψ'. Then, Billy takes this new state and lets L^2 act on it to yield a second new state, Ψ''. Finally, he lets L_z act on Ψ'' to yield Ψ_f. He describes this sequence of operations as the sequence of measurements. He is unable to proceed with his calculation until the interviewer reminds him of the eigenstates and eigenvalues of L_z. He then performs the calculation, carrying the eigenvalues through each transformation so that he ends up with an $\hbar^4$ in his final state. "And from here, we just get more ħ's. Something's weird. Well, I mean, granted we never actually did like, oh, do it, you know, go one after the other in our actual courses, but I'm not used to seeing $\hbar^4$ kind of thing." He's troubled by these factors in his final state and comments that while this seems unfamiliar, it may just be unfamiliar because he doesn't remember having done a sequential measurement calculation like this.

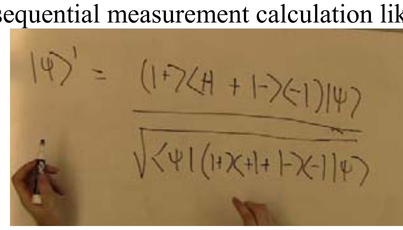

FIGURE 1. Billy's general expression for using a projection operator.

Interestingly, when asked what the state of the particle would be after the first L_z measurement, he says that the state would be your m value times ħ times your state back. This appears to be consistent, in his mind, with the computation he just performed by having the L_z operator act on the initial state Ψ. This is interesting because he seems to be thinking that the initial state is an eigenstate of the operator, even though he recognizes it as a superposition of eigenstates with different eigenvalues. Also, this answer is problematic because this new state includes dimensions from the mħ eigenvalue. Billy indicates that in the initial state, "the m is 1, 0, and 1 and so you just get ħΨ back." When the interviewer asks what happens to the m=0, and Billy says, "Mathematically, it's zero, if we just use the definition. But now I'm thinking it seems like it'd be weird if it was, um, if you could have another state that was not one, but so you'd have like, ah, you know 1ħ, 1ħ, with, I don't know, I don't know if it's, I don't know if 3, 2, 2 is actually allowed or if it's prohibited. I think it is. Then you'd have $2\hbar$. And that seems kind of wild, because it seems like your angular momentum should be a discrete value." Billy doesn't resolve this issue in the interview. In this discussion, Billy seems to be thinking that his calculation is deterministic - having the operator act on the state should indicate which eigenvalue is going to be measured. When he imagines a different case that would result in different (non-zero) eigenvalues showing up in different terms, Billy seems troubled by this. Later in the interview, he makes statements about the operator acting on Ψ not indicating the value that would be measured.

Billy tentatively disagrees with the Operator Is Not a Measurement Statement. "I would say that is False (rising intonation) - it may not, it's not going to give you an observable, necessarily, but when you measure something you are acting on it. Whenever you do an experiment you act on that state which changes the state, which is why quantum mechanics is weird. I would say this is false, but, you don't get any, like ah, the energy value or your eigen...or angular momentum value." Here, Billy anchors his reasoning on the fact that, when a measurement is made, the state changes (i.e. the wave function collapses) which Billy conflates with the mathematical change induced by an operator - "you're acting on it". He also states that "it", the operator acting on Ψ, is not going to yield an eigenvalue, which he refers to as an "observable".

When considering another statement "The operator $\hat{A}$ acting on the wavefunction Ψ is: $\hat{A}\Psi = \Psi'$ ", Billy interprets this as saying, "To me it's saying that like if you have L_z and you operate on to your state vector, your given state vector, for instance L_z is the operator, then you'd get whatever that L_z pulls out, which is some eigenvalue times that state back. So actually…and I guess the way I'm interpreting the Ψ' is that, it's really the same state but with some new constants in front now. Although, in good old linear algebra that would, that could drastically, you wouldn't necessarily just get that back. Like I, the way we've been viewing these is just the eigenvalue equation. That's how we've always been, that's how we've been interpreting all, whenever we do operator acting on some wave function or a state vector. So, how we, the way that looks, you just get, you have your state and the states remain the same but now you have an eigenvalue multiplied by your state. So I would say, it's not like, it's not like your whole state is drastically changed but now there's just some scalar multiple of that state..." This discussion reveals that the eigenvalue equation is now central in Billy's reasoning about measurements. As soon as he mentions that it's the same state with constants out in front, he starts to consider whether that really is the same state. He refers to "good old linear algebra," trying to think what pure math is telling him and realizes that the change could be "drastic". Then he reverts to thinking about the interpretation of the eigenvalue equation. He is struggling in his own mind with how much the state has changed. By this point in the interview, his early references to projections have essentially disappeared and Billy is focused on justifying his eigenvalue equation reasoning that the operator does represent the experiment. One of his last and most emphatic statements in the interview is: "And I would say that when you operate on some wavefunction, then you are "measuring" (makes air quotations) and you get some observable out."

DISCUSSION

In order to make sense of quantum mechanics and make sensible computations, students must understand the role of operators and eigenvalue equations. Students must also coordinate unfamiliar language, symbols, procedural knowledge and conceptual knowledge. In these two interviews, we see students struggling to express themselves verbally and mathematically. Ann, who uses operators correctly in the context of quantum measurements, at least once uses language that doing a measurement requires the operator to "act" on a state. Billy uses the term "observable" to mean "a value of a measurement", and in several interviews we noticed this non-standard use of terminology. It is common for textbooks to say that observables are represented by operators. Here we see Billy (and to a lesser extent Ann) trying to negotiate the extent of this correspondence. The role of the eigenvalue equation seems to add to this confusion. Billy's attempts to make sense of the interview task include referring to his understanding of linear algebra and his experiences with the familiar spin-½ system. Instructional approaches should attend to making these potential sensemaking resources more effective.

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