A learning progression for partial derivatives

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We describe a learning progression for partial derivatives spanning virtually all undergraduate courses in calculus through upper-division physics. Anchored by the concept images of both novices and content experts, our learning progression outlines an idealized trajectory for students to follow. The progression details a particular sequence of experiences designed to build on each other and to foster student learning using various strategies for interactive engagement. We then empirically validate the learning progression by studying student understanding of partial derivatives at key points along the trajectory. We provide examples, in the context of differentials, for different layers of our learning progression: concept image, task analysis, and representations.

I. INTRODUCTION: LEARNING PROGRESSIONS

Learning progressions (LPs) in science, known as learning trajectories (LTs) in mathematics, are possible sequences of increasingly sophisticated understandings of topics. Key features of LPs [1, 2], include: (1) LPs are hypotheses about learning in a given domain; (2) LPs include Upper and Lower Anchors, with the Upper Anchor grounded in societal goals for learning core knowledge and practices in science, and the Lower Anchor grounded in the ideas that students bring to the classroom; and (3) LPs describe ways students may develop more sophisticated ways of thinking in a domain, often with support of specific instructional strategies. A major goal of developing learning progressions is to deepen the focus of science and math education on central concepts rather than on inconsequential topics [3].

As the culmination of 20 years of research and curriculum development, we are identifying and evaluating a learning progression for student understanding of partial derivatives which spans the student experience from lower-division multivariable calculus through multivariable chain rules in upper-division thermodynamics and geometric combinations of partial derivatives in upper-division electro- and magneto-statics.

Our LP extends over a large range of advanced and interrelated curricular content; therefore, we have found it useful to focus, at different phases of the development of the LP, on three different aspects, each with its own theoretical framework: concept images, task analysis, and representations. These aspects overlap somewhat, and it is not necessary to think of them as distinct. Each of these aspects is discussed in a subsequent subsection.

In each subsection, we give examples from our LP, chosen to highlight a major theme which threads throughout our LP: how the concept of differentials is a critical part of the concept image of partial derivatives for physicists. We have identified the following ideas about differentials in the upper and lower anchors of the LP:

Upper Anchor. Experts' rich concept images allow them to choose amongst many solution paths and are strengthened by sense-making, but these paths vary from person to person and from field to field; mathematics experts and physics experts are not the same! When developing the upper anchor of our LP by interviewing faculty experts from mathematics, engineering, and physics [4–6], we found, in particular, that the physicists and engineers had several ways of reasoning about small quantities that were not shared by mathematicians. Mathematicians made an explicit choice in the 1950s not to regard differentials in calculus as infinitesimal quantities [7, 8]. On the other hand, physicists have always used differentials to represent quantities that are either truly infinitesimal or at least "small enough" to yield the desired degree of accuracy; we call either usage infinitesimal reasoning [10]. Physicists are very much aware that what really matters is being in the linear regime, rather than being able to determine limits to zero that may not exist in the real world [10–12]. Infinitesimal reasoning turns (first-order) differential equations into linear relationships between differentials, thus blending related rates, implicit differentiation, differential equations, and, of course, the chain rule into a single, unified concept, thereby revealing the true essence of calculus, which is fundamentally about linearization.

Lower Anchor. Based on research surveys [13], we have found that middle-division students tend to think of the derivative as a slope and that these students are good at relating the derivative to changes and computing derivatives algebraically. However, students need more practice viewing the derivative as a ratio, thinking about small changes, and interpreting when a change is "small enough."

We conclude this paper with a section on what we have learned about the types of research that are possible and/or appropriate to do in deriving and validating an LP at the upper-division undergraduate level.

A. Concept Image

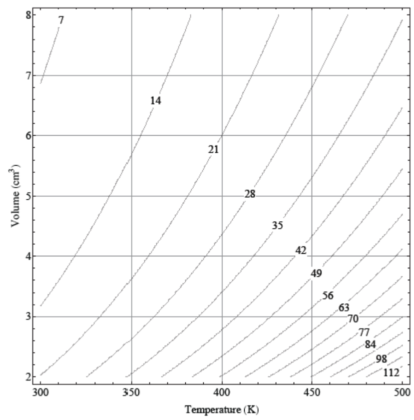
Tall and Vinner [14, p.152] define a concept image as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." As developing professionals, middle-division students need to develop rich concept images rapidly. We use research about the Upper and Lower Anchors, as well as student work in between, to identify a rich concept image of each of the main concepts in our LP.

FIG. 1. Interview Task focused on coordinating different representations

Evaluate $\left(\frac{\partial U}{\partial T}\right)_P$ at $P = 10$ atm., $T = 410$K using the information below.

$P(atm.)$	$T(K)$	$V(cm^3)$
10	300	1.32
10	310	1.44
10	320	1.57
10	330	1.71
10	340	1.85
10	350	2.00
10	360	2.15
10	370	2.32
10	380	2.49
10	390	2.67
10	400	2.86
10	410	3.05
10	420	3.25
10	430	3.47
10	440	3.69
10	450	3.91
10	460	4.15
10	470	4.40

Pressure P, Temperature T, and Volume



Internal Energy $U\,(T, V)$.

Hiebert and Carpenter [15, p.67], suggest that understanding a mathematical idea requires it to be part of an internal network and that "the degree of understanding is determined by the number and strength of the internal connections." In our own research [16], we learned that middle-division students tend not to go back and forth between the elements of the concept image spontaneously. To help students increase the strength of their connections, our LP emphasizes opportunities for students to translate between such representations. Furthermore, students' ability to transfer in these ways is, for us, an important empirical validation of our LP.

The concept image of ordinary derivatives was discussed in detail in a seminal paper by Zandieh [17]. To find derivatives from data, it is necessary to use the ratio layer of Zandieh, namely the part of the concept image that says that derivatives can be thought of as "ratios of small changes." In recent work, we emphasized this element of the concept image by adding an explicit numerical representation [10, 11], and by introducing the notion of "thick derivatives" for ratios that are "good enough" [10, 12]. The finite, but "small enough," changes in both the numerator and the denominator of the ratio, are, of course, exactly what physicists mean by differentials.

We are now extending this concept image to partial derivatives where the idea that the direction in which the derivative is taken (equivalently, what is being held constant) becomes an important element.

To understand more clearly how students reason about thick derivatives, we have conducted problem-solving interviews that require students to take information from both a table of data and a contour map to construct the desired partial derivative and we are exploring the ways in which they transfer from one representation to another [18], see Fig. 1.

B. Task Analysis

Once we have a clear sense of one or more elements of a concept image that we hope to convey to students, we begin to build curriculum. Referring to our anchors as references, we carefully decompose potential student tasks and examine the resulting structure and level of complexity in a cognitive task analysis. "Cognitive task analysis is the extension of traditional task analysis techniques to yield information about the knowledge, thought processes, and goal structures that underlie observable task performance" [19].

When we perform a task analysis, we pay particular attention to the ways in which students engage with the task, using feedback from: small whiteboard questions and other classroom formative feedback; our own pedagogical content knowledge built from years of teaching and office hours; constant conversations with TAs, LAs, and student team members; student work on homework, quizzes, and exams; and student focus groups. Often, this informal research draws our attention to extra steps in the task that we, as experts, did not notice but which need to be addressed in our curricular tasks.

To physicists, differentials describe the small changes necessary to define the derivative from representations of data. A cognitive task analysis has uncovered several ways in which physicists routinely use differentials in thermodynamics and explored how students learn to manipulate differentials over time. Equations found by taking the total differential of an equation (which we will call differentials equations) can be manipulated using simple algebraic rules to solve many thermodynamics problems. Differentials equations are always linear, meaning that substitution is always an effective strategy. Even for complicated equations chain rules can be determined quickly [10, 20].

Physicists can also use differentials equations to "identify" a particular partial derivative as a physical quantity – which has its own operational definition – by comparing physics statements to mathematics statements. For example, a physicist can identify temperature and volume as partial derivatives of enthalpy (Eqns. (3)–(4)) by comparing the definition of enthalpy (derived from the thermodynamic identity via a Legendre transformation, Eqn. (1)) to an equation which is a mathematical identity (e.g. the differentials version of the multivariable chain rule, Eqn. (2)) [21].

$$dH = T\,dS + V\,dp \tag{1}$$

$$dH = \left(\frac{\partial H}{\partial S}\right)_p dS + \left(\frac{\partial H}{\partial p}\right)_S dp \tag{2}$$

$$T = \left(\frac{\partial H}{\partial S}\right)_p \tag{3}$$

$$V = \left(\frac{\partial H}{\partial p}\right)_S \tag{4}$$

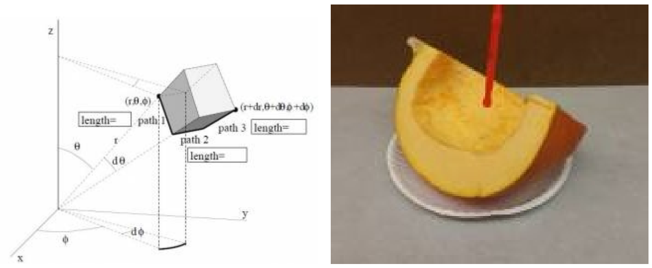


FIG. 2. Representations for the vector differential and volume element in spherical coordinates.

FIG. 3. Students pointing in the direction of the gradient.

C. Representations

A theoretical appreciation of representations is helpful in understanding how students interpret, use, and move between different representations. We draw on the perspective of distributed cognition [22, 23], which provides an account for the role of external entities (including tools, other people, and representations) in cognition. Many of the representations we use are external representations (as opposed to mental representations), and have physical or material features. In the case of written text, the material features include the symbol shapes that make up the text and the surface on which the text appears (like paper or whiteboard). These features both enable and constrain possibilities: written words are stable, thus freeing working memory for other work; paper is two-dimensional, so objects and functions can only be drawn in cross section or perspective, which requires cognitive processing to interpret. In this way, the material features affect the computational power and the pedagogical affordances of the representation. To facilitate student reasoning in three-dimensional cases, using a physical model that can be manipulated (like a pumpkin or a hula hoop) helps students visualize the physical situation and facilitates creating algebraic representations of geometric relationships. In kinesthetic activities, we evoke embodied cognition by asking students to represent the geometric or physical situation with their own bodies.

Differentials of spatial quantities, particularly in E&M contexts, can be combined into the vector differential, which describes a small change in all directions at once. The multivariable chain rule becomes a statement about the gradient, namely

$$\begin{aligned} df &= \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \\ &= \vec{\nabla} f \cdot d\vec{r}. \end{aligned} \tag{5}$$

Equation (5) can be used to find: any directional derivative, the total derivative with respect to a parameter like time, and properties of the gradient (e.g., it is perpendicular to the level curves and its magnitude is the slope in the steepest direction) [7, 24].

At the middle-division level, most students can effectively use different representations but many are nevertheless unable to move smoothly back and forth among these representations. For example, in problem-solving interviews regarding flux [16], we found that if students are prompted with words, they often respond with words; prompted with equations, they respond with equations, etc. But they may not spontaneously draw connections between these representations in the ways that a professional would; that is, they are not "harmonic" reasoners in the sense of Krutetskii [25].

In our LP, we often give students the opportunity to contrast two different representations of the same concept which have different material features. In curricular materials we developed to help students visualize the geometric nature of the vector differential and the gradient, we pair representations several times. Fig. 2 shows both a traditional 2-dimensional textbook figure of the volume element in spherical polar coordinates, which students compare, in an activity, to a 3-dimensional tangible representation of the same volume element which they cut out of a pumpkin. Figure 3 shows a kinesthetic activity in which students point "in the direction of the gradient." For a function of two variables, the gradient is a two-dimensional vector, but the representation in words that "the gradient points in the steepest direction," leads many students to believe that the gradient points upward.

II. RESEARCH:

A central feature of formal learning progressions is that they are based on research, as opposed to selecting sequences of topics and learning experiences based only on logical analysis. However, we have found that it is unrealistic to empirically validate our LP as a whole: It encompasses content from all four years of the undergraduate curriculum and crosses all subdisciplines of physics; our student population comes from diverse backgrounds; individual students flow through the courses differently; n is small (≈ 35 students/year); and there is no reasonable comparator group. Nevertheless, we believe that we have achieved a robust understanding of the ways in which students interact with the LP by collecting a wide range of types of data and by employing a rich variety of research perspectives and methods. We have used both formal and informal research to establish snapshots of student understanding at many points along the LP.

Throughout this paper, we have given examples of the formal research methods and frameworks we have used and referenced the results of both our empirical validation of specific curricular elements of the LP and our tests of the current location of students along the LP. In addition, we have constantly embedded our work in informal research that generates significant knowledge about the LP but may not result in the publication of research papers. Due to the extreme active-engagement nature of our courses and because our group involves teachers, curriculum developers, and researchers (including some individuals who play multiple roles), we are able to triangulate information from many sources and validate the LP through a large variety of scholarly activities.

III. CONCLUSIONS

We have begun the process of building and documenting a learning progression by which students can learn to understand, manipulate and reason with partial derivatives and differentials. This learning progression is a synthesis of research we have performed as well as our experience teaching partial derivatives and differentials, and will be useful as a research-based guide for teaching partial derivatives. Stay tuned as we flesh out the entire learning progression!

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