Graphical Representations Of Vector Functions In Upper-Division E&M

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Abstract. In upper division electricity and magnetism, the manipulation and interpretation of vector functions is pervasive and a significant challenge to students. At CSU San Marcos, using in-class activities adapted from the Oregon State University Paradigms in Physics Curriculum, students' difficulties with vector functions become evident in two types of in-class activities: sketching vector functions and relating vector and scalar functions (e.g., electric field and electric potential). For many students, the cause of these difficulties is a failure to fully distinguish between the components of a vector function and its coordinate variables. To address this difficulty, we implement an additional in-class activity requiring students to translate between graphical and algebraic representations of vector functions. We present our experience with these issues, how to address them, and how in-class activities can provide evidence of student thinking that facilitates curricular refinement.

Keywords: Electromagnetism, Vector Functions, Graphical Representations, Upper Division, Sense-making
PACS: 01.40.Fk, 01.40.gb

INTRODUCTION

A vector field is an assignment of a vector quantity to each point in space. Physical examples from electricity and magnetism (E&M) include the electric field, magnetic field, and magnetic vector potential. Typical upper-level E&M courses require students to use and interpret these fields and the related mathematics of vector calculus. Vector fields can be represented algebraically, through several graphical representations, or even kinesthetically (where students physically represent the field with their bodies). Many upper-level E&M courses emphasize the algebraic representation. The Paradigms in Physics Curriculum developed at Oregon State University includes an explicit focus on graphical representations of vector fields in E&M. In particular, course goals include having students "develop conceptual and geometric understandings of... electrostatic potentials and fields, including geometric understanding of vector and scalar fields" and "be able to move between algebraic and diagrammatic representations of these fields… [1]" Many students find this challenging.

In E&M, common graphical representations of vector fields include vector arrows drawn at some sampling of points in the space or field lines that start and stop at the location of charges and have tangent lines at each point that are parallel to the vector field at that point. Related representations include contour or equipotential plots of scalar functions.

In this paper, we compare different representations of vector fields, what we can learn from students' use of those representations, and how various representations can be used to facilitate student learning. In particular, we note that some of our students' conceptual difficulties with vector fields only become evident when they use certain representations. Furthermore, comparing another representation often helps resolve these difficulties. We interpret this in terms of the affordances and constraints of these representations on student learning and classroom discourse in upper-level E&M courses. We report on student engagement in classroom activities developed and administered at Oregon State University (OSU) and CSU San Marcos [1-3]. Data include videos of these activities recorded at OSU during the fall of 2008 and 2010, as well as recordings of a CSU San Marcos instructor's reflections on these activities. These reflections were recorded in fall 2010 in the few days after the activities were done in class.

DIMENSIONALITY OF THE GRADIENT

The gradient of a scalar field is an important concept for upper level E&M because conservative vector fields can be written as the gradient of a scalar potential field. Understanding how to calculate the gradient and interpret it geometrically is an important goal in these courses. Before using the gradient to relate the electric field and electric potential, students are engaged in an activity sequence where they are given a 2D scalar function describing the elevation of a hill. For a number of specific points in the 2D space, students are asked to find a number of geometric features, including some that require using the gradient. In one kinesthetic activity, students were asked to visualize an elliptical hill in the center of the room. They were then asked to imagine their right shoulders were points on the hill. Using their right arms as vectors, they were asked to point in the direction of the gradient of the function that described the height of the hill with respect to their location on the floor [4]. As seen in Figure 1, many students pointed up towards the ceiling while many others pointed their arms parallel to the ground. (Additionally, many of these students pointed in the direction of the highest point of the hill, while many others pointed in the direction of the steepest slope at their location.)

FIGURE 1. Students indicating the direction of the gradient.

This activity revealed an important issue related to the dimensionality of the gradient. In this case, the height of the hill as a function of location is a 2D scalar field. For such a field, the gradient is also a 2D quantity - it only has components in the coordinate plane. Yet, based on their response to the kinesthetic activity, many students in the class believed the gradient in this case was a 3D quantity. The basis for students' thinking could be the fact that the hill exists in 3D, and it's natural to point parallel to the surface of the hill while standing on the hill. Furthermore, this example differs from most E&M applications because the function (elevation) and the variables (location) share the dimensionality (length). The ambiguous yet common verbal representation "the gradient points in the direction of steepest slope" could also contribute to this thinking.

Importantly, the kinesthetic activity provides an opportunity to explore the dimensionality of the gradient, because the students can represent the gradient vector in a 3D space. Thus, students must make a choice about the vertical component of the gradient vector. Similar activities that ask students to draw the gradient vector on a 2D sheet of paper or whiteboard constrain the students to a 2D representation of the gradient and do not bring out this issue. Overall, these different representations have varying freedom; in the kinesthetic activity, students face the choice of giving a vertical component and only here does the issue become obvious. Constraints in the representation can be useful (if they match the physics/math), in that they can prevent incorrect operations, but they can also prevent student conceptual difficulties from being revealed.

If one were to calculate the gradient from an equation describing the height of the hill, this dimensionality issue also would not arise. A correct computation would result in the gradient field having only two components. In this case, drawing parallels between two representations - an algebraic representation and a 3D geometric representation - may help students resolve the conceptual issues that arise in one of them.

LOCALITY OF THE VECTOR REPRESENTATION

Upper-level students are encouraged to think of the electric field, magnetic field, and other vector fields in a true "field" sense - as vector quantities at every point in space. Furthermore, these fields do not represent geometric quantities such as lengths, but instead their magnitude is associated with their field strength. These features have important consequences for geometric representations of vector fields in E&M. Unlike a scalar field, which can be represented with a continuum of points (for instance a 3D surface can represent the value of a 2D scalar function), representations of vector fields only include vectors at a sampling of points. For instance, students are encouraged to visualize these fields as arrows located at a sampling of points in space, where the base of the arrow is the point where the field is being evaluated, the arrow points in the direction of the vector field, and the length of the arrow indicates the magnitude of the vector field. We call this the vector representation. One feature of this representation is that the arrow represents the field at a single point in space. We refer to this feature as locality. However, in the representation, the arrow has a length that takes up some "space" which doesn't correspond to real space. These features of the vector representation can cause confusion for students.

For example, in the Concept of Flux [4] activity, each student was asked to hold a ruler from the end and told to imagine that the ruler represents an electric field vector evaluated at the location of their hand. This is a direct kinesthetic analog to the graphical vector representation. The instructor had a hoop and explained that the area enclosed by the hoop was a surface. The instructor placed the hoop so that some of the rulers stuck through the hoop but the students' hands were not in the plane of the hoop (see Figure 2). When the instructor asked what the flux through the hoop is, many students said there is a positive flux and some said the flux is zero. However, in this vector representation, the extent of the arrow does not correspond to a physical extent in real space so it can't really "stick through" the surface. Only the points that lie in the surface can contribute to the flux.

FIGURE 2. Students are asked to consider the flux through the hoop. Rulers represent vectors.

This feature of the vector representation also arises in drawing a vector field. In drawing a 2D vector field, we observed students start by drawing a Cartesian grid. They then picked points, identified the *x*- and *y*-coordinates at these points, and plugged those values into the vector field function. For drawing the vectors, the natural thing to do is to use the same scaling as the spatial grid. However, in physics, most vector fields (e.g. electric fields) have dimensions that are not spatial (i.e., units of newtons/coulomb rather than meters). Therefore, the scale of the spatial grid has nothing to do with the magnitudes of the vectors - they have different dimensions! It is the relative lengths of the vectors that matter in this representation.

While the vector representation is emphasized in upper level physics, in lower level physics, the field line representation is emphasized for visualizing vector fields. When drawing the vector representation, we observed some students negotiating how the field lines representation is related to the vector representation. In a small group activity, students were asked to draw the electric field vectors for four positive charges arranged in a square. Most groups started by drawing field lines. The instructor then discussed the difference between field lines and vectors for a single point charge, a case where the field lines and vectors are co-linear. The discussions in this activity were filled with much productive reasoning using superposition and symmetry to draw the field vector arrows.

During a whole class discussion of the activity, one student suggested that you should draw more vectors in the region where the field has a large magnitude. This led to a comparison of how the two representations handle the magnitude of the field. With field lines, the density of the lines represents magnitude (the closer the lines are to each other, the larger the magnitude of the field - though the exact relationship only works if the lines are drawn in a 3D space [5]); in the vector representation, the arrows' lengths represent the magnitude of the field. This student's suggestion may indicate not understanding that the vectors in the vector representation come from an arbitrary sampling of points.

Another student suggested that electric field vectors cannot cross. This idea could come from either his knowledge of electric field lines or equipotential surfaces. This suggestion illustrates a lack of understanding of the locality feature of the representation. The extent of the arrows does not correspond to 3D space and it does not matter if the arrows cross or overlap.

In sum, we have noted two limitations of the vector representation of vector fields. First, the representation only shows the vectors for some sample of points. Second, the vector arrows are represented with a physical extent that does not correspond to a geometric length. The student conceptual difficulties described above seem to arise from a too-literal interpretation of these aspects of the vector representation.

COORDINATES VS. COMPONENTS

In E&M, we are typically interested in functions of spatial coordinates (typically *x*, *y*, *z*) in two or three dimensions. A 3D vector field function has three components, and each component may depend on the three spatial coordinates. Students often struggle to untangle the coordinates from the components. Particularly confusing is the case of "mixed" dependencies, for example when E_x is a function of the *y*-coordinate. Electric fields often vary in the same direction the field points (e.g. the electric field for a positive point charge falls off radially and the field points radially away from the charge). Magnetic fields more often have mixed dependencies (e.g. the magnetic field for an infinite current carrying wire falls off radially but points azimuthally [6]).

Students' difficulties with coordinates and components were evident during an in-class activity where students were given algebraic expressions for 2D vector field functions and asked to sketch them. Students were given expressions such as $\vec{F} = y\hat{i} + x\hat{j}$; $\vec{F} = (\hat{i} + \hat{j})/\sqrt{2}$; $\vec{F} = x\hat{i} - y\hat{j}$; $\vec{F} = y\hat{i}$; and $\vec{F} = x\hat{j}$ [7]. Many students struggled to begin translating the algebraic formula into a graph. After a few minutes, the instructor offered an interpretation of the vector function expression as shorthand for three equations, one for each component of the vector field. He then suggested that students adopt a routine for sketching the functions that involved explicitly picking a point in the plane, evaluating each component of the vector function with the coordinates of that point, sketching the resulting vector, and then considering what changes at different points. Interestingly, after students were able to complete one of these sketches, further examples presented much less difficulty, suggesting that students had a breakthrough in understanding that was consolidated on subsequent items.

Why does this sketching activity both reveal and provide an opportunity to resolve students' confusion about components and coordinates? The graphical vector representation allows separate tracking of components and coordinates by using the vector arrows in addition to the 2D coordinate space. Apparently, in the algebraic representation, the *x*'s and $\hat{i}$'s are indistinct for some students. We conjecture that the understanding students gain through the graphing activity would result in their greater understanding of the algebraic expression; however, we do not have evidence to support this.

DISCUSSION

Different representations have different affordances and constraints. Algebraic representations of vector fields are easy to manipulate but students do not easily interpret differences between components and coordinates. Drawing a 2D vector representation of a field disambiguates components and coordinates but constrains students to thinking and representing in a plane. Kinesthetic activities allows students to use a 3rd dimension to represent vectors, but allows for representing direction of the field more easily than the magnitude or functional dependencies (and is not readily preserved for further analysis). In order to understand and use any representation, the user must understand how information about the phenomenon is presented. We have seen that different representations allow students to consider different choices in expressing their understanding of the situation. These choices can illuminate different aspects of student thinking and give instructors more information about student ideas. This will hopefully lead to more opportunities to address student difficulties and deeper conceptual and analytic learning. Further, different representations can help resolve the conceptual uncertainties that arise during the use of other representations. Thus, we expect that as students become familiar with different representations of vector fields and how to translate from one representation to another, their sense-making skills will become more versatile.

ACKNOWLEDGMENTS

Special thanks to Corinne Manogue for valuable discussions and help with data. Thanks also to Jeff Monroe, Teal Pershing, Andy Svesko for locating many of the examples in this paper. Finally, we would like to thank all of the Paradigms students and TA's over the years who have contributed so much to our understanding. This work was also supported by NSF grants DUE 0231194, DUE 0231032, DUE 0618877 and DUE 1023120.

REFERENCES

1. All of the materials and activities discussed here, as well as a complete description of the Paradigms in Physics Program can be found at: http://www.physics.oregonstate.edu/portfolioswiki
2. C.A. Manogue, et al., *American Journal of Physics* 69, 978-990 (2001).
3. C.A. Manogue and K.S. Krane, *Physics Today* 56, 53-58 (2003).
4. This activity was originally developed as part of the Vector Calculus Bridge Project: http://www.physics.oregonstate.edu/bridge
4. E. van Zee and C. Manogue, *2010 PERC Proceedings* 61-64 (2010), and for examples, see: http://www.physics.oregonstate.edu/portfolioswiki/whitepapers:narratives:start
5. A. Wolf, S. J. Van Hook, E. R. Weeks, *American Journal of Physics*, Vol. 64, No. 6. (1996), pp. 714-724
6. C.A. Manogue, K Browne, T. Dray, and B. Edwards, *American Journal of Physics* 74, 344-350 (2006).
7. This in-class activity was based on a homework problem in H.M. Schey, *Div, grad, curl, and all that* (W.W. Norton, 2005).