Using great circles to understand motion on a rotating sphere

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Motion observed in a rotating frame of reference is generally explained by invoking inertial forces. While this approach simplifies some problems, there is often little physical insight into the motion, in particular into the effects of the Coriolis force. To aid in the understanding of three-dimensional inertial forces, motion on a rotating sphere is considered from the points of view of an inertial observer and of an observer fixed on the sphere. The inertial observer observes the motion to be along a great circle fixed in the inertial frame, in analogy with simple straight-line motion in the two-dimensional case. This simple ''straight-line'' viewpoint of the inertial observer is reconciled qualitatively and quantitatively with the view of the rotating observer that requires inertial forces in order to account for the motion. Through a succession of simple examples, the Coriolis and centrifugal effects are isolated and illustrated, as well as effects due to the curvilinear nature of motion on a sphere.

I. INTRODUCTION

Observation of motion from a rotating frame of reference introduces many curious features. Fundamentally, the acceleration of the rotating observer causes motion that is well behaved as viewed by an inertial observer to become distinctly nonintuitive when viewed by the rotating observer. For example, particles upon which no forces act appear to be deflected from their straight line paths. The general approach taken to account for this strange behavior is to introduce inertial forces. These new ''forces'' then enter the equation of motion in the rotating frame. While this approach is very powerful in some circumstances, the complexity of the vector cross products and of the resultant coupled differential equations generally obscures physical insight into the motion. Upon introduction to inertial forces, students generally spend more time grappling with the mathematics than understanding the motion. The physical origin of the inertial forces is generally only truly understood by viewing the motion in the inertial frame and then relating that to the noninertial view. This approach is common in the case of two-dimensional motion on a frictionless turntable.[1,2] In that example, particles merely travel in straight lines as seen by the inertial observer. The complicated motion seen by the rotating observer is then just a transformation of the simple straight-line inertial motion into the rotating reference frame. The simplicity of the geometry and of the frame transformation helps students focus on the motion rather than the mathematics. Understanding of inertial forces in the two-dimensional case is also aided by a wide variety of lecture demonstrations of turntable motion,[3] as well as the simple exercise of playing catch on a merry-go-round.[4]

Unfortunately, physical insight into three-dimensional inertial forces is harder to come by. There are few treatments that compare the inertial and rotating viewpoints, and there are no simple ways to demonstrate the effects of motion on a rotating sphere, either in the lecture hall or at the amusement park. The most common examples of three-dimensional inertial forces are provided by motion relative to our rotating earth. Unfortunately, the small effects arising from the centrifugal and Coriolis forces on earth are not generally part of the everyday experiences that we use to build up our physical intuition. Effects upon the weather, ocean currents, rivers, and projectile motion are well documented,[4] but motion over very long distances is required for discernible effects. A common pedagogical example of the Coriolis force is a rocket fired from the north pole, which, as shown in Fig. 1, flies south in the inertial frame but does not move east along with the rotating earth, thus landing west of its intended target. While other examples of the utility of the inertial viewpoint in explaining Coriolis effects have appeared in the literature,[5–9] these examples typically emphasize a particular aspect of the motion, and, like the example of Fig. 1, are not easily generalized.

The aim of this article is to provide simple explanations of three-dimensional inertial forces by considering how the inertial motion is viewed in the rotating frame. A general framework is used to analyze motion on a rotating sphere from the point of view of an inertial observer and to show how that motion corresponds qualitatively and quantitatively to the rotating-frame description that invokes inertial forces. We consider the idealized situation of motion on a frictionless, rotating sphere. This ''terrestrial ice hockey'' example is a generalization of the two-dimensional frictionless turntable, with the simple straight-line inertial motion replaced by motion along an inertial great circle. The simplicity of great circle motion together with a judicious choice of initial conditions permits us to isolate the different inertial forces and provide simple qualitative explanations. We then present a quantitative comparison of the motion as viewed in the two frames. While the use of great circles in the context of motion on a sphere may appear obvious, instances of such usage have been infrequent.[10,11]

II. ROTATING REFERENCE FRAMES

Consider two coordinate systems whose axes rotate with respect to one another and whose origins coincide. Assume that one system is an inertial system and let the angular velocity of the rotating system with respect to the inertial system be $\vec{\boldsymbol{\omega}}$. Considering the motion of a particle at position $\vec{\mathbf{r}}$, the relations between the velocities and accelerations as measured in the two coordinate systems are[12]

$$\left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{\text{inertial}}=\left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{\text{rotating}}+\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}, \tag{1}$$

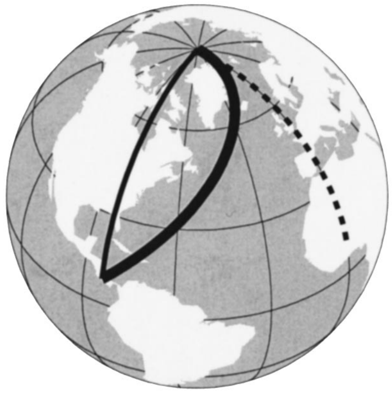

Fig. 1. A rocket launched along the prime meridian from the North Pole continues south (the medium thickness solid line) along a meridian fixed in the inertial frame. The intended earthbound route (the dashed line) moves east with the earth, and the rotating observer sees the rocket follow a curved path (the thickest solid line) that deflects to the right of the intended path.

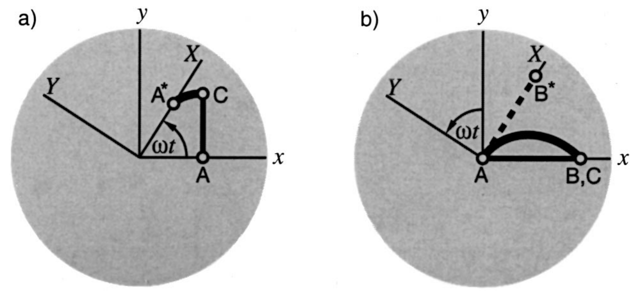

Fig. 2. (a) Motion of a puck released from rest with respect to a rotating turntable from position A. After a time t, the initial position of the puck has rotated from A to A^*. The inertial observer sees the puck follow the straight line from A to C, while the rotating observer sees the puck follow the curved path from A^* to C. (b) Motion of a puck launched with a speed v from the center of the turntable toward a target B on the turntable. After a time t, the target has rotated from B to B^*. The inertial observer sees the puck follow the straight line from A to C, while the rotating observer sees the puck follow the curved path from A to C, which deviates from the straight path from A to B^* as expected if rotation is ignored.

$$\left(\frac{d^2\vec{\mathbf{r}}}{dt^2}\right)_{\text{inertial}} = \left(\frac{d^2\vec{\mathbf{r}}}{dt^2}\right)_{\text{rotating}} + \vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}) + 2\vec{\boldsymbol{\omega}}\times\left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{\text{rotating}} + \frac{d\vec{\boldsymbol{\omega}}}{dt}\times\vec{\mathbf{r}}. \tag{2}$$

The acceleration equation is known as Coriolis' theorem as it was presented by Gaspard Gustave de Coriolis in 1835. Hereafter, we ignore the last term to consider only systems that rotate at a constant angular velocity. In the inertial system, a particle of mass m subject to a force $\vec{\mathbf{F}}$ obeys Newton's equation of motion:

$$m\left(\frac{d^2\vec{\mathbf{r}}}{dt^2}\right)_{\text{inertial}} = \vec{\mathbf{F}}. \tag{3}$$

Substitution of Eq. (2) into Eq. (3) leads to the equation of motion in the rotating frame:

$$m\left(\frac{d^2\vec{\mathbf{r}}}{dt^2}\right)_{\text{rotating}} = \vec{\mathbf{F}} - m\vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}) - 2m\vec{\boldsymbol{\omega}}\times\left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{\text{rotating}}. \tag{4}$$

The rotating observer thus postulates two new forces to explain the motion of the particle. These new forces go by a variety of names: ''inertial'' forces, because they represent the inertia of the body; ''noninertial'' forces, because they arise from being in a noninertial frame; or ''fictitious'' or ''pseudo'' forces, because they are artifacts of being in a noninertial reference frame. The first new term on the right-hand side of Eq. (4) is called the centrifugal force,

$$\vec{\mathbf{F}}_{\text{cent}} = -m\vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\mathbf{r}}), \tag{5}$$

and points away from the axis of rotation. The second new term on the right-hand side of Eq. (4) is called the Coriolis force and is often written as

$$\vec{\mathbf{F}}_{\text{Cor}} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}}_r, \tag{6}$$

where $\vec{\mathbf{v}}_r$ is the velocity relative to the rotating system. The Coriolis force causes deflection perpendicular to the motion in the rotating frame.

These new forces allow one to solve for the motion of a particle in a rotating frame without any reference to the motion in an inertial frame. While this is a convenient and powerful technique, the physical basis for the inertial forces is often masked by such an approach. Our approach is thus to describe the motion in both frames and show how the correspondence between the two can help to understand the inertial forces.

III. QUALITATIVE DISCUSSION

Before addressing the problem of motion on a rotating sphere, it is instructive to consider first the case of motion in a plane. Consider the simplest case of a hockey puck on an icy turntable, where friction can be ignored and the force of gravity is countered by the normal force of the turntable on the puck. Since there is no net force on the puck, it remains at rest or continues with constant velocity in the inertial system. The acceleration of the rotating observer causes this simple inertial motion to be seen as curved motion in the rotating frame, which is explained by invoking inertial forces. Figure 2 shows two simple examples that serve to demonstrate centrifugal and Coriolis forces. All the figures in this section (Figs. 2 and 3) represent time durations long enough for higher-order effects to be evident, but we focus on the lowest-order effects in each case. In Fig. 2(a) the puck is released from rest with respect to the turntable, at a position A, which is a radius R from the rotation axis. An observer on the turntable who is oblivious to the rotation (hereafter referred to as the ''naïve rotating observer'') would expect the puck to remain at rest. An observer in the inertial frame notes that the puck has a tangential velocity component ωR arising from the rotation of the turntable. The inertial observer sees the puck follow the straight path from A to C, while the rotating observer rotates from A to A^* and sees the puck follow the path from A^* to C, taking it to larger radii. The rotating observer aware of the rotation (hereafter referred to as simply as the ''rotating observer'') invokes the centrifugal force to explain why the puck drifts to larger radii, while the inertial observer claims it is a simple consequence of the inertial motion of the puck combined with the acceleration of the rotating observer. In Fig. 2(b) the puck is launched from the origin toward a target B on the turntable. The naïve rotating observer would expect the puck to follow a straight path, shown in Fig. 2(b) as A to B^*, where B^* is the rotated inertial position of the target after some time t. The inertial observer sees the puck follow the straight path from A to C, while the rotating observer sees the puck follow the curved path from A to C, which clearly misses the target B^*. The rotating observer invokes the Coriolis force to explain the deflection, while the inertial observer claims it is a simple consequence of the inertial motion of the puck combined with the acceleration of the target. These two simple examples illustrate that inertial straight-line motion is transformed into more complicated, and in general curved, motion when viewed from the rotating frame.

To illustrate the effects of inertial forces in the three-dimensional case, we choose the simplest example of motion on a rotating sphere. In analogy with the two-dimensional frictionless turntable, we consider particles sliding on a frictionless, rotating sphere of radius R. This is akin to playing ice hockey on a frozen earth, but frozen before the earth started spinning so we can ignore the effect of the rotation on its shape. We require gravity to keep the hockey puck on the surface, but do not need to know the magnitude of the gravitational acceleration. We require that the puck stays on the earth, so we consider only velocities, with respect to the inertial frame, that are less than the orbital velocity. Again, in analogy with the two-dimensional case, we consider the viewpoints of an inertial observer, an earthbound observer, and a ''naïve'' earthbound observer who is unaware of the rotation.

In the inertial frame, the only forces on the hockey puck are the normal force of the ice and the force of gravity. Since both forces are radial, there is no torque about the center of the earth, ensuring conservation of angular momentum. The motion of the puck is therefore in the plane defined by the initial radius vector to the puck and the initial velocity vector of the puck. Since this plane passes through the center of the earth, it intersects the surface of the earth in a great circle. The puck thus follows a great circle path in the inertial frame. To the extent that we consider a great circle as a ''straight line'' on a sphere, the motion of the puck is analogous to the two-dimensional motion of the puck on the frictionless turntable.

The great circle motion of the puck takes place in the inertial frame. A naïve earthbound observer would expect the puck to follow a great circle path with respect to the earth. These two great circles are in general different because the rotation of the earth gives the puck an additional eastward velocity in the inertial frame, resulting in different initial directions of the motion, as viewed in the two frames. The actual motion perceived by the earthbound observer is not a great circle, but rather is the transformation of the inertial great circle into the rotating frame. The essence of the approach in this article is to describe the differences between the path viewed by the earthbound observer and the path expected on a stationary earth. Particular choices of initial conditions permit us to isolate and qualitatively describe the different inertial forces. Since these are dynamical effects, their demonstration and understanding are generally aided by dynamical presentation. Animations of the figures in this section (both turntable and sphere) are available for viewing on the World Wide Web.[13]

To illustrate some basic features of great circle motion, it is instructive to first consider motion on a stationary sphere. A puck with an initial velocity to the east follows the great circle path shown in Fig. 3(a). Some time after leaving the initial position A, the puck arrives at a position B, which is

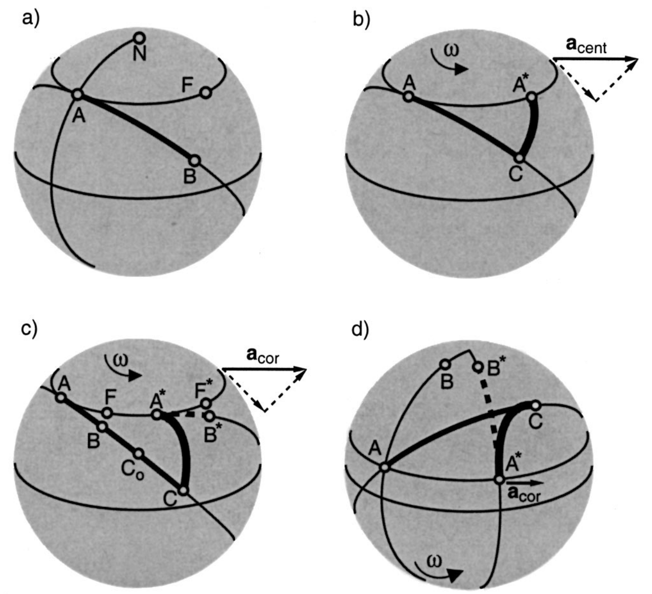

Fig. 3. (a) Motion of a puck launched to the east on a stationary earth. The puck travels along a great circle path from A to B. On a flat earth, the puck would always travel east and end up at F. (b) Inertial view of the motion of a puck released from rest with respect to the rotating earth. The puck travels along the inertial great circle path from A to C in a time t. After the time t, the earthbound observer has moved along the original line of latitude from A to A^*, and has seen the puck move along the path from A^* to C. (c) Motion of a puck that is launched eastward with respect to the rotating earth. The puck is launched from position A at time $t=0$ toward a target located at position B. After a time t, the launch site and the target have rotated to positions A^* and B^*, respectively. The target on a flat earth would be at F (rotated to F^* after time t). The inertial observer sees the puck travel along the great circle from A to C, while the earthbound observer sees the puck move along the path from A^* to C, missing the intended target at B^*. The puck would end up at C_0 if the initial velocity with respect to the earth were zero. (d) Motion of a puck that is launched to the north with respect to the rotating earth. The description of the motion is the same as in (c) above. The flat earth target F is coincident with the target B.

clearly east of A, but also south of the original line of latitude. This appears counterintuitive since we are generally accustomed to considering the earth as locally flat and to using latitude and longitude as rectilinear coordinates, in which case a puck thrown to the east along a line of latitude would continue heading east and would arrive at F, a point on the same line of latitude. However, a line of latitude is not a ''straight line'' on a sphere; this is most obvious near one of the poles. Thus, instead of following a line of latitude to the east, the puck travels along a great circle or geodesic on the sphere with no deflections from that ''straight line'' path. This great circle path is also related to the line of sight from A toward B. Imagine constructing a vertical tower at position B tall enough for the observer at A to see above the horizon. The observer at A sees the tower when looking directly east, meaning that a straight line tangent to the circle of latitude at A intersects the tower. This line projected radially down to the earth's surface coincides with the great circle path from A to B. Hence, we also refer to this path as the line of sight from A to B. Thus, if we call the final position of the puck (B) along the great circle the ''target,'' then in simple terms we can say that the earthbound observer saw a target to the east, launched the puck to the east, and hit the target. The puck clearly misses the target F expected on a flat earth. We call the difference between these two targets the curvilinear correction, since it arises from the inherent curvilinear nature of a great circle on a sphere and the mismatch between the great circle and the supposedly straight lines used in the latitude–longitude coordinate system. A similar effect has previously been pointed out in the two-dimensional turntable problem when the rotating observer uses a curvilinear coordinate system.[3] Another counterintuitive aspect of this effect on the earth is that if an observer sees a landmark by looking directly east, then the observer is not directly west of the landmark. Note that if the puck is launched toward the north pole [N in Fig. 3(a)], then it continues heading north along the original meridian. Since the meridians are great circles, there is no curvilinear correction for this motion.

Now consider a rotating earth and a puck that is released from rest (with respect to the rotating earth) by an earthbound observer at a northern latitude λ_{start}. A naïve earthbound observer would expect the puck to remain at its initial location on the earth. An inertial observer notes that the puck has an eastward speed equal to the speed of the surface of the earth at the initial latitude, $v_{\text{earth}} = \omega R \cos \lambda_{\text{start}}$. The puck travels along the inertial great circle shown in Fig. 3(b) and after some time arrives at position C. During that time, the earthbound observer travels along the original line of latitude from the initial position of the puck A to a new position A^* in the inertial frame. Thus the earthbound observer sees the puck follow the path from A^* to C (not a great circle). The inertial observer explains this relative motion as the difference between the great circle path of the puck and the fixed latitude of the earthbound observer. The earthbound observer explains the southward motion by invoking the centrifugal force [Eq. (5)], which points away from the axis of rotation and has upward and southward components, as shown in Fig. 3(b). The upward component simply reduces the normal force and hence the apparent weight of the puck. The southward component is responsible for the southward displacement of C from A^*. Note that the westward displacement evident in Fig. 3(b) is a higher-order effect caused by the Coriolis force due to the acquired southward motion (see the discussion of north–south motion below), and will not be evident when we focus on small times in the quantitative analysis later.

Next consider the case where the earthbound observer (at the same northern latitude) gives the puck an eastward velocity v_E so that the inertial speed of the puck (i.e., as measured by the inertial observer) is $v_E + v_{\text{earth}}$. The puck follows the same inertial great circle path as in Fig. 3(b), with an increased speed along the path. Figure 3(c) shows the path of the puck and the initial and final inertial positions of the earthbound observer (A and A^*), the target (B and B^*), the target expected on a flat earth (F and F^*), and the final inertial position of the puck (C). The puck moves along the inertial great circle from A to C, which in this case is coincident with the earthbound great circle that the puck would follow on a stationary earth since the initial velocity is to the east in both frames. After some time, the expected earthbound path (A to B) has rotated (it appears as A^* to B^*) and the inertial path from A to C transformed into the earth frame appears as the path from A^* to C (which is not a great circle), taking the puck south of the flat-earth target F^* and south of the spherical-earth target B^*. Once again, the inertial observer explains the southward relative motion (F^* to C) as the difference between the great circle path of the puck and the original circle of latitude. The earthbound observer attributes the southward displacement to three effects. The curvilinear correction [see Fig. 3(a)] accounts for the difference between the flat-earth target F^* and the spherical-earth target B^*. The centrifugal effect alone (i.e., when $v_E = 0$) would cause the puck to end up at C_0, which corresponds to position C in Fig. 3(b). The additional displacement in latitude to C is attributed to the Coriolis force that the earthbound observer invokes due to the puck's motion in the rotating frame. The Coriolis force [Eq. (6)] for eastward motion points away from the axis of rotation and has upward and southward components, as shown in Fig. 3(c). The upward component again reduces the normal force and hence the apparent weight of the puck. The southward component is responsible for the displacement of the puck, which is to the right of the velocity in the rotating frame (in the Northern Hemisphere).

Note the similarity of the effects depicted in Figs. 3(a)–(c). In each case the puck finishes south of the original circle of latitude due to its motion along the great circle. Thus the inertial observer treats the three cases similarly. On the other hand, the earthbound observer credits the curvilinear corrections of Figs. 3(a) and (c) to the motion of the puck, the centrifugal deflections of Figs. 3(b) and (c) to the rotation of the earth, and the Coriolis deflection of Fig. 3(c) to the combination of the puck's motion and the earth's rotation. The notion that the three effects described by the earthbound observer are treated as a single effect by the inertial observer will become more evident in the later quantitative analysis.

Finally, consider a puck that is launched to the north with respect to the earth. The naïve earthbound observer expects the puck to follow the meridian toward the north pole, as shown in Fig. 3(d) (A to B; rotated to A^* to B^* after a time t). In the inertial frame, the rotation of the earth imparts an eastward velocity component to the puck, causing it to follow the great circle path from A to C shown in Fig. 3(d). The earthbound observer sees the puck head north and then curve to the east (A^* to C), ending up south and east of the target (B^*). The earthbound observer attributes the southward deflection to the centrifugal force, just as in the previous examples, and the eastward deflection to the Coriolis force, which is solely to the east for northward velocities. Once again, the inertial observer explains the deflections as the difference between the great circle motion of the puck and the motion of the intended target. For small times, the inertial velocity of the puck is constant in magnitude and direction (this is true for all times in the two-dimensional turntable example). Since the target moves in a circle (B to B^*) along its line of latitude, it has inertial displacements both parallel and perpendicular to the original meridian of longitude [line AB in Fig. 3(d)]. The perpendicular displacement of the target is less than that of the puck since the local earth speed is smaller at the target. In other words, the difference in the azimuthal speeds of the puck and the target causes the puck to move east of the target. The displacement of the target parallel to the meridian AB causes the puck to end up south of the target. If the target had moved only perpendicularly to AB and had the same speed as the puck's easterly speed, then the puck would have hit the target; but both conditions are not true, leading to two effects. The eastward deflection can equivalently be viewed as a consequence of the conservation of angular momentum.[9] As the puck moves northward, its distance from the axis of rotation decreases, so

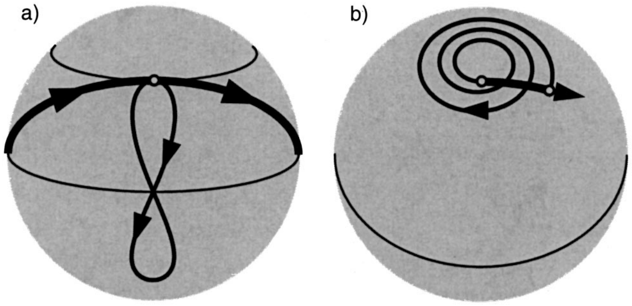

Fig. 4. (a) Motion of a puck launched to the east with an inertial speed equal to the speed of the earth at the equator. The inertial observer sees the puck follow the great circle shown (the thickest line), while the earthbound observer sees the puck follow the figure-eight path. (b) Motion of a puck launched to the west with a speed slightly less than the local speed of the earth. The inertial observer sees the puck move east along the great circle (the thickest line), while the earthbound observer sees the puck follow the westward path spiraling away from the pole.

its angular velocity must increase in order to conserve angular momentum. Since the angular velocity of the target (fixed to the earth) remains constant, the puck leads the target longitudinally.

These simple examples of the correspondences between the inertial and rotating viewpoints facilitate a qualitative understanding of inertial forces and also make it possible to explain some other interesting and more general situations. For example, consider a puck launched to the east from a northern latitude such that the inertial speed is equal to the speed of the earth at the equator (ωR). The puck follows an inertial great circle and takes one day to return to its inertial starting point, at which time the earthbound observer also returns, as shown in Fig. 4(a). The earthbound observer sees the puck follow a figure-eight-shaped path, always turning to the right in the Northern Hemisphere and to the left in the Southern Hemisphere, which is attributed to the combined effect of the centrifugal and Coriolis forces. This boomerang-type path can be obtained with any initial heading, as long as the inertial speed of the puck is ωR. Next consider a puck launched to the west with a speed slightly less than the local earth speed. The inertial observer sees the puck travel east very slowly, such that the earth rotates many times before the puck travels once around the inertial great circle. The earthbound observer sees a path headed west and slightly south of the original line of latitude, resulting in a spiral around the pole as shown in Fig. 4(b). In this case, the earthbound observer sees the puck continually turning to the right (with respect to the naïve earthbound great circle), since the Coriolis force dominates the centrifugal force.

As a final qualitative note on this hockey puck example, we relax the requirement of a sphere and discuss the consequences of the oblateness of the real earth. A rotating deformable earth takes on an oblate spheroidal shape because the centrifugal force pushes material toward the equator [see Fig. 3(b)]. The resultant surface gives rise to a normal force that is no longer purely radial but is tipped slightly toward the pole, with a component that tends to cancel the surface component of the centrifugal force. An exact analysis must also account for the change in gravitational acceleration due to the equatorial bulge, which causes the bulge to be approximately twice as large as the centrifugal effect alone would imply.[11] Nonetheless, the shape of the earth is such that particles at rest with respect to the rotating earth remain at rest. For the earthbound observer, this means that there is no net force on the stationary hockey puck, and hence the centrifugal deflections shown in Fig. 3 are not present on an oblate earth. For the inertial observer, the net force on a hockey puck at rest with respect to the oblate earth is toward the axis of rotation, causing the puck to travel in a circle along its line of latitude; in contrast to the net radial force on a spherical earth and the resultant great circle path. The net centripetal force on the puck is matched to the rotation frequency ω of the earth, but is not matched should the puck rotate at a different frequency. This concept provides a common explanation of the Coriolis force from the inertial viewpoint.[3,9] A hockey puck launched to the east has an inertial velocity faster than that of the local surface of the earth. The net centripetal force that kept it at rest before is now not sufficient to keep the puck traveling in a circle at this speed, so the puck moves to a larger radius where there is a smaller centripetal acceleration (v^2/r). The eastward-launched puck thus moves southward, explaining the rightward Coriolis deflection (in the Northern Hemisphere). A puck launched to the west is traveling too slowly and moves to a smaller radius where there is a larger centripetal acceleration. The westward-launched puck moves northward, again to the right (in the Northern Hemisphere). In the rotating frame, the deflection of the puck from its intended target is due only to the Coriolis force. This is why discussions of the effects of rotation upon the weather, ocean currents, and rivers on our oblate earth invoke only the Coriolis force.

IV. QUANTITATIVE ANALYSIS

Great circles remain the focal point in our quantitative analysis of inertial forces on a rotating sphere. While the concept of a great circle is commonly appreciated, the equations describing a great circle are seldom documented, so we begin with a presentation of the necessary equations. The motion of the hockey puck on the frozen spherical earth is then analyzed using the formalism of great circles.

A. Great circles

To describe a general great circle, we use two coordinate systems as shown in Fig. 5. Both coordinate systems are fixed with respect to the sphere. The unprimed *xyz* coordinate system has its origin at the center of the sphere, with the *z* axis through the North Pole. Points on the sphere are described using the latitude λ, measured as positive (negative) for the Northern (Southern) Hemisphere, and the longitude ϕ, measured counterclockwise from the prime meridian, which lies in the *xz* plane. The equator is the great circle in the *xy* plane, and is described simply by $\lambda=0$. Any other general great circle is considered as the equator in a primed $x'y'z'$ coordinate system, which is obtained by rotating the unprimed system first about the z axis by an angle ϕ_0 and then about the new y' axis by an angle $\lambda_{\max}$. All possible great circles can be accessed using rotation angles $0\leqslant\phi_0\leqslant 2\pi$ and $0\leqslant\lambda_{\max}\leqslant\pi/2$. In the primed coordinate system, the equation of the great circle is simply $\lambda'=0$, where λ' and ϕ' are the latitude and longitude, respectively, as measured in that system. In the unprimed coordinate system, this general great circle reaches a maximum latitude $\lambda_{\max}$ at a longitude ϕ_0. By transforming the equation $\lambda'=0$ back to the unprimed frame or by requiring that the normal vector to the great circle plane be perpendicular to

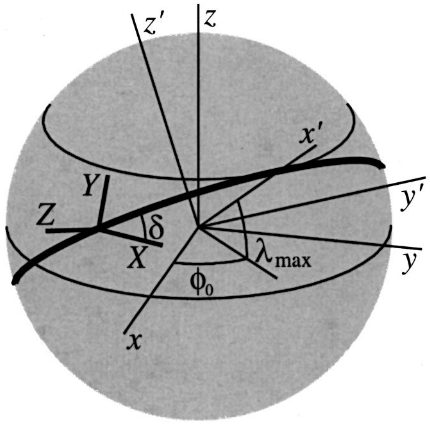

Fig. 5. Coordinate systems for the description of a great circle. The general great circle (the thickest line) lies in the $x'y'$ plane. Also shown are the equator in the xy plane and the line of latitude corresponding to the maximum latitude $\lambda_{\max}$ reached by the great circle. The maximum latitude is reached at a longitude of ϕ_0. The earthbound observer uses the XYZ coordinate system with origin at the initial position of the puck. The initial inertial position of the puck is latitude λ_{start} and longitude ϕ_{start} with initial heading δ, measured counterclockwise from local east.

any general vector in that plane, it is straightforward to show that the equation of the general great circle in the unprimed coordinate system can be written as

$$\tan\lambda = \tan\lambda_{\max}\cos(\phi-\phi_0). \tag{7}$$

For a dynamical description of motion along a great circle, we also need parametric equations. Motion at a constant speed v along the great circle, beginning at $\phi'=\phi'_{\text{start}}$ at time $t=0$, can be described simply by $\phi'(t)=\phi'_{\text{start}}+\Omega t$, $\lambda'(t)=0$, where the angular speed $\Omega=\pm v/R$ and the plus (minus) sign denotes motion that is counterclockwise (clockwise) when viewed looking down from the positive z' axis. In the unprimed system, the great circle path can be written parametrically as

$$\sin\lambda(t) = \sin\lambda_{\max}\cos(\phi'_{\text{start}}+\Omega t), \tag{8}$$

$$\tan(\phi(t)-\phi_0) = \frac{\tan(\phi'_{\text{start}}+\Omega t)}{\cos\lambda_{\max}}. \tag{9}$$

To generate the complete great circle, π must be added to the longitude $\phi(t)$ obtained by solving Eq. (9) for part of the path, since the inverse trigonometric functions have limited principal values. This is not a problem when solving Eq. (7) or (8) for λ.

The above equations describe the great circle and the motion along it in terms of the quantities $\lambda_{\max}$, ϕ_0, and ϕ'_{start}, whereas most problems are posed in terms of the initial position λ_{start}, ϕ_{start} and the initial heading, which we denote by the angle δ with respect to local east. Figure 5 shows an earthbound reference frame with the origin at the initial position of the puck and X coincident with local east, Y with local north, and Z with local up (along the radius vector). As shown in Fig. 5, the initial heading δ is measured as positive in the counterclockwise sense toward the north. The equations relating these two sets of great circle parameters are

$$\cos\lambda_{\max} = |\cos\delta\cos\lambda_{\text{start}}|, \tag{10}$$

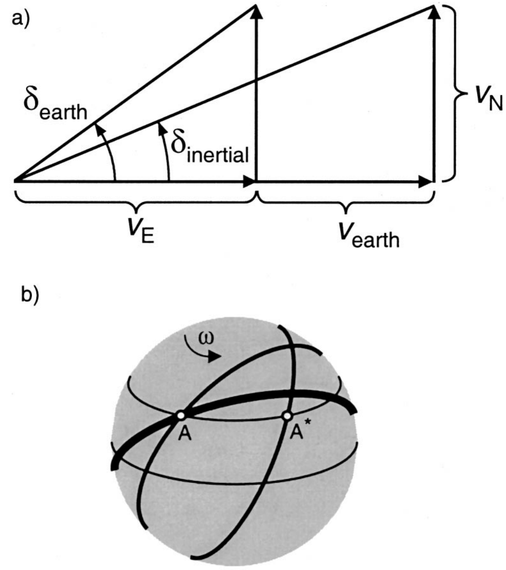

Fig. 6. (a) Initial heading of the puck as measured in inertial and rotating (earth) frames. The puck has velocity components v_E and v_N to the east and north, respectively, as measured in the earth frame. The speed of the earth at the initial puck position is $v_{\text{earth}}=\omega R\cos\lambda_{\text{start}}$. (b) Great circle path of the puck as seen by an inertial observer (the thickest line). The puck is launched from position A, which rotates to position A^* after a time t. The great circle path that the puck would follow on a nonrotating earth is shown both at the time of launch and after the earth has rotated.

$$\tan(\phi_0-\phi_{\text{start}}) = \frac{\tan\delta}{\sin\lambda_{\text{start}}}, \tag{11}$$

$$\tan\phi'_{\text{start}} = -\frac{\tan\delta}{\tan\lambda_{\text{start}}\sqrt{1+\tan^2\delta}}, \tag{12}$$

and have been written with the intent of solving them for the parameters $\lambda_{\max}$, ϕ_0, and ϕ'_{start}. The absolute value in Eq. (10) ensures that $\lambda_{\max}$ is within the range $0-\pi/2$ for all starting conditions. Since the principal values of the inverse trigonometric functions are limited, one must add or subtract π when solving Eqs. (11) and (12) to find $\phi_0-\phi_{\text{start}}$ and ϕ'_{start} for the case $\lambda_{\text{start}}<0$. Equation (12) can be written more simply, but the form shown ensures that ϕ'_{start} takes on the proper values (for $\lambda_{\text{start}}>0$) when the equation is inverted to find ϕ'_{start} in terms of λ_{start} and δ.

B. Terrestrial ice hockey

We now apply these great circle equations to the problem of a hockey puck sliding on an icy, spherical earth that rotates about the z axis (see Fig. 5) with an angular velocity ω with respect to the inertial frame. The great circle equations derived above are used to describe great circles in both the inertial and rotating frames. We work primarily with the latitude–longitude coordinate description of the motion, which makes the transformation from one frame to the other simple—only the longitudinal difference ωt due to the rotation is required.

The motion of the sliding hockey puck is along a great circle that is fixed in the inertial frame. A naïve earthbound observer would expect the puck to follow a great circle that is fixed with respect to the earth. The initial headings of these two great circles are shown in Fig. 6(a). The heading of the naïve earthbound great circle is given by

$$\tan \delta_{\text{earth}} = \frac{\nu_N}{\nu_E}, \tag{13}$$

where ν_E and ν_N are the east and north components of the initial velocity of the puck relative to the earth. The rotation of the earth causes the inertial observer to measure the puck's initial heading as

$$\tan \delta_{\text{inertial}} = \frac{\nu_N}{\nu_E + \nu_{\text{earth}}} = \frac{\nu_N}{\nu_E + \omega R \cos \lambda_{\text{start}}}. \tag{14}$$

Figure 6(b) shows the inertial great circle along which the puck moves, and the naïve earthbound great circle at the time of launch and at a later time after the earth has rotated. In general, the inertial and earthbound great circles have different initial headings, and so reach different maximum latitudes. The two headings [Eqs. (13) and (14)] are coincident only when $\nu_N = 0$, i.e., for motion that is initially east or west, or when $\lambda_{\text{start}} = \pm \pi/2$, i.e., for motion from the poles. The inertial observer notes that the puck has an angular velocity along the inertial great circle of

$$\Omega = \pm \frac{\nu}{R} = \pm \frac{\sqrt{(\nu_E + \omega R \cos \lambda_{\text{start}})^2 + \nu_N^2}}{R}. \tag{15}$$

The great circles in Fig. 3 used to facilitate the qualitative discussion were drawn using the parametric Eqs. (8) and (9) for a great circle. To make quantitative comparisons between the rotating and inertial descriptions of the motion we expand the equations describing the inertial motion to second order in the small quantity Ωt. This yields terms of the same order as the lowest-order calculations of noninertial effects in the rotating frame. It is these lowest-order terms that we compare. These expanded great circle equations giving the inertial position of the puck are

$$\phi(t) = \phi_{\text{start}} + \Omega t \frac{\cos \delta_{\text{inertial}}}{\cos \lambda_{\text{start}}} + \Omega^2 t^2 \frac{\sin \lambda_{\text{start}} \sin \delta_{\text{inertial}} \cos \delta_{\text{inertial}}}{\cos^2 \lambda_{\text{start}}}, \tag{16}$$

$$\lambda(t) = \lambda_{\text{start}} + \Omega t \sin \delta_{\text{inertial}} - \tfrac{1}{2} \Omega^2 t^2 \tan \lambda_{\text{start}} \cos^2 \delta_{\text{inertial}}. \tag{17}$$

Expressing these in terms of parameters measured by the earthbound observer results in

$$\phi(t) = \phi_{\text{start}} + \omega t + \frac{\nu_E t}{R \cos \lambda_{\text{start}}} + \frac{\nu_E \nu_N t^2 \sin \lambda_{\text{start}}}{R^2 \cos^2 \lambda_{\text{start}}} + \frac{\omega \nu_N}{R} t^2 \tan \lambda_{\text{start}}, \tag{18}$$

$$\lambda(t) = \lambda_{\text{start}} + \frac{\nu_N}{R} t - \frac{1}{2R^2} (\nu_E + \omega R \cos \lambda_{\text{start}})^2 t^2 \tan \lambda_{\text{start}}. \tag{19}$$

In order to illustrate the physical significance of each term in these expansions, we consider a succession of specialized cases with simple initial conditions, as was done in Fig. 3, and then finish with the general case. We will see that the first-order terms represent the expected motion of the puck on a stationary flat earth plus the angular displacement of the rotating earth, while the second-order terms represent the effects of inertial forces and the curvature of the earth.

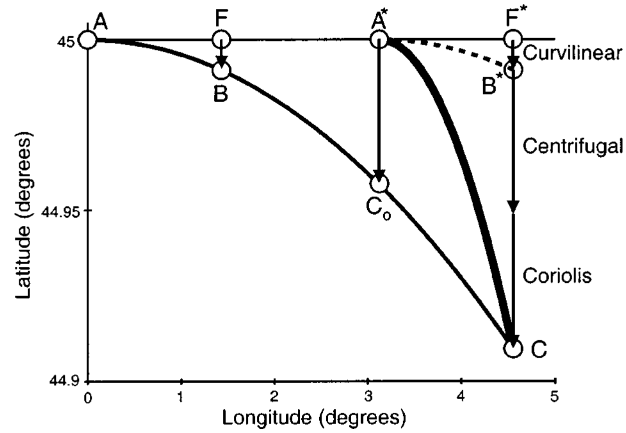

Fig. 7. Motion of a puck that is launched eastward with respect to the rotating earth for short times. The description of the motion from launch site to target is the same as in Fig. 3(c). The arrows indicate the curvilinear displacement of the target from the original line of latitude and the calculated centrifugal and Coriolis displacements of the puck from the target.

First consider the case where the puck is given a velocity ν_E in the eastward direction by the earthbound observer. The motion of the puck was depicted in the spherical plot of Fig. 3(c). An expanded plot of the path of the puck for short times is shown in Fig. 7 with an equirectangular projection (simple latitude versus longitude) and notation equivalent to Fig. 3(c). Since there is only an eastward initial velocity in this case, the expanded equations for the inertial position of the puck are

$$\phi(t) = \phi_{\text{start}} + \omega t + \frac{\nu_E t}{R \cos \lambda_{\text{start}}}, \tag{20}$$

$$\lambda(t) = \lambda_{\text{start}} - \frac{1}{2R^2} (\nu_E + \omega R \cos \lambda_{\text{start}})^2 t^2 \tan \lambda_{\text{start}}, \tag{21}$$

$$\lambda(t) = \lambda_{\text{start}} - \frac{1}{2R^2} \nu_E^2 t^2 \tan \lambda_{\text{start}} - \frac{1}{2} \omega^2 t^2 \sin \lambda_{\text{start}} \cos \lambda_{\text{start}} - \frac{\omega \nu_E}{R} t^2 \sin \lambda_{\text{start}}. \tag{22}$$

The inertial observer credits the longitudinal displacement to the initial velocity of the puck, with contributions from the rotating earth's velocity (ωt, corresponding to A to A^*) and the launch speed with respect to the earth (term $\propto \nu_E t$, corresponding to A^* to F^*). The inertial observer credits the latitudinal displacement to the inertial motion along the great circle, which takes the puck south of the original line of latitude. Since the inertial motion is composed of the motion of the puck with respect to the earth and the motion of the earth, the squared term of Eq. (21) gives rise to three terms, as shown in Eq. (22). Thus, what the inertial observer credits to a single effect, the rotating observer credits to three effects, which, in order of appearance in Eq. (22), correspond to the three effects demonstrated in Figs. 3(a), (b), and (c), respectively. The first term $[\propto (\nu_E/R)^2]$ corresponds to the curvilinear effect described above whereby the target seen to the east is at a lower latitude. This is depicted in Fig. 7 by the arrow between the flat earth target F and the spherical earth target B. Note that this curvilinear correction becomes zero on a flat earth $(R \to \infty)$, as one would expect.

The second term $(\propto \omega^2)$ corresponds to the centrifugal force, which was depicted in Fig. 3(b). The earthbound observer notes that the surface component of the centrifugal force points to the south and results in an acceleration of

$$a_{Y,\mathrm{cent}} = -\omega^2 R \sin\lambda \cos\lambda \tag{23}$$

at latitude λ. After a time t, this acceleration results in a displacement

$$\Delta Y_{\mathrm{cent}} = -\tfrac{1}{2}\omega^2 t^2 R \sin\lambda_{\mathrm{start}} \cos\lambda_{\mathrm{start}}, \tag{24}$$

where the displacement is assumed to be small enough that the latitude can be taken as a constant equal to the starting latitude. This southward displacement calculated by the earthbound observer is equivalent to the displacement calculated by the inertial observer $(\Delta Y = R\Delta\lambda)$ using the second correction term in Eq. (22) and is represented in Fig. 7 by the arrow from the final position of the earthbound observer A^* to the final position the puck would have if it were released from rest $(v_E = 0)$, which is labeled C_0.

The third displacement term in Eq. (22) $(\propto \omega v_E)$ corresponds to the Coriolis force, which was depicted in Fig. 3(c). The surface component of the Coriolis force results in an acceleration of

$$a_{Y,\mathrm{Cor}} = -2\omega v_E \sin\lambda \tag{25}$$

at latitude λ. After a time t, this acceleration results in a displacement

$$\Delta Y_{\mathrm{Cor}} = -\omega v_E t^2 \sin\lambda_{\mathrm{start}}, \tag{26}$$

where again λ is assumed constant for small t. This southward displacement again matches the inertial term and is shown as the arrow ending at C in Fig. 7. The arrows depicting the other two effects are duplicated at the final longitude to show that all three latitude correction terms contribute to explain the displacement of the puck (C) from the flat earth target (F^*). Thus the lowest-order displacements calculated by a rotating observer using inertial forces agree with the calculation of the inertial observer for short times, and so the two observers agree on the motion of the puck but not on the physics behind the motion. Note that the higher-order westward displacement discussed earlier in regard to Fig. 3(b) is not evident for the short times shown in Fig. 7.

Next consider a puck launched directly north with a speed v_N with respect to the earth. In this case, the motion of the earth's surface leads the inertial observer to measure a heading given by

$$\tan\delta_{\mathrm{inertial}} = \frac{v_N}{v_{\mathrm{earth}}} = \frac{v_N}{\omega R \cos\lambda_{\mathrm{start}}}, \tag{27}$$

in contrast to the earth heading of $\delta_{\mathrm{earth}} = 90°$. The great circle motion of the puck on the earth was shown in Fig. 3(d). An expanded view of the motion for small times is shown in Fig. 8, with notation equivalent to Fig. 3(d). Since there is only a northward initial velocity (in the earth frame) in this case, the expanded equations for the inertial position of the puck are

$$\phi(t) = \phi_{\mathrm{start}} + \omega t + \frac{\omega v_N}{R} t^2 \tan\lambda_{\mathrm{start}}, \tag{28}$$

$$\lambda(t) = \lambda_{\mathrm{start}} + \frac{v_N}{R} t - \frac{1}{2}\omega^2 t^2 \sin\lambda_{\mathrm{start}} \cos\lambda_{\mathrm{start}}. \tag{29}$$

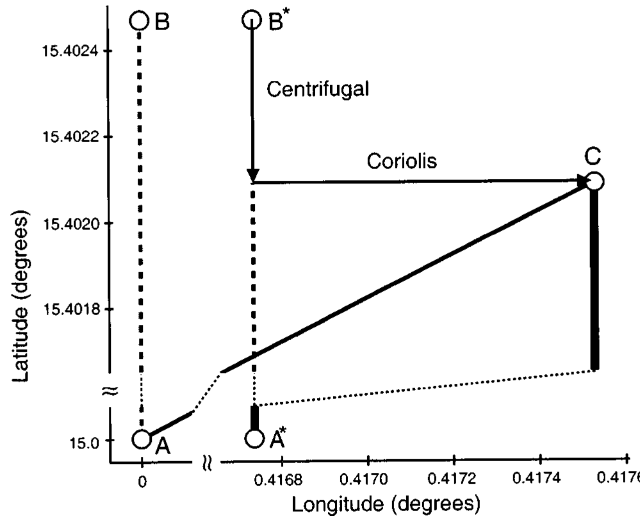

Fig. 8. Motion of a puck that is launched to the north with respect to the rotating earth. The description of the motion from launch site to target is the same as in Fig. 3(d). Both axes are broken in order to show launch and target locations at both times and the small corrections. As shown in the spherical plot of Fig. 3(d), the paths from A to C and A^* to C are curved, but in this plot all the curvature is hidden in the broken region of the plot (thin dotted lines). The arrows indicate the calculated centrifugal and Coriolis displacements of the puck from the target.

The expected northward range of the puck [term in Eq. (29) $\propto v_N$] is reduced by the same southward centrifugal deflection calculated above [Eq. (24)]. The longitude exhibits the rotation of the earth (ωt) and a correction term that corresponds to the Coriolis force. For a northward velocity, the Coriolis force is solely to the east [to the right again, as shown in Fig. 3(d)] and causes an acceleration of

$$a_{X,\mathrm{Cor}} = 2\omega v_N \sin\lambda, \tag{30}$$

which results in an eastward displacement after a time t (assumed small) of

$$\Delta X_{\mathrm{Cor}} = \omega v_N t^2 \sin\lambda_{\mathrm{start}}. \tag{31}$$

Both displacements calculated by the rotating observer are shown as arrows in Fig. 8, and again agree with the inertial description of the motion for small times. Note that there is no curvilinear correction in this case since the expected path along the meridian is a great circle (coincident with the line of sight to the north).

The general case, where a puck has both eastward and northward velocities, is depicted in Fig. 9. The puck follows the inertial great circle from A to C, while the earthbound observer sees the puck follow the path from A^* to C. The expected path on a stationary earth is from A^* to B^*, and the path expected on a flat earth is from A^* to F^* (constant heading). For short times, as shown in the main plot of Fig. 9, the puck (C) ends up south and east of the target (B^*). The full inertial position Eqs. (18) and (19) required here include all the correction terms discussed in the east and north cases and one new term in the longitude $(\propto v_E v_N)$, which is another curvilinear correction and is shown as the arrow ending at B^* in Fig. 9. The deflection of the puck from the target (for small times) is a combination of the previously discussed terms and includes the southward centrifugal de-

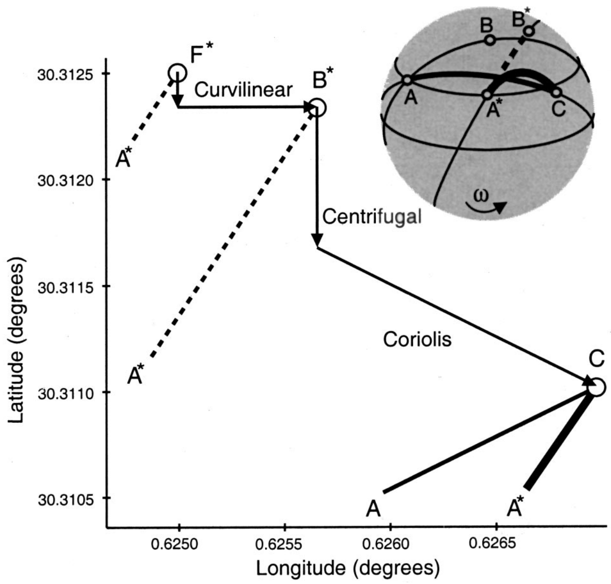

Fig. 9. Motion of a puck that is launched to the northeast with respect to the rotating earth. Short times are shown in the main plot and longer times are shown in the spherical plot. The heading for this case is $\delta_{\mathrm{earth}}=60°$, with a speed (in the earth frame) of v_{earth}, which gives an inertial heading of $\delta_{\mathrm{inertial}}=30°$. The description of the motion from launch site to target is the same as in Fig. 3(c). Only the final positions of the puck (C) and the intended target (B^* on a spherical earth and F^* on a flat earth) are shown in the main plot; the labels A and A^* denote the origins of the respective paths. The two curvilinear arrows indicate the difference between the constant heading path from A^* to F^* and the great circle path (or line of sight) from A^* to B^*. The other arrows indicate the calculated centrifugal and Coriolis displacements of the puck from the target.

flection [Eq. (24)] and both southward [Eq. (26)] and eastward [Eq. (31)] Coriolis deflections (which are combined into one arrow in Fig. 9).

V. SUMMARY

In summary, we have presented an analysis of motion on a rotating sphere from both the inertial and rotating viewpoints. This analysis is grounded in the simple geometry of great circles and is applied to an idealized terrestrial ice hockey example to provide particularly straightforward demonstrations of the physical origin of inertial forces. The Coriolis and centrifugal forces are isolated and illustrated by analyzing specific simple initial conditions. The exact inertial equations are expanded to show that the lowest-order corrections agree with the rotating observer's analysis that invokes inertial forces. The analysis also brings forth effects due to the curvilinear nature of the motion that are not commonly discussed. The recent advent of powerful symbolic manipulation software makes possible the effective dynamical presentation of these often hard-to-visualize dynamical effects.[13]

This approach should be particularly well suited for use in an intermediate undergraduate mechanics course, which is usually students' first introduction to inertial forces. The simplicity of the geometry and of the frame transformations involved should allow students to focus on the physics of the motion rather than the mathematical complexity of the inertial forces and the resultant equations of motion. The rotating frame viewpoint may be more appropriate for real world examples like atmospheric winds and ocean currents, but the common pedagogical examples used to introduce students to inertial forces are simple enough to be analyzed in the manner presented here.

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