Mapping Mathematical Tools to Physical Models: An Evaluation of the ACER Framework.

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Table of Contents

Abstract ---------------

The ACER is an analytic framework designed to investigate/analyze student reasoning compared to professional reasoning when mapping mathematical tools to physical models. The ACER framework developed at the University of Colorado, Boulder, has been designed to be implemented by non-PER (physics education researchers) while still leveraging concepts in PER to assist in analytics. This paper describes the implementation of the ACER framework at Oregon State University for a canonical junior level Coulomb's Law problem. This paper describes the process, difficulties, and benefits of implementation of the ACER framework as carried out by undergraduate physics major at Oregon State University. The ACER framework gave a non-PER physics undergraduate a richer understanding of student work. This richer understanding allowed the undergraduate researcher to get a better understanding of possible student thinking which enabled them a wider perspective when assisting students in future encounters.

Chapter 1 - Introduction

1.1 Motivation and Objective

One of the goals for all instructors is to assist student understanding. If an instructor can identify student difficulties, then the instructor can help the student overcome these difficulties. While this is trivial to state, understanding clearly what goes on in a student's mind is a daunting task. Most of an instructor's insight into student reasoning comes in the form of written work. The interpretation of this work allows a glimpse into student reasoning and possibly their difficulties.

Instructors typically use some form of rubric to assess student's written work. Rubrics done well can highlight student difficulties and clearly signal these difficulties to an instructor. Unfortunately, even an excellent rubric only gives a sense of general student difficulties.

One of the first investigative steps in assessing student specific difficulties is posing a question to students that may highlight a specific difficulty. As an instructor designs their question, they are also thinking about or even designing the rubric in parallel. Depending on the type of student difficulty the instructor is investigating, this design framework for the investigation can span the spectrum from very simple to quite complex.

In the realm of physics, one typical student difficulty is taking the mathematical language (*i.e. Coulomb's Law*) and correctly applying it to a given physical situation *(i.e. potential due to a ring of charge*). Students understanding of how the mathematical language is translated to a physical situation is one of the keys to their success. As a student goes higher in their undergraduate physics education, this mathematical language becomes more and more replaced with the mathematical tools (i.e. *finding the curl of an electrostatic field is zero*) behind the mathematical language. This is taking the mathematical language and putting it into a physics context.

In the upper division of college physics, a common student difficulty is not being able to correctly map mathematical tools onto a corresponding physical model. This type of student difficulty is not only common in physics, but in chemistry, biology, and engineering. Most college instructors are not formally trained in strategies that would allow them to easily build such a rubric that would help instructors investigate this difficulty. The physics education research (PER) faculty University of Colorado Boulder saw a need for a framework that would allow instructors to produce rubrics that would highlight student difficulties in mapping mathematical tools to physical models. With that in mind, they produced the Activation Construction Execution Reflection (ACER) framework[1]. The goal of the ACER framework is to assist non-PER trained instructors in building rubrics that highlight student difficulties in mapping mathematical tools to physical models.

This paper will use the ACER to qualitatively assess upper-division physics student's ability to select, map, and use appropriate mathematical tools for a problem for a junior level electromagnetism course. This paper will also examine the strengths and shortcomings of the ACER framework.

1.2 The ACER

1.2.1 - What is the ACER Framework?

The primary goal of the ACER is to highlight student difficulties in mapping mathematical tools to physical models[1]. This is done by giving instructors a framework to build an ACER rubric. The ACER rubric is broken down into four parts:

A. Activation of mathematical tools
C. Construction of mathematical models
E. Execution of the mathematics
R. Reflection on the result

These categories represent a generalized structure for solving upper-division physics problems. While these categories are not an attempt to approximate students' solutions, they do help setup the analysis of student work[1]. Ideally problem solving starts with activation of a mathematical tool, then construction of a model that is based on the activated tool and the parameters of the problem, then the execution of the mathematics of the constructed model, and finally reflection on the result to see if the outcome of the execution passes any checks that seem relevant to explore. It should be stressed it is not necessary for these steps to happen in this specific order. It is just as likely after construction of the model that one can reflect on the resulting model and then activate a new resource that leads to an updated model. These categories are meant to help isolate the different steps of the problem solving process.

Activation of the tool generally relies on the problem statement. The problem statement can cue the activation of resources for specific tools (*e.g. find the electric field due to a point charge*). Unfortunately, the resources that students activate are not necessarily the resources that are intended. The resources that an individual leverages is dependent on that individual and their mindset during the task.[1]

Construction of the model takes what has been activated and starts putting it together. Models can range from canonical equations (*e.g.* $V = kq/r$) to drawing diagrams. Then these constructed models have the physical quantities of the situation mapped onto them (*e.g. point charge 'Q', find V at 'D' meters away;* $V = kQ/D$). This category gets at how the individual is applying the activated tool to the physical situation.

Execution of the mathematics takes the constructed model (*e.g. unevaluated integrals*) and takes the model and turns it into simplified mathematical expressions (*e.g. evaluated integrals*)[1]. This category covers how the mathematical tool is understood on a mathematical level.

Reflection on the result takes what has been executed and attempts to test the result. These tests can be as simple as doing unit analysis on a final expression. This is the step that professionals generally do without prompting. Hopefully, this professional habit starts building during the sophomore and junior years for most physics students.

While these categories for the ACER provide a generalized framework, the steps inside each category need to be operationalized in order to give a rubric specific to a given problem. The first step in this operationalization requires the instructor to design a question that targets a specific mathematical tool. The example in this paper uses the integral form of Coulomb's Law as the tool to be investigated. Once the instructor designs the question around the specific mathematical tool, the instructor then consults a "content expert". A content expert is, well, an expert in the content of the question being posed. The content expert will then solve the question completely and in a discussion with the instructor will lay out the important aspects of the problem solving process. Once the important aspects have been agreed upon, these aspects are mapped to framework and the ACER rubric is complete. Examples of a completed ACER rubric are given in the following section.

This rubric framework allows for a deeper evaluation for student reasoning. The ACER focuses on what may cause students to activate use of, for example, Gauss' Law versus the integral form of Coulomb's Law. How do students then construct the mathematical model to fit the physical situation activated by the question (*e.g. Did they choose limits inconsistent with the physical model?*). The student's execution of the mathematics from the constructed model, did they correctly integrate? Finally, reflection on the final result, did students check limiting cases or units of their final answer?

This detailed rubric allows non-PER instructors to "systematically investigate how students integrate mathematics with their conceptual knowledge to solve complex physics problems." The results of this investigation will reveal possible gaps in student understanding.

1.3 Research Environment

1.3.1 The Researcher

Michael Goldtrap is a physics undergraduate at Oregon State University (OSU). He has completed the paradigm in physics sequence at OSU and is currently a senior. In the past he has worked in multiple capacities (camera operator, data entry, etc.) for the Paradigms in Physics project. Michael is currently working under Professor Corinne Manogue.

1.3.2 Paradigms in Physics at Oregon State University

The upper division physics courses at OSU are heavily reformed. This reform is called the Paradigms in Physics. The Paradigms in Physics is a classroom reform primarily focuses on the junior year physics courses for physics majors. During each term, students take 3 unique paradigm courses in succession. Each course is subject specific and lasts three weeks. Each week students have 7 instructor contact hours. During these contact hours, instructors primarily use small group activities, lab, and lecture in varying degrees. The primary goal is to promote active student learning/engagement. At the end of each three week paradigm students are given a final exam.

Chapter 2 – Methods

2.1 Building the ACER Rubric

This section will describe the methods used to build an ACER rubric using the ACER framework and how the analysis was carried out.

2.1.1 The Mathematical Tool To Investigate

Are students using the integral form Coulomb's Law for charge distributions that do not favor the use of Gauss's Law?

2.1.2 Forming the Question

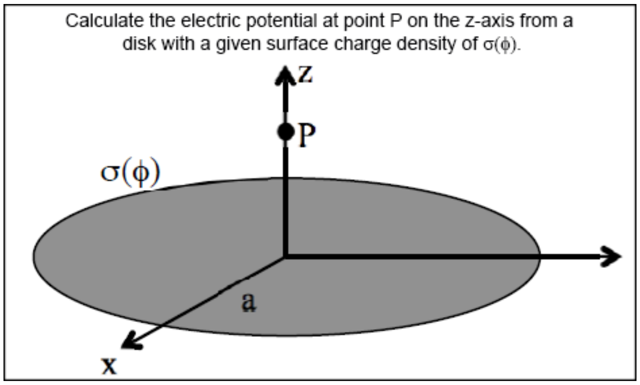

figure 2.1 - "An example of the canonical exam problem on continuous charge distributions."[1]. This was the primary example question used to help shape/form the question given to for OSU physics students.

In their initial ACER paper, Wilcox et al. used, as an explicit example, the electrostatic potential above the center of a charged disk, figure 2.1. The published problem statement asked students to "Calculate the electric potential at point P on the z-axis from a disk with a given surface charge density of $\sigma(\varphi)$". Students were also given a diagram of the situation.

4. Suppose there is a thin disk with a radius R and a surface charge density that changes with angle $\sigma = \sigma(\varphi)$. (a) Find the total charge of the disk. (b) Find the electric potential at a distance D above the exact center of the disk.

figure 2.2 - Problem statement given to OSU students

In this paper, the researcher, in consultation with the PER faculty of OSU modified the previously published question. This modified problem statement can be seen in figure 2.2. Both the Colorado and the OSU problem statements specify a charge density $\sigma(\varphi)$ which depends on angle but is otherwise unspecified. The finite size of the disk and the φ dependent charge density both make the use of Gauss's law inappropriate. There were changes in: wordage and notation, parts of the question, and the diagram. The wordage and notation changes were due to keeping the question more in line with the class norms. The breaking down into multiple parts allowed isolation of student reasoning when building dA twice. Finally, the removal of the diagram would allow data gathering on spontaneous student diagram drawing.

2.1.3 Interviewing OSU Content Expert

As specified by the ACER protocol, we interviews a content expert to help us construct the ACER rubric for the OSU version of the disk question. The content expert used was the current faculty instructor for Vector Fields. The researcher interviewed the instructor by posing the problem statement as previously seen in figure 2.2. The content expert solved the problem while discussing the reasoning behind each step she took. Once the content expert completed the problem, the researcher asked follow up questions to confirm and clarify the content expert's reasoning. The researcher and content expert then discussed the published CU-Boulder's ACER rubric for a similar problem. After that discussion the ACER rubric for OSU was finalized, which can be seen in figure 2.3.

2.1.4 ACER Rubric

Oregon State University ACER Rubric	CU-Boulder ACER Rubric
A1: The problem asks for the electric potential. **A2**: The problem gives a charge distribution. **A3**: The charge distribution does not have appropriate symmetry to use Gauss's Law effectively. **A4**: The charge density is not constant, it is necessary to find the charge on an infinitesimal piece of area and integrate.	**A1**: The problem asks for the electric potential. **A2**: The problem gives a charge distribution. **A3**: The charge distribution does not have appropriate symmetry to use Gauss's Law effectively. **A4**: Direct calculation of the potential is more efficient than starting with the electric field.
C1: Use the geometry of the charge distribution to select a coordinate system. **C2**: Express the differential charge area element dA in the selected coordinates. **C3**: Select integration limits consistent with the differential charge area element and the extent of the physical system. **C4**: Express the difference vector, r-r', in the selected coordinates.	**C1**: Use the geometry of the charge distribution to select a coordinate system. **C2**: Express the differential charge element dq in the selected coordinates. **C3**: Select integration limits consistent with the differential charge element and the extent of the physical system. **C4**: Express the difference vector, r-r', in the selected coordinates.
E1: Maintain an awareness of which variables are being integrated over. (*e.g. r' vs. r*). **E2**: Execute (multivariable) integrals in the selected coordinate system **E3**: Maintain an awareness that $\sigma(\varphi)$ is an unknown varying surface charge density.	**E1**: Maintain an awareness of which variables are being integrated over. (*e.g. r' vs. r*). **E2**: Execute (multivariable) integrals in the selected coordinate system **E3**: Manipulate the resulting algebraic expression into a form that can be readily interpreted.
R1: Verify that the units are correct. **R2**: Check the limiting behavior to ensure it is consistent with the total charge and geometry of the charge distribution.	**R1**: Verify that the units are correct. **R2**: Check the limiting behavior to ensure it is consistent with the total charge and geometry of the charge distribution.

figure 2.3 - Left column: Oregon State University ACER rubric. Right column: CU-Boulder ACER rubric. Both rubrics are made for the disk problem and are in their code forms. The highlighted codes are difference between OSU and CU-Boulder.

The ACER rubric is broken down into activation, construction, evaluation and reflection steps as shown in figure 2.3. These steps reflect the important aspects of each category of the ACER. Each step is then given a code and description. For example, the code C2 for OSU is described as: *express the differential charge area element dA in the selected coordinates*. This code describes an important aspect of problem solving inside the construction category. It should be noted these steps are descriptive and not explicitly written as statements that have a clear yes-no answer for a given student. This distinction will be discussed further in the analysis and results sections.

While the majority of the codes remain the same between OSU and CU-Boulder, there are three code descriptions that changed: A4, C2, and E3. Code A4 for OSU, *the charge density is not constant, it is necessary to find the charge on an infinitesimal piece of area and integrate*, highlights OSU's focus on finding and building dA rather than dq. Code A4 for CU-Boulder, *direct calculation of the potential is more efficient than starting with the electric field*, possibly gives too much credit to the student rationalisation. If a student is asked to calculated electric potential, they will mostly likely activate resources for electric potential. That electric potential activation does not necessarily mean that the student knows that path is more efficient. The difference in code descriptions for C2 are in OSU's focus on dA rather than a focus on dq, this difference is discussed further in a later section. Finally, the code description of E3 for OSU, *maintain an awareness that σ(φ)is an unknown varying surface charge density*, was changed to focus on how students maintained an awareness of $\sigma(\varphi)$ since it is an undefined function with a φ dependence. This replaced the E3 description from CU-Boulder, *manipulate the resulting algebraic expression into a form that can be readily interpreted*. The question, figure 2.2, after students successfully complete their integrals cannot be simplified further. This made CU-Boulder's code description for E3 irrelevant for the OSU problem.

2.2 Data Collection

2.2.1 How the Data was Collected

The question, figure 2.2, was posed to 39 students on the final exam of Vector Fields during Fall 2015. Of the 39 students in the Vector Fields class, only 36 agreed to be a part of this research. The final exam had 6 questions and students were allotted 120 minutes to complete the test. See Appendix A for the materials given to students. The completed the exams for students who had agreed to participate in research on the Paradigms were scanned, anonymized, and given to the researcher.

2.2.2 Information on subjects

The subjects are students 6 weeks into their junior year of physics at Oregon State University. These students have completed two paradigm courses: Symmetries and Vector Fields. These two paradigm courses are taught back to back in the fall term of student's junior year. This totals 6 weeks of class, giving students a minimum of 42 instructor contact hours. These courses cover mathematical tools/concepts such as: Superposition principle, integral forms of Maxwell's equations, understanding and application of |r-r'|, translating between coordinate systems, power series approximation, Gauss's and Ampere's' Law, divergence/curl/gradient, boundary conditions, and using/interpreting multiple representations of all concepts learned.

During the junior year of physics students are taught the integral forms of the canonical electromagnetism equations they were introduced to in their sophomore level physics classes. In the junior year, looking at small bits of charge is always thought of as a small bit of length/area/volume ($dl, dA, d\tau.$) multiplied by a linear/surface/volume charge density. It is rare that dq is discussed in class explicitly as "dq". Typically dq is discussed as a $\lambda(\vec{r}')dl', \sigma(\vec{r}')dA', or\ \rho(\vec{r}')d\tau'.$

2.3 Data Analysis

2.3.1 Application of the OSU ACER Rubric

This section will describe how the OSU ACER rubric was applied to the OSU student data set. OSU student data will serve as examples of how the coding applied. The researcher started by looking at individual category *(e.g. Activation)* and then analyzed the data set for each code. After the researcher completely applied all the codes in a single category *(e.g. A1-A4 for the category of activation),* he held discussions with his advisors to discuss any confusion when applying codes during the analysis of student work. The researcher completed this process for each of the four categories for the OSU ACER.

2.3.2: Activation Analysis Methods

A1: The problem asks for the electric potential
A2: The problem gives a charge distribution.
A3: The charge distribution does not have appropriate symmetry to use Gauss's Law effectively.
A4: The charge density is not constant, it is necessary to find the charge on an infinitesimal piece of area and integrate.

figure 2.4 - Activation codes

The codes for Activation can be seen in figure 2.4. A1-A2 are identified inside the problem statement (figure 2.2). A1, *the problem asks for the electric potential,*" is satisfied by part b of the problem statement, "(b) Find the electric potential at a distance D above the exact center of the disk." Code A2, *the problem gives a charge distribution*, are satisfied by the opening problem statement, "Suppose there is a thin disk with a radius R and a surface charge density of that changes with angle $\sigma = \sigma(\varphi)$" The last two codes: A3, *the charge distribution does not have appropriate symmetry to use Gauss's Law effectively,* and A4, *the charge density is not constant, it is necessary to find the charge on an infinitesimal piece of area and integrate*, are both identified in student data. Students successfully activated A3 if they do not attempt to use Gauss's Law; and A4 they activate building dA in relation to the surface charge density $\sigma(\varphi)$.

2.3.3: Construction Analysis Methods

C1: Use the geometry of the charge distribution to select a coordinate system.
C2: Express the differential charge area element dA in the selected coordinates.
C3: Select integration limits consistent with the differential charge area element and the extent of the physical system.
C4: Express the difference vector, r-r', in the selected coordinates.

figure 2.5 - Construction codes

The codes for Construction can been seen in figure 2.5. All the construction steps C1-C4 are only identified inside student data. Identifying C1, *use the geometry of the charge distribution to select a coordinate system*, was done by looking for any signals of a selection of cylindrical coordinate system and by looking for equations for dA that contained $rdrd\varphi$, or $\hat{\varphi}$, and looking for words or pictures that show this selection, as shown in figure 2.6.

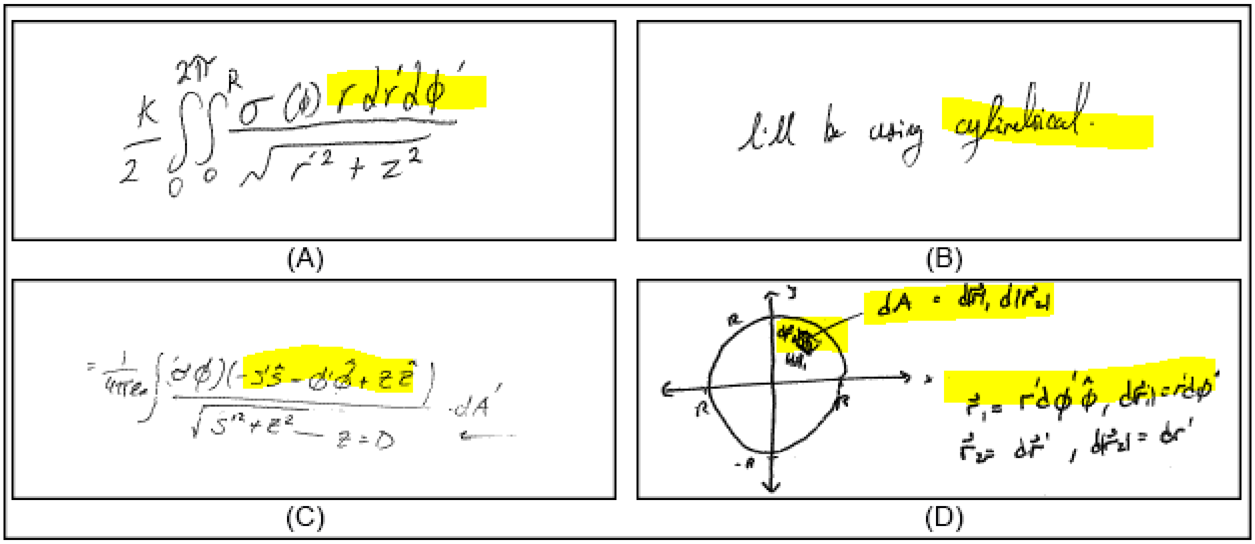

figure 2.6 - Examples of students successfully showing construction C1, *use the geometry of the charge distribution to select a coordinate system*. The author has highlighted the relevant text. A) Shows a transformation into cylindrical coordinates from dA'. B) Shows a student stating 'I'll be using cylindrical'. C) Shows an incorrect attempt at generating a generic vector in cylindrical coordinates. D) Shows a diagram that is constructing dA from $d\vec{r}_1$ & $d\vec{r}_2$ in cylindrical coordinates.

C2, *express the differential charge area element dA in the selected coordinates*, was identified by looking at dA and how it relates to $\sigma(\varphi)dA$, as shown in figure 2.6a. The unpacking of $\sigma(\varphi)dA$ must have $\sigma(\varphi')r'dr'd\varphi'$ or any combination of symbols and primes to still have an appropriate units of charge in cylindrical coordinates. Identifying C3, *select integration limits consistent with the differential charge element and the extent of the physical system*, this focused on the limits of integration matching the proportions of the disk given in the problem statement. Integral limits matched the physical properties of the disk. C4, *express the difference vector, r-r', in the selected coordinates*, was identified by looking how the student found their initial r-r'. Students showing the long form in cylindrical with and without simplification were both considered correctly achieving C4.

2.3.4: Execution Analysis Methods

> **E1**: Maintain an awareness of which variables are being integrated over. (*e.g. r' vs. r*).
> **E2**: Execute (multivariable) integrals in the selected coordinate system
> **E3**: Maintain an awareness that $\sigma(\varphi)$is an unknown varying surface charge density.

figure 2.7 - Execution codes

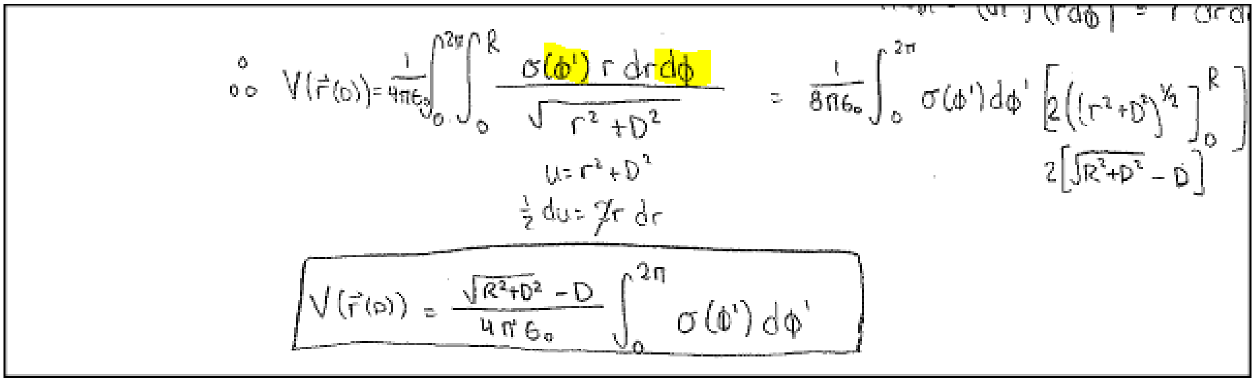

figure 2.8 - Student unsuccessfully executing E1, by not maintaining an awareness of which variables are being integrated over. This can be seen in the first step the student has a $\sigma(\varphi')$ *and* $d\varphi$. Student successfully executes E2 and E3.

Execution codes E1-E3 are the broad expert level necessities to successfully compute what was "constructed" in the previous step. This is the, "turning the crank on the mathematics" steps. Coding for E1, *maintain an awareness of which variables are being integrated over* (*e.g. r' vs. r*), was only successfully if students were consistent in their prime notation. If the student was inconsistent with their notation at any point during their solution it was considered unsuccessfully executing E1, see figure 2.8. Coding for E2, *execute (multivariable) integrals in the selected coordinate system*, was successful if the student integrated correctly, again see figure 2.8.

Coding for E3, *maintain an awareness that σ(φ)is an unknown varying surface charge density,* the function $\sigma(\varphi)$ has an unknown dependence on angle φ which makes it impossible to integrate with respect to $d\varphi$.

$$\text{(eq. 2.1)}\, V(D) = \frac{1}{4\pi\epsilon_0}\left(\sqrt{R^2 + D^2} - D\right) \int_0^{2\pi} \sigma(\varphi')\, d\varphi'$$

For a student to get this final equation (2.1), students must have completed the problem correctly. This is the final statement after completing the *dr'* integration. Since there is no further simplification to be done at this point.

2.3.5: Reflection Analysis Methods

R1: Verify that the units are correct.
R2: Check the limiting behavior to ensure it is consistent with the total charge and geometry of the charge distribution.

figure 2.9 - Reflection codes

Coding R1, any student indication of unit analysis of their final answer would be considered successful reflection. Coding for R2, student indication of looking at limiting behavior would be considered successful reflection.

Chapter 3 - Results

3.1 Results

This section describes the number of students whose results were given particular by the analysis of the OSU ACER rubric codes and describes common and/or interesting student difficulties. For Activation we provide a breakdown of students choosing to electric field rather than electric potential. For construction, we describe student difficulties producing $|\vec{r} - \vec{r}'|$,and student usage of diagrams. For execution there will be examples and discussion of how students incorrectly evaluated $\sigma(\varphi)$. Finally for Reflection, we found that none of our students reflected in a manner consistent with professionals. This shows a strong disconnect between the explicit professional level reflections versus the type of reflections students are signaling.

As previously discussed, 36 student tests were analyzed using the ACER rubric. Of these 36 students, two students showed activation of resources that did not lead them to construct anything that would fall under the application of this ACER rubric beyond the Activation stage. For example, one student activated Ampere's Law. This ACER rubric does not assist in analysis of this work beyond the fact the student activated Ampere's Law. While these cases show interesting student resource activation, the complete analysis with this ACER rubric did not yield significant insight into these student activation missteps.

3.1.1 Activation

Over three-quarters (75%, N=36) of OSU students successfully activated electric potential as their primary approach. Of the remaining students, seven (19%) activated electric field and only one of those students attempted Gauss's Law. There is one student that started with electric field, then rejected that approach and switched to electric potential mid problem. Finally, two students left work that was unable to be analyzed using this framework.

These results in activation show OSU students are far more likely to activate the integral form of Coulomb's Law for this question rather than Gauss's Law. The analysis of the student data doesn't explicitly show that students considered Gauss's Law and rejected it for this problem. The data does support the notion if students happened to consider Gauss's Law they would mostly likely reject using it for this problem. For the students that were activating electric field over electric potential, most chose to attempt to solve the problem by starting with the integral form of electric field due to a charge source

$$\text{(eq. 3.1)} \qquad \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\prime)d\vec{\tau}\prime}{|\vec{r}-\vec{r}\prime|^2}\widehat{(\vec{r}-\vec{r}')}$$

While this not an incorrect approach to solve the problem statement they were given, it is a far more mathematically difficult path to take to find the electric potential. These students could be activating resources that are chronologically fresher in their minds.

3.1.2 Construction

Nearly all students (94%) showed work that could be considered in the category of construction. For code C1, *use the geometry of the charge distribution to select a coordinate system*, all the students that attempted construction signaled the use of cylindrical coordinates as their preferred coordinate system. Code C2, *express the differential charge area element* dA *in the selected coordinates*, nine students (25%) had difficulty expressing the infinitesimal area element dA. These students made one of two errors. They expressed dA with the incorrect dimensions (e.g. $dA = dr'd\varphi'$). Otherwise, these students took their answers from part A of the question (Solve for the total charge) and directly substituted that answer into their equations for charge in part B, as seen in figure 3.1.

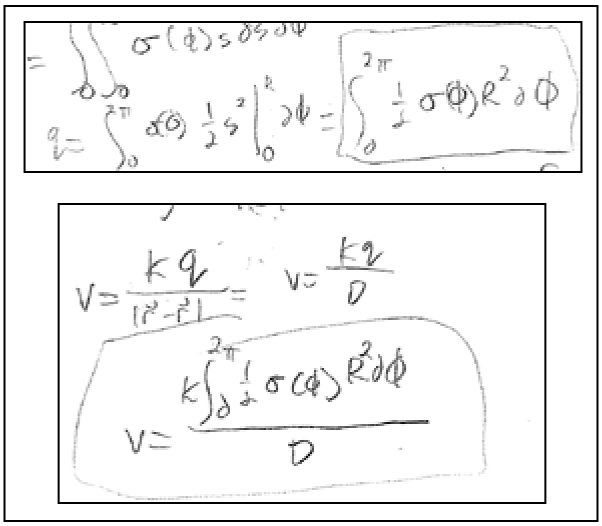

Figure 3.1 - Student 29 solves for total charge ‘q’ for part A and later substitutes their solution for ‘q’ when trying to find the electric potential at point D.

This error is classified as a problem with constructing the infinitesimal area element dA. Student 29, seen in figure 3.1, when dealing with total charge produces a correct dA and also correctly finds the total charge of the disk. However, student 29 falsely believes that they can take the total charge with the evaluated integrals and simply substitute that into their electric potential equation.

Next for code C3, *select integration limits consistent with the differential charge area element and the extent of the physical system*, 6 students made errors in selecting these limits. There was no single unifying error among these students. Finally, two students produced indefinite integrals. Later in execution, these same two students did not attempt to integrate their expressions.

Finally for code C4, *express the difference vector, r-r’, in the selected coordinates*, 13 students (36%) had difficulty expressing $\vec{r} - \vec{r}'$. One of the more common issues for OSU students was building $|\vec{r} - \vec{r}'|$ in cylindrical coordinates. As shown in figure 3.2, a number of students had a factor of φ or φ' in the final form of $|\vec{r} - \vec{r}'|$. The student depicted in figure 3.2 starts with the integral form of Coulomb potential due to a surface charge distribution. They then go on to build dA' with cylindrical coordinates in mind.

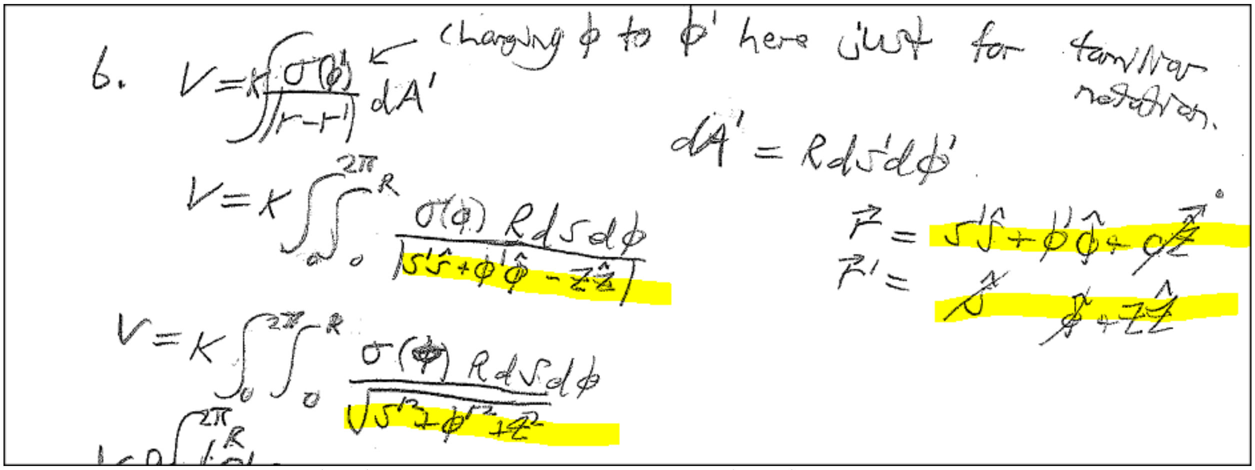

Figure 3.2 - Errors building $|\vec{r}-\vec{r}'|$. The student attempts to build $\vec{r}$and $\vec{r}'$ in cylindrical coordinates. They seem to be mapping their knowledge for building a vector in Cartesian coordinates directly to making a vector in cylindrical coordinates. [emphasis added].

$$\text{(eq. 3.1)} \quad dA' = R\,ds'd\varphi'$$

The student then turns their attention to building $|\vec{r}-\vec{r}'|$. Defining $\vec{r}$ incorrectly as:

$$\text{(eq. 3.2)} \quad \vec{r} = s'\hat{s} + \varphi'\hat{\varphi} + 0\hat{z}$$

Then striking out the $0\hat{z}$. They follow by defining $\vec{r}'$,also incorrectly, as:

$$\text{(eq. 3.3)} \quad \vec{r}' = \hat{s} + \hat{\varphi} + Z\hat{z}$$

Then striking out the $\hat{s}$ and $\hat{\varphi}$ to denote the values for both to be zero.

The student has made multiple errors in building these incorrect $\vec{r}$ and $\vec{r}'$. First, they are mixing their prime notation in multiple ways. The student defines $\vec{r}$ with s' and φ' which strongly suggests they want $\vec{r}$ to be descriptive of where the charges are on the disk They then define $\vec{r}$' as a vector pointing to the point to be evaluated. This evidence points towards the student being unsure in what they want the prime notation to denote. Second, the student incorrectly evaluates a vector based in cylindrical coordinates. The student could also be employing resources for the more familiar Cartesian coordinate system and then incorrectly attempting to directly map that resource to build $\vec{r}$ and $\vec{r}'$ in cylindrical coordinates.

$$\text{(eq. 3.4)} \quad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \rightarrow \vec{r} = s\,\hat{s} + \varphi\,\hat{\varphi} + z\,\hat{z}$$

$$\text{(eq. 3.5)} \quad \vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z} \rightarrow \vec{r}' = s'\,\hat{s} + \varphi'\,\hat{\varphi} + z'\,\hat{z}$$

The errors to produce equations (3.4) and (3.5) reside in the Cartesian understanding that $\hat{x}\ is\ \hat{x}$ everywhere in space, therefore $\hat{s}\ is\ \hat{s}$ everywhere in space as well. That thought process is most likely much deeper than what the students were actually consciously thinking. The more likely reasoning is direct pattern matching between the two coordinate systems. The same train of thought would go for how the student defined $\hat{\varphi}$. Again in figure 3.3 and also seen in equations (3.2) and (3.3), the student does not prime either $\hat{s}$ or $\hat{\varphi}$ in either $\vec{r}$ or $\vec{r}'$. This further illustrates student's incorrect idea that $\hat{s}$ and $\hat{\varphi}$ are the same everywhere in cylindrical coordinates.

The student then goes back to their previously defined generic definition for electric potential. They attempt to evaluate $|\vec{r} - \vec{r}'|$and replace dA', $\vec{r}$, and $\vec{r}'$ with the student's derived quantities.

$$\text{(eq. 3.6)} \quad V = K \int_0^{2\pi} \int_0^R \frac{\sigma(\varphi) R ds d\varphi}{|s'\hat{s} + \varphi'\hat{\varphi} - Z\hat{z}|}$$

The current point of interest in equation (3.6) is the $|\vec{r} - \vec{r}'|$evaluation:

$$\text{(eq. 3.7)} \quad |s'\hat{s} + \varphi'\hat{\varphi} - Z\hat{z}|$$

To reach the conclusion seen in equation (3.7), the student took $\vec{r}$ from equation (3.2) and subtracted $\vec{r}'$ from equation (3.3) using the same incorrect idea that all $\hat{s}$ and $\hat{\varphi}$are the same in cylindrical coordinates.

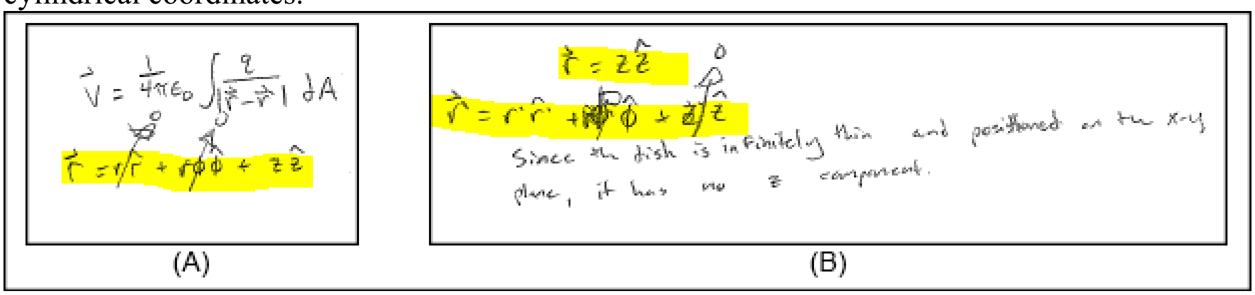

figure 3.3 - Errors building $|\vec{r} - \vec{r}'|$. The student attempts to build $\vec{r}$and $\vec{r}'$ in cylindrical coordinates.

The student in figure 3.3 also makes a similar error in building $\vec{r}$ and $\vec{r}'$. The student incorrectly evaluates r in cylindrical coordinates:

$$\text{(eq. 3.8)} \quad \vec{r} = r\,\hat{r} + r\varphi\,\hat{\varphi} + z\,\hat{z}$$

In equation (3.8) the term $r\varphi\,\hat{\varphi}$ shows the student has some mindfulness of the units that φ should have associated with it, but does not recognize that this term should not be present at all. The student strikes out what quantities in $\vec{r}$ should be zero due to the physical properties of the problem, describes their reasoning in words, and then simplify to:

$$\text{(eq. 3.9)} \quad \vec{r} = z\,\hat{z}$$

Which leads them fortuitously to a correct expression for $\vec{r}$. The student then attempts to build $\vec{r}'$. They start with the same basics as equation (3.8) but with some modifications. While the student has scribbled out some work, equation (3.10) is probably what they started out using.

$$\text{(eq. 3.10)} \quad \vec{r}' = r'\,\hat{r}' + r'\varphi'\,\hat{\varphi} + z'\,\hat{z}$$

In figure 3.3 part B, a written statement referencing equation (3.10) is as follows:

"*Since the disk is infinitely thin and positioned on the x-y place, it has no z component.*"

Their simplified $\vec{r}'$ was then defined as:

$$\text{(eq. 3.11)} \quad \vec{r}' = r'\,\hat{r}'$$

Again, a correct statement that started off in the wrong place. To get to the equation (3.11), the student zeros out both the $\hat{\varphi}$ and $\hat{z}$ components of their $\vec{r}'$. At this point the student has distinguished between $\hat{r}$ *and* $\hat{r}'$. Later it appears they go back over the $\hat{\varphi}$ component and scribble something out. That scribble could have been produced while the student was writing $\vec{r} - \vec{r}'$ as seen in figure 3.4. Some resource was possibly triggered during this time as the student erases and rewrites information inside the three lines to produce $\vec{r} - \vec{r}'$ and they applied that resource to $\vec{r}'$.

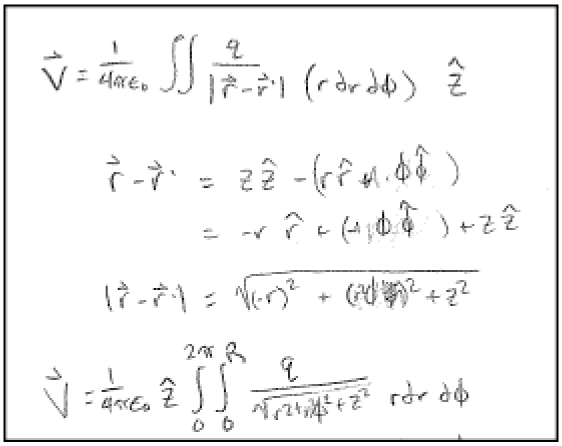

figure 3.4 - Student evaluates $|\vec{r} - \vec{r}'|$ incorrectly. The student abandons their previous $\vec{r}'$and replaces it with a vector in an incorrect invoking of cylindrical coordinates.

In figure 3.4 the same student is evaluating $|\vec{r} - \vec{r}'|$, but they abandon what they had originally defined as $\vec{r}'$ in equation (3.10). Specifically they seem to abandon the idea of $\hat{r}'$ being different from $\hat{r}$. What caused the student to change their mind is unknown. This results in the student going back to basics of what they defined for a generic $\vec{r}$ n their incorrect version of cylindrical coordinates. Where the student defines $\vec{r} - \vec{r}'$ as :

$$\text{(eq. 3.12)} \quad \vec{r} - \vec{r}' = z\,\hat{z} - (r\,\hat{r} + \varphi\,\hat{\varphi})$$

This implies the student has now redefined $\vec{r}'$ to be:

$$\text{(eq. 3.13)} \quad \vec{r}' = r\,\hat{r} + \varphi\,\hat{\varphi}$$

This newly defined $\vec{r}'$ shows that they have removed the r factor that was associated with $r\varphi\,\hat{\varphi}$ that showed up in both cases when the student initially defined $\vec{r}$ *and* $\vec{r}'$. The final equation (3.14) shows us an addition to their $|\vec{r} - \vec{r}'|$evaluation.

(eq. 3.14) $\vec{V} = \frac{1}{4\pi\epsilon_0}\hat{z}\int_0^{2\pi}\int_0^R \frac{q}{\sqrt{r^2 + r^2\varphi^2 + z^2}} rdrd\varphi$

As seen in figure 3.4, the student erases, replaces, and/or adds information from equation (3.12) through equation (3.14). At this point the student could have reflected on a lone $\varphi^2$and was dissatisfied at the unit analysis of their $|\vec{r} - \vec{r}'|$evaluation in equation (3.14). This may have lead the student into adding the factor of r^2 to equation (3.14) in order for the unit analysis to come out more mathematically sound.

As shown, the individual students depicted in figure 3.2 and figure 3.3-3.4 both made very similar errors. The root difficulty stems from their misunderstanding of basis vectors in cylindrical coordinates. The student in figure 3.2 is possibly mapping the Cartesian fact that $\hat{x}\ is\ \hat{x}$ everywhere, therefore in cylindrical coordinates $\hat{s}\ must\ be\ \hat{s}$ everywhere as well. This thinking is probably deeper than what is actually occurring. While the student in figure 3.3 initially denotes a difference between $\hat{r}\ and\ \hat{r}'$in cylindrical coordinates, but later this difference is ignored for the more familiar Cartesian coordinate reasoning. This Cartesian sense making is clearly more familiar/comfortable to both, if not most, students.

3.1.3 Execution

For E1, *maintain an awareness of which variables are being integrated over* (*e.g. r' vs. r*), a large percentage of students (88%) of students did not maintain awareness of the variables being integrated over. These errors mostly consisted of not attempting to use prime notation or just dropping the prime notation after the initial construction setup. It should be noted of the five students who correctly answered the problem, only two students correctly maintained the prime notation throughout their solutions.

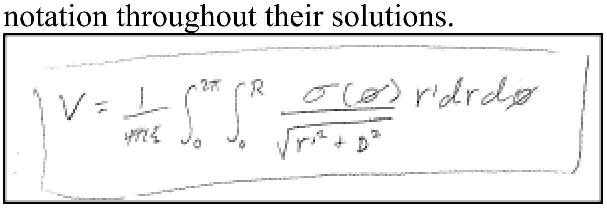

Figure 3.5 - Student 36 boxes their final solution without attempting any integration.

For E2, *execute (multivariable) integrals in the selected coordinate system,* again a large percentage of students (83%) of student did not integrate their expressions in their selected coordinate systems correctly. Six students just setup the integrals but did not attempt to do them, as seen in figure 3.5. The rest of the student errors during integration did not fall into categories. Those errors range from dropping factors of two, incorrect 'u substitution', and minor math errors. There is also overlap in students that did not correctly handle E3.

For E3, *maintain an awareness that σ(φ)is an unknown varying surface charge density*, roughly 42% of students correctly maintained an awareness that $\sigma(\varphi)$ is a function of surface charge density that varies with φ in unknown way. Therefore, this function cannot be evaluated completed. Again, six students did not evaluate their final integrals as seen in figure 3.5. Aside from the non-evaluators, there were three major student errors that occurred for E3. Students (25%) either incorrectly treated $\sigma(\varphi)$as a constant, incorrectly invoked the fundamental theorem of calculus, or showed strong confusion over function notation as presented in this problem. The rest of the student errors did not fall into categories.

Students that incorrectly treated $\sigma(\varphi)$ as a constant all had one common factor in relation to their notation, they wrote σ in their work instead of the full form $\sigma(\varphi)$. This is in stark contrast to the students that correctly maintained the awareness of $\sigma(\varphi)$. Students who correctly treated $\sigma(\varphi)$ as a function dependent on φ always wrote $\sigma(\varphi)$in it's full function form. Students not being attentive to the notation for $\sigma(\varphi)$ possibly didn't know what to do with an undefined function and just ignored the function notation and treated $\sigma(\varphi)$ as just a constant σ. Perhaps they were just being careless. The most troubling reason would be that students did not recognize or understand basic function notation.

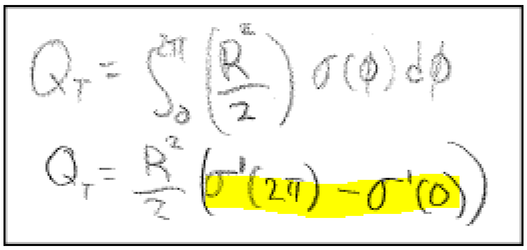

Figure 3.6 - Student 16 incorrectly tries to use the fundamental theorem of calculus in an attempt to evaluate the integral of $\sigma(\varphi)\ d\varphi$ from 0 to 2π.

The second common error came from students attempting to leverage the fundamental theorem of calculus to help them evaluate an integral they couldn't evaluate. As a quick reminder the fundamental theorem of calculus is given by:

$$\text{(eq. 3.15)} \qquad \int_a^b g(x)\, dx \;=\; G(b) - G(a)$$

Where g(x) is a continuous function and the function G is the antiderivative of the function g with respect to x. Student 16, as seen in figure 3.6, attempts to use the fundamental theorem of calculus but makes a significant error in their notation. Just focusing on how the student has dealt with $\sigma(\varphi)$ aspect of the work in figure 3.6, this relationship shows itself more clearly.

$$\text{(eq 3.16)} \qquad \int_0^{2\pi} \sigma(\varphi) d\varphi \;=\; \sigma'(2\pi) - \sigma'(0)$$

Equation 3.16 shows us the student has denoted the integral of $\sigma(\varphi)$ using the prime notation. This shows the student has a misunderstanding of what the prime notation when applied to functions typically denotes a derivative of a function not a function's antiderivative. Let's look at another student who made, what appears to be, a similar error.

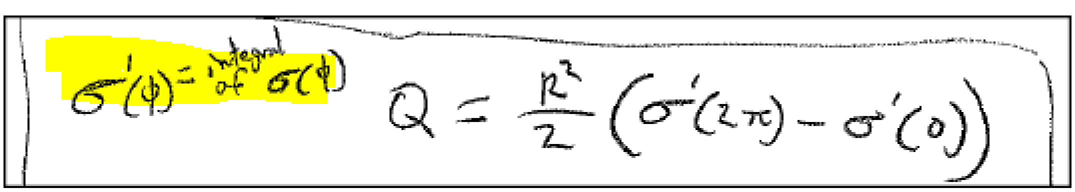

figure 3.7 - Student 35 appears to repeat the same mistake as Student 16 (figure 3.6). The important difference is the note student 35 leaves on the left hand side of his equation: "$\sigma'(\varphi) = integral\ of\ \sigma(\varphi)$".

Seen in figure 3.7, it would appear student 35 erred in the same fashion as student 16. However, student 35 makes a note of what they are defining the prime notation to mean:

$$"\sigma'(\varphi) \ = \ integral\ of\ \sigma(\varphi)"$$

Student 35 redefines what prime notation denotes. That note makes their entire statement correct, if misleading. It is unknown if the previous student, student 16, had the same idea but failed to record the redefining of the prime functionality.

The final common error for dealing with code E3 comes from students viewing $\sigma(\varphi)$ as $\sigma * \varphi$. Instead of seeing σ as a function of φ, students saw $\sigma = \sigma * \varphi$. This can be seen most clearly in three cases shown in figure 3.8.

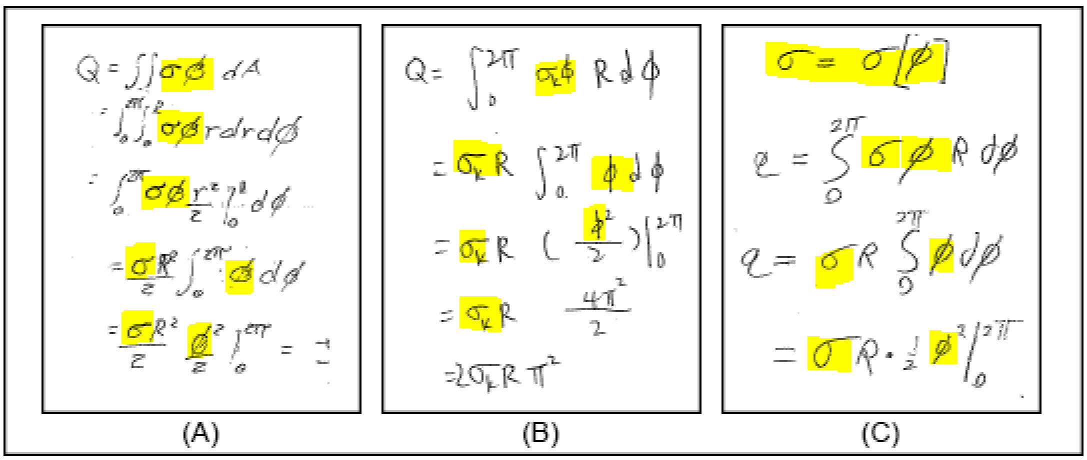

Figure 3.8 - In (A),(B), and (C) three individual students incorrectly treat $\sigma(\varphi)$not as a function but as $\sigma\ multiplied\ by\ \varphi$. All three students then carry out the integration on φ with respect to $d\varphi$ while treating σ as a constant.

In all three cases, these students treat σ as a constant being multiplied by the angle φ. While that interpretation of $\sigma(\varphi)$ is reasonable, it was not the intended interpretation. One of the possible reasons for this misunderstanding of function notation could come from the first line of the problem statement: “Suppose there is a thin disk with a radius R and a surface charge density that changes with angle $\sigma = \sigma(\varphi)$.” This misunderstanding could stem from just a lack of confidence, knowledge, or experience with function notation. Another explanation could be that students at this level don’t have much experience in dealing with functions that are not explicitly defined. Students could have also been considering what they believed the instructor would give them on a final. Students could think that an undefined function “would not be fair to give on a final” and therefore consider the parenthesis not denoting a function but clearly separating the variables for multiplication.

3.1.4 Reflection

While our students did not reflect as an expert would, as the ACER codes suggest, they did do other types of reflection which were not anticipated in the codes. These reflections are typical of student work. When a final solution given was incorrect, notes such as “I know this is wrong” were stated next to solutions. This type of reflection shows that there is an understanding of when something ‘feels’ wrong. A building of physics intuition that may be the first step toward expert level reflection.

Chapter 4 – Discussion

What method did students activate?	75% Electric Potential 19% Electric Field 6% Unable to analyze
C1: Use the geometry of the charge distribution to select a coordinate system.	94% Cylindrical coordinate system 6% Unable to analyze
C2: Express the differential charge area element *dA* in the selected coordinates.	69% Correct *dA* 25% Incorrect *dA* 6% Unable to analyze
C3: Select integration limits consistent with the differential charge element and the extent of the physical system.	71% Correct integration limits 17% Errors in limit selection 6% Indefinite Integrals 6% Unable to analyze
C4: Express the difference vector, r-r', in the selected coordinates.	58% Correct r-r' 36% Incorrect r-r' 6% Unable to analyze
E1: Maintain an awareness of which variables are being integrated over. (e.g. r' vs. r).	6% Maintained awareness 88% Did not maintain awareness 6% Unable to analyze
E2: Execute (multivariable) integrals in the selected coordinate system	14% Correct integration 63% Incorrect integration 17% No integration evaluation 6% Unable to analyze
E3: Maintain an awareness that $\sigma(\varphi)$ is an unknown varying surface charge density.	42% Maintained awareness 35% Did not maintain awareness 17% No integration evaluation 6% Unable to analyze
R1: Verify that the units are correct.	0% Verified Units
R2: Check the limiting behavior to ensure it is consistent with the total charge and geometry of the charge distribution.	0% Checked limiting behavior

figure 4.1 - The major results from the analysis of the OSU ACER rubric.

The main goal of the ACER rubric applied to this student data set was to evaluate if student were appropriately choosing the integral form of Coulomb's Law. The bulk of the results are summarized in table 4.1, but let's look at a few specific results. For Activation, 75% of students activated the integral form of Coulomb's Law. While only a single student, out of the total of 36, is choosing to use Gauss's Law. For Construction, 27% of students incorrectly constructed da and 36% of student incorrectly constructed $|\vec{r} - \vec{r}'|$in cylindrical coordinates. For Execution, only 14% of students correctly integrated their expressions and 42% of students correctly dealt with $\sigma(\varphi)$ as an unknown varying function. Finally for Reflection, not a single student showed evidence of reflection as outlined by the OSU ACER rubric.

My interpretation of these results would be as follows: Are students correctly activating the right tools? It seems a majority of our students are correctly going for Coulomb's Law when it comes to approaching this problem. Once they have activated these tools, are students correctly constructing them with the physical variables given in the problem? Again, it seems a majority of our students are correctly choosing an appropriate coordinate system, building *dA* cylindrical coordinates, and choosing appropriate limits of integration when it comes to this disk of charge problem. About half of OSU students have difficulty when it comes to constructing |r-r'| in cylindrical coordinates for this problem. This is the first red flag. It should also be noted that only 44% of students correctly constructed codes C1 (coordinate system),C2 (*dA*),C3 (integral limits) and C4 (r-r'). This problem requires that students correctly construct codes C1, C2, C3, and C4 in order to successfully solve this problem. Now on to Execution. Only 14% of students correctly integrated their constructed expressions. This means that less than half of students that correctly constructed the problem were successful at integrating their constructed models. Finally starting junior level OSU physics students are not explicitly showing professional reflection. To me, this is expected of a junior level student as they will attempt to give you what your prompt asks of them. My final assessment would be: That about half of our students are still struggling with r-r' in cylindrical coordinates; Over half of our students are having difficulties executing integration even with the correct setup. Armed with the assessment I could modify lesson plans to assist students in these perceived gaps. The ACER Framework has allow me to identify many issues with student work and my analysis left me with two large student issues that I have identified.

Also over the course of my research, I've started to frame my thinking about student thinking in terms of the ACER categories of Activation, Construction, Execution, and Reflection. This categorization has been professionally useful in my capacity as a physics tutor and teaching assistant. I tended to think about these sorts of categories in the terms of student thinking, but I never had a label to put with it. For instance, I was assisting a student who was having difficulties with a car on a track problem. They were confused about where to start. After reading the question I can clearly see a method to get to a solution. Then comes the task of assisting the student down the road of understanding. One of the cardinal sins on this road tutors is to do the work for your students. I rarely sin in this matter anymore, but I still must be mindful of the possibility. The ACER categorizations can helped me be more mindful about what is going on each step of the way as I assist in student problem solving. Has the student activated the correct tool? If not, how can I get them to reflect in a way to go to the correct tool? They executed the mathematics correctly but constructed an incorrect model, how can I get them to reflect on their constructed model to see the error? In this context, I could be talking a wide variety of questions and all which are useful lines of inquiry. As a non-PER educated undergraduate in physics, the ACER framework has given me additional tools in assisting in my own understanding of how to frame student thinking.

The ACER framework has given me a new way of framing student thinking that I did have before, but did the ACER rubric give me more analytic information than an general analytic rubric (see Appendix A)? I do not believe it did. I believe if you were to build an emergent rubric from the OSU data set, the two big insights would still be: that students have having problems with integration and they are incorrectly build r-r' in cylindrical coordinates. Of course, I have only used the ACER framework to analyze one set of data for one junior level physics course. This far too small of a data set to make a larger generalizations to the ACER framework and the rubrics built with the ACER framework to any other type of rubric. In this instance, I do not believe it lent to a deeper insight into student difficulties with Coulomb's Law.

However, I don’t think that’s a negative aspect of the framework. For someone like myself, I believe the ACER framework is an excellent tool novices when it comes to building rubrics and interpreting student data. My discussion with a content expert afforded comparison of my problem solving method versus theirs. This discussion hammers out the different possible avenues for problem solving. Then finding the consensus for the important steps allows for a clear description of the said important steps. This consensus allowed for investigation of multiple points of view that I hadn’t considered initially. This research project has been far more educational and informative than just strictly developing an analytic rubric would have ever been.

In the future, I would like to use the ACER framework to analyze a different question for thermodynamics or electronics. The outcomes and possible insights that could be gathered for either of those subjects would be interesting to evaluate. A larger data consisting of multiple years’ worth of students would be another variation I’d like to investigate. The results of each year’s cohort of students analyzed together and separately would be another aspect for further investigation.

Appendix A --

Three Styles of Rubrics

Rubrics can be used in nearly any classroom. While a certain style of rubric may work in the realm of mathematics, the same style of rubric may not be appropriate to use in the realm of writing. There are nearly as many styles of rubric as there are classroom settings. With that in mind, this document will describe only 3 specific styles: holistic, analytic, and emergent.

Holistic rubrics are designed to evaluate the assignment as a whole and give general guidelines for differing levels of success. This style of rubric can be attractive to instructors since it is an evaluation of the assignment as a whole. The downside of holistic rubrics are, they only look at the assignment as a whole. For example, two students work on a multi-step physics problem. Each student makes an error during integration. Student A integrates over the incorrect variables. Student B integrates over the incorrect limits. While these are different errors, both students 'integrated incorrectly'. The holistic rubric would give the students the same evaluation since they both 'integrated incorrectly' but completed the rest of the problem successfully. The holistic rubric loses fidelity for the sake of a wide evaluation net.

Analytic rubrics are designed to evaluate the different steps of the assignment and give guidelines for the differing levels of success for each step. This style of rubric allows the instructor to choose the step granularity for an assignment. The instructor could evaluate 'integration' as one step or evaluate: 'choosing correct limits', 'integrating over the correct variables', etc. as multiple steps to be individually evaluated for differing levels of success. The evaluation of 'integration' as one step still gives the instructor more choices for giving feedback than the holistic rubric. Again using students A and B, the instructor can then choose what level of success each student had for the integration step. The analytic rubric allows for a finer level of detail, but it still loses fidelity when it comes to the differing levels of success for a given step. The levels of success will not cover every possible error a student can make placed in them.

Emergent rubrics are designed with the data. Emergent rubrics are not pre-designed based off the question, rather these rubrics are created around the answers given. This style of rubric is very flexible as it allows the instructor to set the granularity. The instructor may choose to give each individual student mistake or success its own category. Once finished, the instructor may choose to lump different mistakes together and categorize them as one. In this way, the instructor can see the students' difficulties as a whole.

All rubrics require a description for differing levels of success. These differing levels of success are usually categorized with a description such as: 'poor', 'average', or 'excellent'. Commonly these categories will be labeled just as discrete point values (*i.e.* '1', '2', or '3'). Each category will have a description of what is required to be successful for that category. While the number of categories remain constant throughout the rubric, their descriptions will depend on the step they are evaluating. Choosing an odd or even number of categories changes how the step will be evaluated. In choosing an odd number of categories (*i.e.* '1', '2', '3', '4', or '5') the instructor has an easier time evaluating 'average' work. However, if an even number of categories is chosen then the instructor must evaluate what is a better than average success and a worse than average success.

References

[1] B. R. Wilcox, M. D. Caballero, D. A. Rehn, and S. J. Pollock, *Analytic framework for students use of mathematics in upper-division physics*, Phys. Rev. ST Phys. Educ. Res. **9**, 020119 (2013), URL http://link.aps.org/doi/10.1103/PhysRevSTPER.9.020119