Solution method and error evolution of student responses to chain rule problems within a thermodynamics course

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Chain rules are critical to the process of solving many thermodynamics-related partial derivatives. This study evaluates the solution method and error evolution of students' responses to a pair of chain rule problems in an upper-level undergraduate thermodynamics course. Students' responses were categorized by solution method. Students' solution methods included implicit differentiation, substitution, differential algebra, and chain rule diagrams. In addition to categorizing students' solution methods, students' errors were sorted and analyzed. In particular, many students did not know how to hold the appropriate variable(s) constant while evaluating partial derivatives. Students also had difficulties identifying partial derivatives, and reading and building chain rule diagrams. These results could be used to improve student understanding of partial derivatives and chain rules by adjusting the order, portrayal, and intensity of course material.

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CHAPTER 1 - INTRODUCTION

1.1. Motivation

This project's purpose is to further the understanding of how students algebraically approach thermodynamics-related chain rule problems. Student success in thermodynamics is critically dependent on a thorough understanding of the algebraic workings and manipulations of partial derivatives, the building blocks of chain rules. The subjects of this study are students in the Paradigms in Physics program at Oregon State University (OSU). The Paradigms in Physics program at OSU is a constantly evolving upper-division undergraduate physics program.

The project's primary purpose is to help improve the learning experiences of students in OSU's thermodynamics Paradigm. The course's designer will use the results to gain insight into the effectiveness of current teaching strategies. Result implementation will take the form of adjusted order, portrayal, and intensity of relevant discussions, activities, and assignments. Via these means, students' preparedness to solve chain-rule problems will be improved.

This project resembles the work of **Kustusch *et al.* [3]**. Both studies have similar evaluations of subjects' understandings of thermodynamics. **Kustusch *et al.* [3]** evaluates expert understanding as well as student understanding. This project focuses only on the later. As will be seem in **Section 2.2**, this project's prompts are virtually parallel to the prompt in **Kustusch *et al.* [3]**. Due to these similarities, the results from this project will provide some level of independent verification of the validity of the results from **Kustusch *et al.* [3]**.

1.2. Literature Review

This project is not the first of its kind in PER. The following are short summaries of three relevant works cited in later sections. **Thompson *et al.* [1]** and **Kustusch *et al.* [3]** are both thermodynamics-related PER studies. This project bears great resemblance to the latter of the two. An introduction to **Tuminaro and Redish [2]** is necessary for an understanding **Kustusch *et al.* [3]**.

The work of **Thompson *et al.* [1]** is based on the collaborative research between the University of Maine and Iowa State University regarding student understanding of thermodynamics. The paper contains an analysis of data representing student use of partial derivatives, particularly Maxwell relations and Clairaut's Theorem (equality of mixed partial derivatives), throughout the course. Thompson ***et al.*** conclude by discussing student successes and difficulties observed within the analysis.

Student successes include: knowing the difference between derivatives of systems with one or more independent variables, understanding (either conceptually or functionally) that partial derivatives treat some variables as constants, verbally expressing partial derivatives, and successfully creating Maxwell relations. Derivatives of systems with multiple independent variables are better known as partial derivatives. The observed difficulties include: treating differential expressions algebraically, applying partial derivative relations to a physical situation, and being unsure of when to apply Maxwell relations to a physical situation.

Tuminaro and Redish [2] design and develop an analysis device, for education research, known as epistemic games. An epistemic game is "a coherent activity that uses particular kinds of knowledge and processes associated with that knowledge to create knowledge or solve a problem." The paper analyzes 11 hours of video recorded in an open physics homework lab, where students were encouraged to discuss the problems. Tuminaro and Redish identify six epistemic games. In order of decreasing difficulty, the games are: mapping meaning to mathematics, mapping mathematics to meaning, physical mechanism game, pictorial analysis, recursive plug-and-chug, and transliteration to mathematics. These games are very general; more specific / situational games, such as those observed in **Kustusch *et al.* [3]**, can be categorized as one (or a combination) of the six Tuminaro and Redish games. The outline of each of these epistemic games can be seen in **Figure 1.1**. From top-left to top-right: mapping meaning to mathematics, mapping mathematics to meaning, physical mechanism game, pictorial analysis, and transliteration to mathematics. Bottom: recursive plug-and-chug.

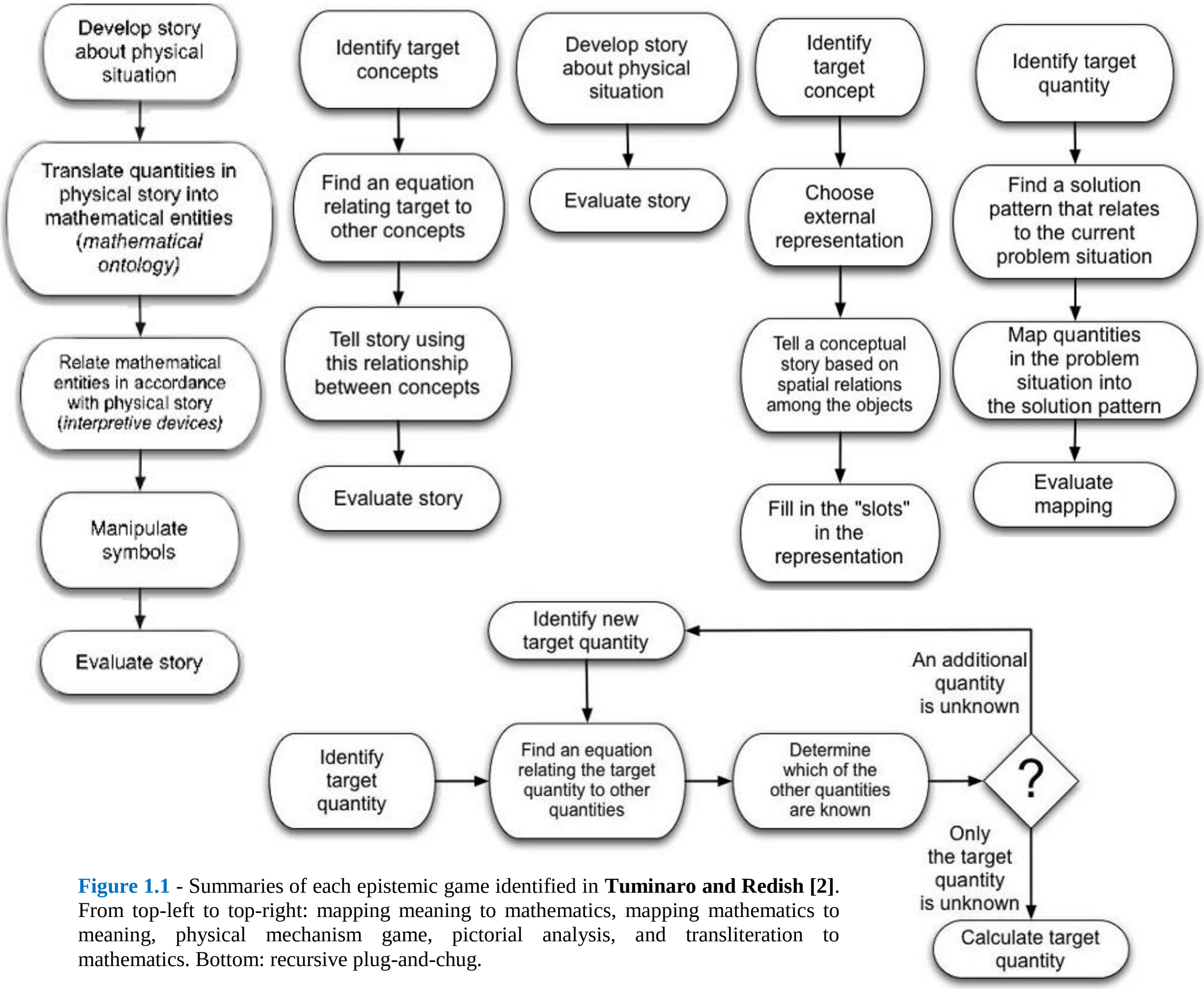


Figure 1.1 - Summaries of each epistemic game identified in **Tuminaro and Redish [2]**. From top-left to top-right: mapping meaning to mathematics, mapping mathematics to meaning, physical mechanism game, pictorial analysis, and transliteration to mathematics. Bottom: recursive plug-and-chug.

Kustusch *et al.* [3] compare and analyze methods used by physics faculty and upper-division undergraduate physics students to solve a thermodynamics problem (shown below).

$$Find\left(\frac{\partial U}{\partial p}\right)_S for\ a\ van\ der\ Waals\ gas, given\ the\ following\ equations\ of\ state:$$

$$p = \frac{NkT}{V - Nb} - \frac{aN^2}{V^2}, \tag{1.1}$$

$$S = Nk\left(\ln\left(\frac{(V - Nb)}{N}T^{\frac{3}{2}}\right) + \frac{5}{2}\right) \tag{1.2}$$

Kustusch *et al.* [3] provide an in-depth discussion of how the physics faculty approached the prompt. The study also makes interesting observations about how expert and student understandings differ. The analysis identifies three epistemic games, used by students and experts alike: Substitution, partial derivatives, and differentials.

The substitution game uses substitution to change the set of quantities (the variables) describing a different quantity (the function). For example, solve $z = z(x, y)$ for $x = x(y, z)$ and substitute $x = x(y, z)$ in to $U = U(x, y, z)$, forming $U = U(y, z)$. The quantity "U" is now represented by the quantities y and z instead of x, y, and z. The partial derivatives game encompasses using the syntactical and semantical functions of partial derivatives to equate the desired partial derivative to an expression in terms of easily solvable partial derivatives. The differentials game involves algebraically manipulating the total differentials of functions in order to obtain an expression for the desired partial derivative. Three variants of these methods were observed: applying information about variables being held constant, using a differential of a thermodynamic identity, and dividing by a differential. Applying any of these variants to the above methods slightly changes the resulting process.

CHAPTER 2 – METHODS

This chapter begins, in **Section 2.1**, by introducing the subject population. **Section 2.2** discusses the prompts and (briefly) the data collection methods. **Section 2.3** defines mathematical terms necessary for an understanding of the prompt's possible solutions and their variations. These possible solutions are described in **Section 2.4**. An instructional timeline of the prompts' assignments and solution methods is included in **Section 2.5**, and **Section 2.6** outlines analysis methods.

2.1. Student Background

The subjects of this study are twenty-nine students from the Spring-term 2016 thermodynamics course ("Energy and Entropy") of the Paradigms in Physics program at OSU. Students should have entered the class with most of the mathematical background necessary to understand the prompts and their possible solution methods; most students had taken integral and vector calculus classes, as well as an introductory course in differential equations. Additionally, students had experience in applying their mathematical knowledge to content in other Paradigms courses. Namely, most students learned and applied partial derivatives in both the "Symmetries and Idealizations" and "Static Vector Fields" Paradigms courses. See **Appendix A** for a full list of relevant material in other Paradigms courses.

2.2. The Prompts

This project studies student responses to two prompts. The prompts were assigned as two quizzes and a final exam question. Quiz prompts were posted online a few days before the quizzes were assigned in class, giving students a very generous chance to prepare for each quiz. Both quizzes had the same prompt:

$$Given\ the\ definitions\ below, evaluate\ the\ requested\ partial\ derivative.$$

$$U = x^2 + y^2 + z^2 \tag{2.1}$$

$$z = \ln(y - x) \tag{2.2}$$

$$Find\ \left(\frac{\partial U}{\partial z}\right)_y \tag{2.3}$$

The first quiz was assigned, graded, and handed back to students on the second Friday of class. This quiz will be referred to as Quiz 4. Later that day, Quiz 4's responses were recognized by the professor as viable research material. On the Monday afterwards, students were asked to hand back their responses. Only seven quizzes were received and scanned. Therefore, statistics involving Quiz 4 were mostly excluded from the analysis. Also on that Monday, the professor showed the class most of the possible Quiz 4 solution methods. The second assignment studied in this project was Quiz 14. Quiz 14 was assigned two weeks after Quiz 4, on the Friday of the fourth and last class week. This time, the graded quizzes were scanned before being handed back to students. There were twenty-nine responses to Quiz 14. Quiz 14's possible solution methods were not discussed in class.

The other prompt studied in this project was given on the final exam, and will be referred to as Final 4.b. Final 4.b's prompt is as follows:

$$Given\ the\ definitions\ below, evaluate\ the\ requested\ partial\ derivative.$$

$$S = NK_B\left(\ln\left(\frac{\mathrm{N - Vb}}{NC}T^{\frac{3}{2}}\right) + \frac{5}{2}\right) \tag{2.4}$$

$$U = \frac{3}{2}NK_BT - \frac{aN^2}{V} \tag{2.5}$$

$$Find\ \left(\frac{\partial U}{\partial V}\right)_S \tag{2.6}$$

In Final 4.b, **Equation 2.4** was designed to make solving for T a nontrivial process. As you will see later, this design has consequences on how students approach the problem. The final exam was given on the class day after Quiz 14 was assigned; there was only a weekend between Quiz 14 and the final exam. Twenty-seven students responded to Final 4.b. Responses to the final exam were scanned after being graded. **Appendix E** contains anonymized student responses from each prompt.

These prompts are less complex than, but virtually parallel to, the prompt in **Kustusch *et al.* [3]**. Both studies' prompts ask the student to evaluate a partial derivative. Each prompt provides two equations with overlapping variables. The prompt in **Kustusch *et al.* [3]** has explicit thermodynamics context due to its variables, as does Final 4.b's prompt. However, the prompt from Quiz 4 / 14 has no such context.

2.3. Terminology

These prompts are difficult to evaluate. In order to understand the prompts' solutions and their variations, some additional mathematical background is required. This section will familiarize the reader with necessary concepts and introduce our terminology.

Direct and Indirect Partial Derivatives

In this study, two types of partial derivatives must be distinguished. The first kind can be evaluated with a single function. The function must have the same variable space as the desired partial derivative; the partial derivative and the function must have the same variables. The function may need to be reorganized in order to evaluate the partial derivative. This kind of partial derivative will be referred to as a direct partial derivative. Consider the following examples of direct partial derivatives:

Calculating $\left(\frac{\partial U}{\partial z}\right)_y$ from $U(y, z)$ is a direct partial derivative because the function and the partial derivative are both in terms of U, y, and z.

Calculating $\left(\frac{\partial U}{\partial y}\right)_{x,z}$ from $U(x, y, z)$ is a direct partial derivative because the function and the partial derivative are both in terms of U, x, y, and z.

The second kind of partial derivative *does not* have the same variable space as the function it acts on. Thermodynamics-related expressions often have many variables, which are not all independent. This can result in needing to evaluate partial derivatives of functions that contain more variables than the partial derivative. To evaluate such a partial derivative, one must know the additional (constraint) equations relating the expression's variables. This type of partial derivative will be referred to as an indirect partial derivative. Consider this example of an indirect partial derivative:

Calculating $\left(\frac{\partial U}{\partial z}\right)_y$ from $U(x, y, z)$ is an indirect partial derivative because the function is in terms of U, x, y, and z, whereas the partial derivative is only in terms of U, y, and z.

Any solution method for finding an indirect partial derivative either changes the indirect partial derivative to a direct partial derivative or creates a chain rule in terms of other partial derivatives. Any indirect partial derivatives in the chain rule must also either be changed to direct partial derivatives or expressed as chain rules. This process continues until the originally desired indirect partial derivative can be expressed by direct partial derivatives. Note that both prompts require the solver to evaluate an indirect partial derivative.

Solution Methods

There exist only a small number of unique solutions to the prompts shown in **Equations 2.1** through **2.6**. The possible solutions have been divided into categories, which will be referred to as solution methods. Distinct solution methods contain at least one logical step not observed in any other solution method, whereas variations contain the same logical process carried out in a slightly different manner. Before discussing each solution method, in **Section 2.4**, it is useful to know where the methods have room for variation.

Constant Variables

Variation can occur in solution methods depending on where the solver chooses to apply information about variables being held constant. Applying variable information early in a solution method can cause some terms to reduce to zero. This simplification makes the solution process faster and leaves less room for algebraic errors. However, the solver loses physical insight about the original expression. Setting a variable as constant removes one degree of freedom from the physical system, thus restricting the system's solution space. Applying constant-variable information is an epistemic game variation observed in **Kustusch *et al.* [3]**.

Forms of Total Differentials

Solution methods can also vary by the solver's choice of total differential form. The two distinct forms of total differentials are defined as follows.

Given a function $D(a,b,c) = a^2 + b^2 + c^2$, the general form of the total differential of D is:

$$dD = \left(\frac{\partial D}{\partial a}\right)_{b,c} da + \left(\frac{\partial D}{\partial b}\right)_{a,c} db + \left(\frac{\partial D}{\partial c}\right)_{a,b} dc \qquad (2.7)$$

The general form of the total differential applies information only about what variables describe a function; the actual expression defined as being the function, $a^2 + b^2 + c^2$ in this case, is irrelevant here. This form's partial derivatives are left unevaluated, and therefore have no explicit functional dependence. **Equation 2.7** is a true statement for any function existing in a space of three variables.

The function's expression is applied to the general total differential to obtain the specific total differential. The specific total differential is shown as a general total differential with each partial derivative evaluated, thus introducing explicit functional dependence. The applicability of the resulting total differential is restricted to the function capable of producing such a partial derivative combination. For example:

Given a function $D(a,b,c) = a^2 + b^2 + c^2$, the specific form of the total differential of D is:

$$dD = 2ada + 2bdb + 2cdc$$

The practical differences between the two forms of total differentials are minor; using the specific form instead of the general form only requires that a few direct partial derivatives be calculated.

2.4. Possible Solution Methods

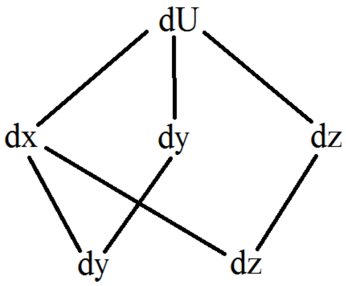
The solution methods for finding the prompts' indirect partial derivatives are described below. Each description provides an example where the respective solution method is applied to the Quiz 4 / 14 prompt (**Equations 2.1** through **2.3**). Solution methods will later be referred to by their abbreviated names, which are included in quotation marks in each method's label. For example: Solution Method 3 will be referred to as Diff RE.

The following solution methods were not written to be epistemic games. Instead, the solution methods were written to represent whole solutions. Each solution may be composed of one or more games. Most of these solution methods can be described by the epistemic games developed in **Kustusch *et al.* [3]**. Due to these equivalencies, the student solution methods observed in this project provide some independent verification of the results in **Kustusch *et al.* [3]**. Each solution method's description will note the respective counterpart in **Kustusch *et al.* [3]**.

Solution Method 1 – Chain Rule Diagram – "CRD"

Use a chain rule diagram of the initial function to build an unevaluated chain rule for the desired partial derivative.

1. Build a chain rule diagram for the desired partial derivative.



2. Construct a chain rule from the chain rule diagram.

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

The above chain rule is read from the chain rule diagram. However, the chain rule diagram innately applies variable information; the chain rule has already been simplified by applying $dy = 0$ and $dz/dz = 1$. The unsimplified chain rule for this problem would read as follows:

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial U}{\partial y}\right)_{x,z}\left(\frac{\partial y}{\partial z}\right)_y + \left(\frac{\partial U}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial z}\right)_y$$

3. Compute the chain rule's partial derivatives in order to obtain an expression for the desired partial derivative.

$$\left(\frac{\partial U}{\partial z}\right)_y = 2x(-e^z) + 2z$$

A chain rule diagram is a tool used to graphically represent differential interdependence. This tool allows the user to quickly construct a chain rule for whatever partial derivative is desired. This method has no explicit equivalent from **Kustusch *et al.* [3]**.

Solution Method 2 – Differential Division – "Diff Div"

Divide the initial function's total differential by a carefully chosen differential.

1. Calculate the specific total differential of $U = U(x, y, z)$.

$$U = x^2 + y^2 + z^2$$

$$dU = \left(\frac{\partial U}{\partial x}\right)_{y,z} dx + \left(\frac{\partial U}{\partial y}\right)_{x,z} dy + \left(\frac{\partial U}{\partial z}\right)_{x,y} dz$$

$$dU = 2xdx + 2ydy + 2zdz$$

2. Divide both sides of $dU = dU(x, y, z)$ by dz. This forms an incomplete mathematical statement that could represent one of two possible derivatives:

$$\left(\frac{\partial U}{\partial z}\right)_y \text{or} \left(\frac{\partial U}{\partial z}\right)_x$$

Given $dy = 0$, it is clear that we desire an expression for the former of these two partial derivatives.

$$\frac{dU}{dz} = \frac{2xdx + 2ydy + 2zdz}{dz}$$

$$\frac{dU}{dz} = 2x\frac{dx}{dz} + 2y\frac{dy}{dz} + 2x\frac{dz}{dz}$$

3. Mindfully transform the differential ratios into partial derivatives.

$$\left(\frac{\partial U}{\partial z}\right)_y = 2x\left(\frac{\partial x}{\partial z}\right)_y + 2y\left(\frac{\partial y}{\partial z}\right)_y + 2x$$

4. Calculate the remaining partial derivatives to obtain an expression for the desired partial derivative.

$$\left(\frac{\partial U}{\partial z}\right)_y = 2x(-e^z) + 2z$$

Diff Div is contained within the "Mathematically Illegal" epistemic game variant observed in **Kustusch *et al.* [3]** where the student or expert divides by a differential to get to the desired partial derivative.

Solution Method 3 – Differential Re-expression – "Diff RE"

Change the differentials of the initial function's total differential via substitution and then identify the desired partial derivative.

1. Obtain an expression for dx. Either solve $z = z(x, y)$ for $x = x(y, z)$ and then take the specific total differential of $x = x(y, z)$, as shown, or calculate the specific total differential of $z = z(x, y)$ and isolate dx.

$$z = \ln(y - x)$$

$$e^z = y - x$$

$$x = y - e^z$$

$$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$dx = 1dy - e^z dz$$

2. Calculate the specific total differential of $U = U(x, y, z)$.

$$U = x^2 + y^2 + z^2$$

$$dU = \left(\frac{\partial U}{\partial x}\right)_{y,z} dx + \left(\frac{\partial U}{\partial y}\right)_{x,z} dy + \left(\frac{\partial U}{\partial z}\right)_{x,y} dz$$

$$dU = 2xdx + 2ydy + 2zdz$$

3. Substitute $x = x(y, z)$ into $dU = dU(x, y, z)$.

$$dU = 2x(1dy - e^z dz) + 2ydy + 2zdz$$

4. Factor out differentials where possible.

$$dU = [2x + 2y]dy + [-2xe^z + 2z]dz$$

5. Identify the contents of the brackets multiplied to dz as the desired partial derivative.

$$\left(\frac{\partial U}{\partial z}\right)_y = -2xe^z + 2z$$

Diff RE is equivalent to the method observed in **Kustusch *et al.* [3]** where the student or expert starts in the differential game and changes to the substitution game, using differentials instead of variables.

Solution Method 4– Implicit Differentiation – "Imp Diff"

Directly calculate the desired partial derivative by using implicit differentiation (taking into account all relevant possible variable interdependencies) on the initial function.

1. Directly calculate the desired partial derivative using implicit differentiation.

$$\left(\frac{\partial U}{\partial z}\right)_y = 2x\left(\frac{\partial x}{\partial z}\right)_y + 2y\left(\frac{\partial y}{\partial z}\right)_y + 2z\left(\frac{\partial z}{\partial z}\right)_y$$

2. Compute the chain rule's partial derivatives in order to obtain an expression for the desired partial derivative.

$$\left(\frac{\partial U}{\partial z}\right)_y = 2x(-e^z) + 2z$$

This solution method is also known as "using the chain rule." This method has no explicit equivalent from **Kustusch *et al.* [3]**.

Solution Method 5 – Variable Re-expression – "Var RE"

Change the initial function's variables via substitution. Then, either directly calculate the desired partial derivative or identify the desired partial derivative from the function's specific total differential.

1. Solve $z = z(x, y)$ for $x = x(y, z)$.

$$z = \ln(y - x)$$

$$e^z = y - x$$

$$x = y - e^z$$

2. Substitute $x = x(y, z)$ in to $U = U(x, y, z)$, forming $U = U(y, z)$.

$$U = x^2 + y^2 + z^2$$

$$U = (y - e^z)^2 + y^2 + z^2$$

3. Directly calculate the desired partial derivative. Alternatively, calculate the specific total differential of $U = U(y, z)$ and identify the desired partial derivative.

$$\left(\frac{\partial U}{\partial z}\right)_y = 2(y - e^z)(-e^z) + 2z$$

This method is the only solution that changes an indirect partial derivative into a direct partial derivative. All other solutions instead create a chain rule. Var RE is also the only solution method not explicitly discussed in class. Changing variables is not possible for every function; Var RE is not a sure-fire solution method for solving thermodynamics-related chain rule problems. Thusly, Final 4.b was designed to make this solution method exceedingly difficult. The design was to drive students towards more widely applicable methods. Var RE is equivalent to the substitution epistemic game from **Kustusch *et al.* [3]**.

2.5. Instructional Timeline

An instructional timeline of solution methods, relevant quizzes, and lecture topics was extracted from classroom videos. This timeline provides a basis for analyzing why student solution methods changed throughout the term. The timeline also improves the understanding of how well students were prepared for the prompts. The full timeline of relevant materials is provided in **Appendix B**. **Table 2.1** contains a brief version of this timeline. Note that Var RE was not discussed in class.

Table 2.1 – A brief timeline of the thermodynamics course's relevant assignments and discussions.

BRIEF INSTRUCTIONAL TIMELINE	
DATE	**DIRECTLY RELEVANT MATERIAL**
4/20/2016	Imp Diff discussed
4/21/2016	Diff RE (general total differential) and CRD discussed
4/29/2016	Quiz 4 assigned, 7 responses collected
5/2/2016	Quiz 4 solutions discussed: CRD, Imp Diff, Diff Div, Diff RE (specific total differential)
5/13/2016	Quiz 14 assigned, 29 responses collected
5/16/2016	Final exam assigned, 27 responses collected

2.6. Coding Scheme Development

As is standard in physics education research, a coding scheme was created to simplify data analysis. Coding schemes are analysis tools used in qualitative research. Coding schemes are generally developed on a project-by-project basis. To develop a coding scheme, one starts by observing the data. Categories are then formed based on what similarities are seen between each piece of data. Categories are adjusted until they encompass the data. The code developer then writes instructions (code descriptions) for placing data in these categories. At this stage, the coding scheme must go through interrater reliability testing (IRT).

IRT requires an individual to act as the interrater reliability tester (IRTr). This person cannot be directly involved with the coding scheme development. The IRTr uses code descriptions to encode a small dataset. The IRTr then compares their results to the code developer's. The IRTr and code developer discuss the differences. Should the agreement be less than 80%, the code developer then revises the coding scheme and another round of testing is done.

The initial coding scheme, which was later abandoned, was created with Dr. Corinne Manogue's oversight. Mike Vignal, a graduate student colleague, acted as both interrater reliability tester and data anonymizer. The scheme is a language with syntax based on how students manipulate pieces. "Pieces" are individual mathematical relations containing a single variable, differential, or derivative equated to an expression in terms of other variables, differentials, or derivatives. Pieces are named according to the isolated term. For example, **Equations 2.1**, **2.2**, **2.4**, and 2.5 would be respectively referred to as U, z, S, and U. The coding syntax has two fundamental moves, or steps: creating new pieces from known pieces, and substituting known pieces into other known pieces; synthesis and substitution, respectively.

This coding scheme succinctly expresses student solutions (ideally) without losing information about student errors and variable information application. It is far easier to identify similarities between coded responses than it is to do so for raw student work. After the third round of IRT, the initial scheme's agreement was better than 80%.

Ultimately, the initial scheme proved to be a supreme distraction from the data; far more time was spent synonymizing coded data with raw data than was spent thinking about the data's meaning. Thus, coding was reverted to simply sorting student solutions according to their methods and errors. The initial coding scheme's final state can be seen in **Appendix C**.

The final coding scheme was a vastly simplified version of the initial coding scheme. The final scheme's coding had already been verified in the initial scheme's IRT. Therefore, no further IRT was performed. In the final coding scheme, student responses were sorted by solution method, and later by the errors they contained. Some student responses did not reflect any solution method. A blanket term, "Other," was created to encompass these responses.

"Other" can be broken down into three more specific categories. The first category is indeterminate responses. If it is hard to tell what the students were doing or thinking, the response is indeterminate. No indeterminate responses were observed in the data. The second category is dead-end methods. Dead-end methods are methods that cannot obtain acceptable solutions to the prompts but also contain no syntactical errors. Acceptable solutions do not contain indirect partial derivatives. The last category is flawed methods. Flawed methods are methods that depend on one or more conceptual errors. Making conceptual errors at certain points in any given solution method can create new, unique methods by significantly changing the initial solution method's process.

The final coding scheme's output is in **Appendix D**. **Chapter 3** discusses and analyzes solution methods observed in the data. Data from Quiz 4 was very limited, so the analysis of its student method statistics was mostly excluded. Unfortunately, this restricts the observation of student response evolution to Quiz 14 and Final 4.b. Flawed methods and dead-end methods will be discussed in **Section 4.3** and **Chapter 5** respectively.

Although this project does not focus on student answer correctness, each student's final answer was labeled as correct, partially correct, or incorrect. Student errors were categorized as either mathematical or conceptual. Algebraic, arithmetic, and sign errors are examples of mathematical errors, which are not a focus in this project. Conceptual errors are broadly defined here as errors pertaining to the differential and partial derivative manipulations. Student conceptual errors were sorted into categories for further analysis. Further discussion and analysis of student errors is present in **Chapter 4**.

CHAPTER 3 – STUDENT SOLUTION METHODS

Students used various solution methods to respond. Success rates and solution method prominence varied per assignment.

Each prompt’s student method distribution is shown in **Section 3.1**. **Section 3.2** contains an analysis of how students changed their responses between Quiz 14 and Final 4.b. Quiz 4’s data was incomplete. Therefore, Quiz 4 was mostly excluded from this analysis. Some student responses had work reflecting two correct methods. These responses were marked and counted as containing both methods.

3.1. Student Method Distribution

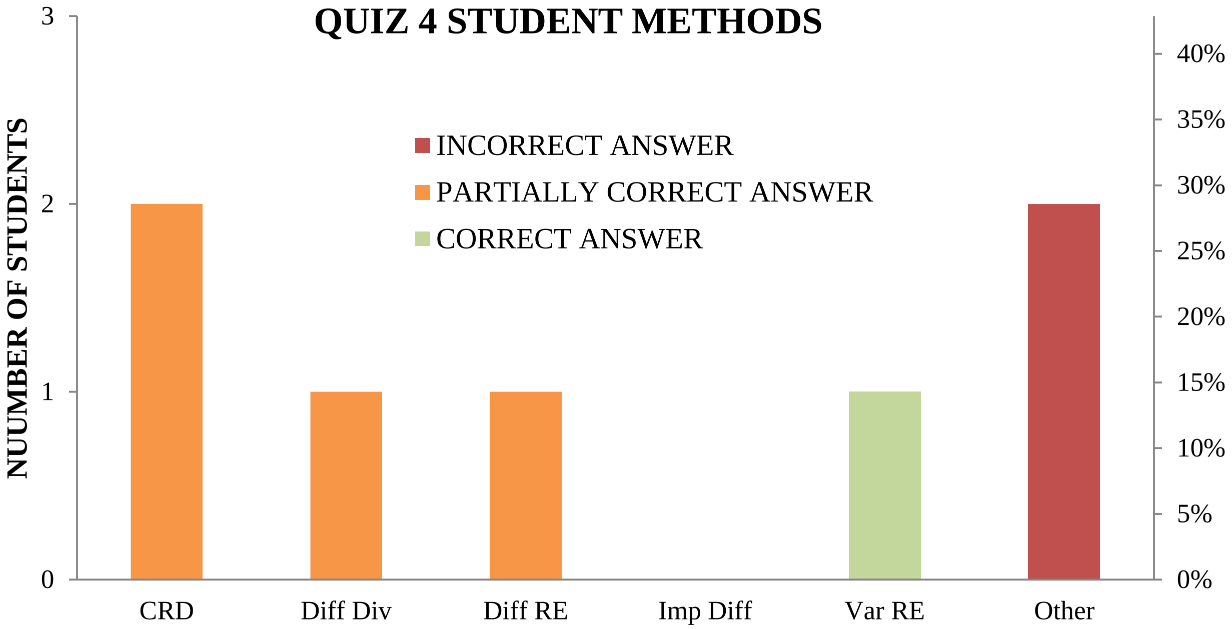


Figure 3.1 – Quiz 4’s student method distribution. Only seven Quiz 4 responses were available for analysis.

As seen in **Figure 3.1**, students used a variety of methods to respond to Quiz 4. However, students were generally unsuccessful at solving Quiz 4’s prompt. Only one response contained a complete solution. Students were struggling with the solution methods, with the possible exception of Var RE. When students were assigned Quiz 4, they had been shown three solution methods in class: Imp Diff, Diff RE, and CRD. However, Var RE was present in one of Quiz 4’s responses. It is not known whether the student transferred this method from their previous mathematics courses, or discovered the method during the first two course weeks. The instructor presented the class with every possible solution method, save for Var RE, the class day after Quiz 4 was assigned. Note that only seven responses to Quiz 4 were available for analysis. With so few data points for Quiz 4, it is difficult to make specific statements about how Quiz 4’s method distribution compares to those in Quiz 14 and Final 4.b.

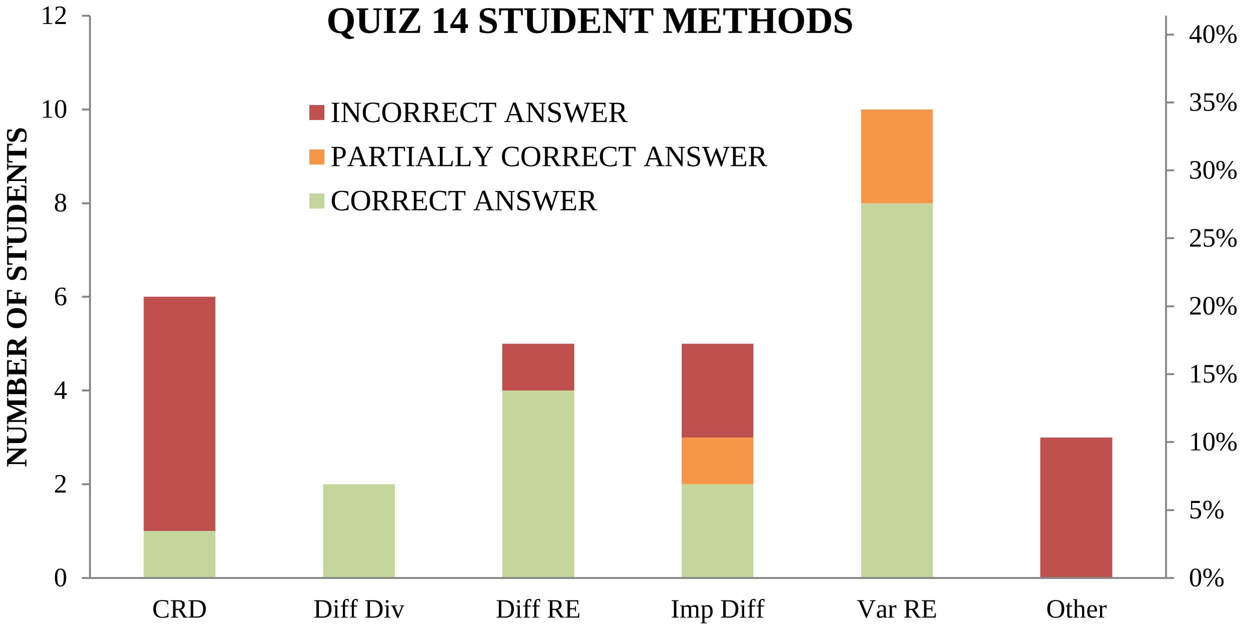


Figure 3.2 – Quiz 14's student method distribution. One student was counted as both Diff Div and CRD, and another student was counted as both Diff Div and ImpDiff.

See **Figure 3.2** for Quiz 14's student method distribution. Note that one student was counted as both Diff Div and CRD, and another student was counted as both Diff Div and ImpDiff. Students used various methods to respond to Quiz 14's prompt. Var RE was by far the most prominent student method. About 70% of students obtained a correct or partially correct answer. The success rate could have risen due to former student experience with the prompt. Students had already responded to the same prompt in Quiz 4, which the instructor then solved in class via every method but Var RE. Increased aptness of students to study for quizzes could also explain the improvement. Given that Var RE was not shown in class, it is curious that so many students used Var RE successfully. One or more students could have transferred or discovered this solution method and shared it with their classmates.

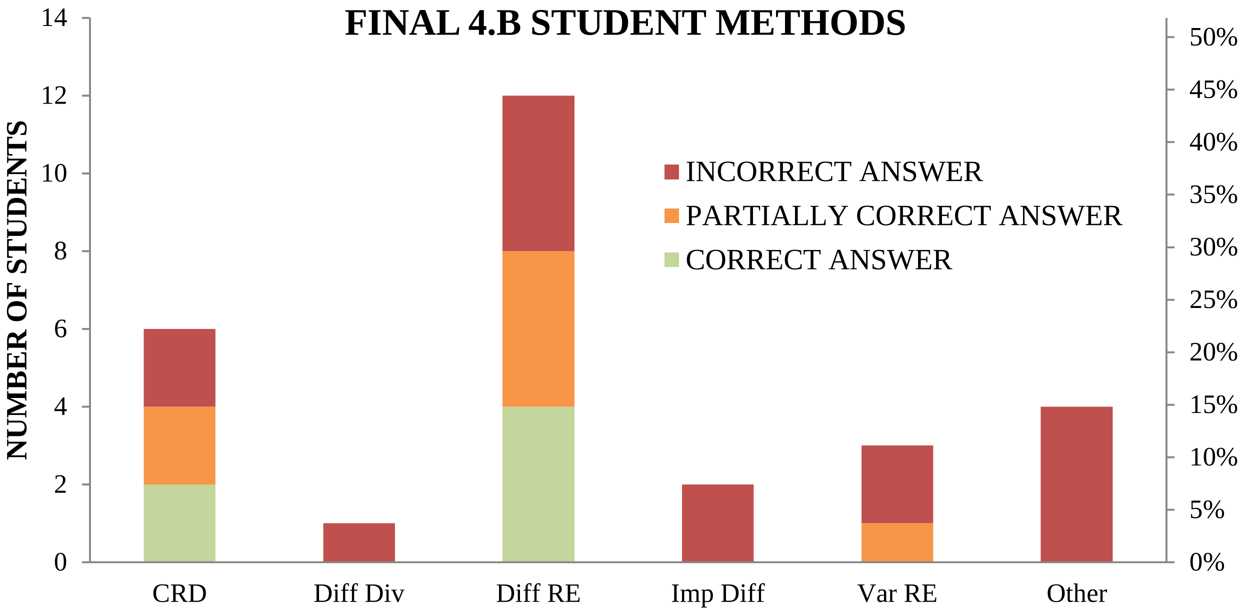


Figure 3.3 – Final 4.b's student method distribution. One student was counted as both CRD and Diff Re.

See **Figure 3.3** for Final 4.b's student method distribution. One student was counted as both CRD and Diff RE. Similar to Quiz 14, students used many methods to respond to Final 4.b. Only ~45% of students obtained a correct or partially correct answer on Final 4.b. Diff RE became the most popular solution method, in place of Var RE on Quiz 14. Final 4.b proved to be a much greater challenge for students than Quiz 14 had. This is likely due to the greater mathematical complexity of Final 4.b's prompt equations. This suspicion will be confirmed in **Chapter 4**. There was a redistribution and convergence of students' method choices from Quiz 14 to Final 4.b. This reduction in diversification will be further discussed in **Section 3.3**. Var RE and Imp Diff were all but abandoned in the final exam. It appears that most students who used Var RE in Quiz 14 moved to either CRD or Diff RE on Final 4.b. This suspicion will be validated in **Section 3.3**.

3.2. Student Method Migration

From Quiz 14 to Final 4.b, there was a redistribution and convergence of students' method choices. As seen in **Figures 3.2** and **3.3**, both Imp Diff and Var RE dwindled while Diff RE became more prominent. Students realized that Var RE was not a good method to use on Final 4.b, and chose other methods instead. Final 4.b was designed to make Var RE a difficult solution method. Most students subsequently abandoned Var RE and instead used Diff RE or CRD.

The numbers in **Tables 3.1** and **3.2** do not add up perfectly. Two students did not attempt to respond to Final 4.b. Also note that students who appeared to use multiple methods, or whose work resembled more than one method, were counted as each relevant method. In Quiz 14, one student was counted as both Diff Div and CRD, and another student was counted as both Diff Div and ImpDiff. One student was counted as both CRD and Diff RE in Final 4.b.

Table 3.1 - Students changing methods between Quiz 14 and Final 4.b.

STUDENT METHOD MIGRATION

		FINAL 4.B					
		CRD	DIFF DIV	DIFF RE	IMP DIFF	VAR RE	OTHER
QUIZ 14	CRD	4	0	2	0	1	0
	DIFF DIV	0	0	1	1	0	0
	DIFF RE	0	0	4	1	0	0
	IMP DIFF	0	1	1	1	0	1
	VAR RE	2	0	5	0	1	1
	OTHER	0	0	0	0	1	2

Table 3.1 shows the change of students' method choices from Quiz 14 to Final 4.b. **Table 3.2** summarizes the data shown in **Table 3.1**. **Table 3.2**'s columns are left to right as follows: responses per method in Quiz 14, number of students who used the same method in Final 4.b (S, for same), number of students who changed from the method in Final 4.b (O, for outflow), number of students who changed to the method in Final 4.b (I, for influx), and the responses per method in Final 4.b.

When asked to respond to a difficult chain-rule problem, most students who abandoned Var RE instead used Diff RE or CRD. From these tables, we can see that CRD and Diff RE retained the most students, and that most students who changed methods moved to Diff RE.

Little more can be said without discussing the errors students made while using these methods. This discussion will take place in **Chapter 4**.

Table 3.2 – Summarization of data shown in Table 4.2, showing the influx (I) and outflow (O) of students from each method, as well as how many students kept the same (S) method, between Quiz 14 and Final 4.b.

STUDENT METHOD MIGRATION SUMMARY

METHOD	QUIZ 14	S	O	I	FINAL 4.B
CRD	6	4	2	2	6
DIFF DIV	2	0	2	1	1
DIFF RE	5	4	1	9	12
IMP DIFF	5	0	4	2	2
VAR RE	10	1	9	2	3
OTHER	3	2	1	2	4

CHAPTER 4 – STUDENT ERRORS

Many conceptual and mathematical errors were present in student responses. Mathematical errors, such as algebraic, arithmetic, and sign errors, are not a focus of this project. Errors pertaining to the manipulation of differentials and partial derivatives, and the use of chain rule diagrams are considered to be conceptual errors. Conceptual errors are this project's primary focus.

The error distribution observed in the data is shown and discussed in **Section 4.1**. **Section 4.2** contains information about conceptual error categories. **Section 4.2** also discusses correlations between method use evolution and error occurrence. Flawed method case studies are provided in **Section 4.3**. Flawed methods are methods that depend on one or more conceptual errors. In flawed methods, an error(s)'s occurrence dictates the steps that follow. Had the error(s) not been made, the response would contain different steps.

4.1.Student Error Distribution

The conceptual error distribution was shown alongside mathematical errors in **Figure 4.1**. Mathematical errors are included here to help explain the drop in success rates between Quiz 14 and Final 4.b, as observed in **Figures 3.2** and **3.3**. Students who made both conceptual and mathematical errors are counted in both columns. Two students made both conceptual and math errors on Quiz 14. Please consider the number of observed responses to each prompt during any following comparisons of Quiz 4, Quiz 14, and Final 4.b. These assignments had 7, 29, and 27 responses respectively.

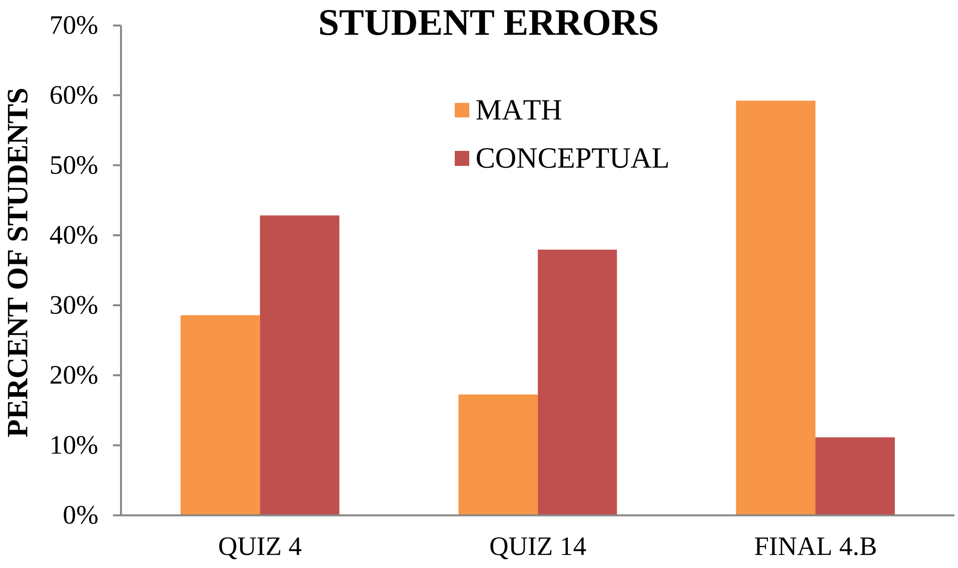


Figure 4.1 – The error distribution observed in student responses. Two students made both conceptual and math errors in Quiz 14, and are included in both respective columns.

As is reflected in **Figure 4.1**, there was a significant decrease in mathematical errors between Quiz 4 and Quiz 14. This trend was not upheld in the final exam. The poor success rate observed in **Figure 3.3** is due to the numerous mathematical errors present in the responses to Final 4.b. The sudden jump in mathematical errors could be explained by the greater extent of algebraic manipulations required to correctly solve Final 4.b; Final 4.b's prompt simply requires more algebra than the prompt in Quiz 4 and Quiz 14. Conceptual errors declined over the course of the thermodynamics class, most significantly so between Quiz 14 and Final 4.b. Very few students made conceptual errors on Final 4.b. Students must have done a great deal of studying over the two days between these assignments.

4.2. Common Conceptual Errors

Most observed conceptual errors took one of five forms, all of which are described below. These conceptual errors will be later referred to by capitalizing the first letter of any word form of the error's shortened name. For example, the words Equated, Equating, and Equation all refer to Error 1, but the words equated, equating, and equation do not.

Error 1 – Erroneous Direct Equating of Partial Derivatives – "Equating"

The student directly equated two unequal partial derivatives. The only two cases of this error in the dataset were observed in the responses to Final 4.b. Both cases took the following form:

$$\left(\frac{\partial U}{\partial V}\right)_S = \left(\frac{\partial U}{\partial V}\right)_T$$

Error 2 – Erroneous Choice of Chain Rule Partial Derivatives – "Choice"

The student did not choose the correct partial derivatives when building a chain rule. Given the Quiz 4/14 prompt, the correct chain rule for the desired partial derivative is:

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

The following are incorrect versions of the above chain rule, each containing one or more erroneous partial derivative choices:

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_y + \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial x}\right)_{y,z}\left(\frac{\partial z}{\partial y}\right)_x + \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial y}\right)_{x,z}\left(\frac{\partial y}{\partial x}\right)_z + \left(\frac{\partial x}{\partial y}\right)_z$$

Error 3 – Creation of Incomplete Chain Rule – "Incomplete"

The student built a two-dimensional chain rule that was missing one or more partial derivatives; the chain rule was not in the proper format for a two-dimensional chain rule. This error is only concerned with the chain rule's partial derivative layout; the correctness of the chain rule's partial derivatives is irrelevant. Given the Quiz 4/14 prompt, the correct chain rule for the desired partial derivative is:

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

The following are incorrect versions of the above chain rule, each missing a partial derivative:

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial x}\right)_{y,z} + \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial z}\right)_y$$

Error 4 – Differential Expression Equated with a Partial Derivative – "Differential"
The student equated a partial derivative with an expression containing a differential. The following is the most prominent example of this error in the dataset.

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial z}\right)_{x,y} dz\,;\ \left(\frac{\partial U}{\partial z}\right)_y = 2zdz$$

Error 5 – Erroneous Identification of a Partial Derivative – "Misidentification"
The student incorrectly identified a term or set of terms as a partial derivative. The following is the dataset's most prominent example of this error.

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial z}\right)_{x,y} dz\,;\ \left(\frac{\partial U}{\partial z}\right)_y = 2zdz$$

Table 4.1 shows the appearances of the above errors in the dataset. The table's cells contain the student numbers of students who made the corresponding error on the corresponding assignment. When reading this table, recall that there were 7, 29, and 27 responses to Quiz 4, Quiz 14, and Final 4.b respectively.

Table 4.1 – The student numbers of students who made conceptual errors in their responses.

COMMON CONCEPTUAL ERRORS			
ERROR	**QUIZ 4**	**QUIZ 14**	**FINAL 4.B**
EQUATING	-	-	4, 28
CHOICE	-	13, 17, 20, 23, 24	16
INCOMPLETE	13	17, 23, 28, 29	16
DIFFERENTIAL	11	4, 11, 16, 28	-
MISIDENTIFICATION	4, 11	4, 11, 16, 28	-

Four students made Differential and Misidentification errors simultaneously. This correlation may be causation. The simultaneity of Differential and Misidentification errors could have occurred due to the presence of the dz differential in the corresponding term of U's total differential. Consider the following:

Note the highlighted term of the general total differential of **Equation 2.1**'s expression for U:

$$dU = \left(\frac{\partial U}{\partial x}\right)_{y,z} dx + \left(\frac{\partial U}{\partial y}\right)_{x,z} dy + \left(\frac{\partial U}{\partial z}\right)_{x,y} dz$$

Misidentifying this term as the partial derivative in **Equation 2.3** led to that partial derivative being equated to an expression containing dz:

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial z}\right)_{x,y} dz$$

Every common conceptual error that occurred in Quiz 14 became less prevalent on the final exam. No Misidentification or Differential errors were present in the Final 4.b responses. Interestingly, the two students who made the Equating error in the final exam had made both the Misidentification and Differential errors in their responses to Quiz 14. These students' understanding must have evolved similarly. By observing **Tables 3.1**, **3.2**, and **4.1**, we can see the evolution of relations between errors and methods.

CRD shows significant improvement between Quiz 14 and Final 4.b. Students had great difficulty performing CRD on Quiz 14. In Quiz 14, students tended to make Choice and Incomplete errors while using CRD. Students who made Choice errors often created chain rule diagrams that did not match the desired partial derivative. These students did not correctly link the differentials in their diagrams. Specifically, the students chose the wrong differential to express in terms of other differentials. Students who made Incomplete errors while using CRD had varying technical misunderstandings of the workings of chain rule diagrams.

Since Quiz 14 was assigned on the last day of the class, one could assume that students have difficulty obtaining a fully functional understanding of chain rule diagrams within the thermodynamics course's timeframe. However, of the six students who used a chain rule diagram on Final 4.b, only one made a conceptual error. Four of the five students who made a conceptual error with CRD on Quiz 14 made no conceptual error while using CRD on Final 4.b. Over the weekend between Quiz 14 and the final exam, these four students suddenly became masters of the chain rule diagram! None of the possible solution methods were discussed during the final exam review session, which occurred on the same day as Quiz 14. These students' improvements on CRD must have been due to out-of-class studying, likely prompted by the upcoming final exam and students' poor success with CRD on Quiz 14. The only student who made a conceptual error in CRD on Final 4.b was student 16. Student 16 did not use the CRD method in Quiz 14.

Student 16's response to Quiz 14 is the only case in the dataset where a student made a conceptual error during Diff RE. Furthermore, the four students who successfully used Diff RE on Quiz 14 also used it on final 4.

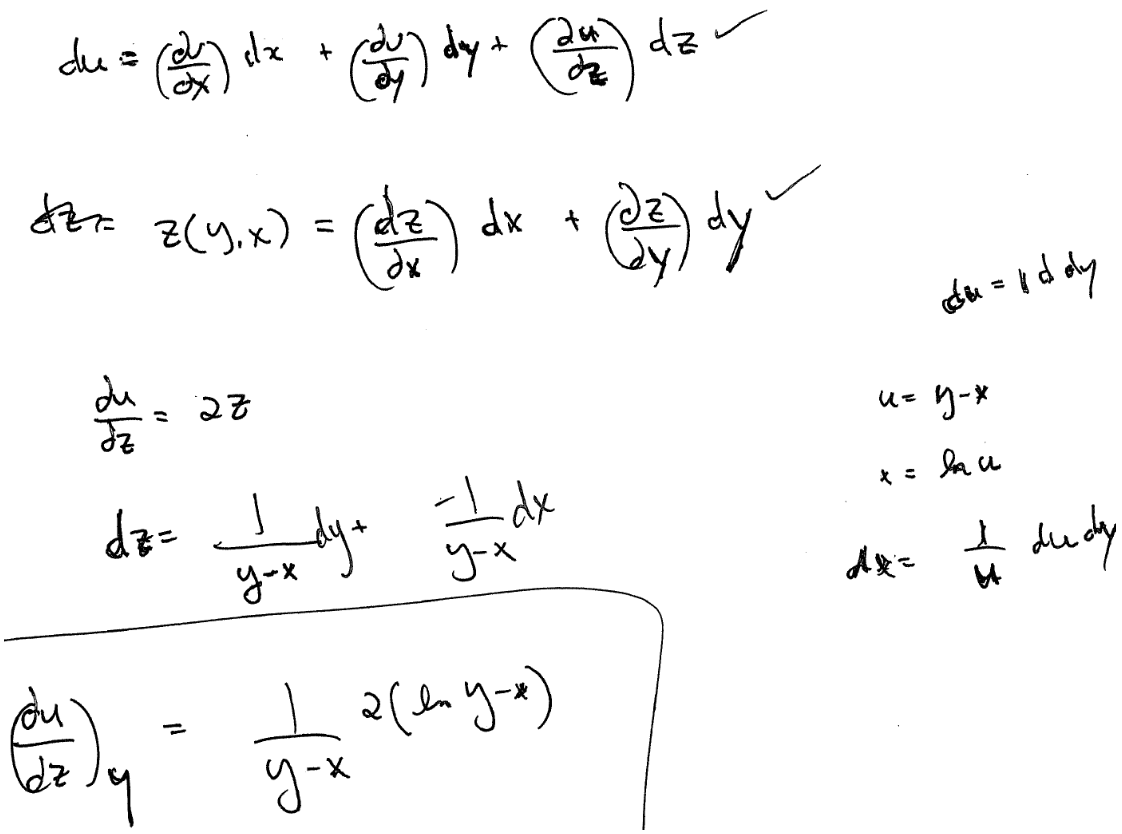
Many student responses contained flawed methods. **Section 4.3** contains case studies of every student whose response contained a flawed method.

4.3. Flawed Methods

Multiple flawed methods were observed in the dataset. Flawed methods are dependent on their contained conceptual errors; in each flawed method, the occurrence of one or more conceptual errors dictates the steps that follow. Student responses that contain flawed methods will be described in detail. Doing so serves two purposes: to distinguish the flawed methods from mis-performed solution methods and to provide conceptual error case-studies.

Students who made Choice and Incomplete errors while using CRD are not included in these case studies. Although Choice and Incomplete errors are considered conceptual for the purpose of this project, they result from technical misunderstandings of CRD use. Additionally, errors made while using CRD cannot lead to flawed methods since diagram use is what distinguishes CRD as a unique method; no matter what errors the student makes while using CRD, the fact remains that they were trying to create and read a chain rule diagram. Any student who used a chain rule diagram was using the CRD method.

The figures on the following pages portray some common student errors and the resulting flawed methods. **Figures 4.3**, **4.4**, **4.5**, and **4.8** show the work of students operating based on their misconceptions of what it means to hold a variable constant. **Figures 4.2**, **4.6**, and **4.7** show flawed methods resulting from student misidentification of partial derivatives.



$du = \left(\frac{\partial U}{\partial x}\right) dx + \left(\frac{\partial U}{\partial y}\right) dy + \left(\frac{\partial u}{\partial z}\right) dz$ ✓

$dz = z(y,x) = \left(\frac{dz}{dx}\right) dx + \left(\frac{\partial z}{dy}\right) dy$ ✓

$du = 1\, d\, dy$

$\frac{du}{\partial z} = 2z$

$u = y - x$

$x = \ln u$

$dz = \frac{1}{y-x} dy + \frac{-1}{y-x} dx$

$dx = \frac{1}{u} du\, dy$

$\left(\frac{\partial u}{dz}\right)_y = \frac{1}{y-x} 2(\ln y - x)$

Figure 4.2 - Student 4's response to the Quiz 4 prompt.

As seen in **Figure 4.2**, student 4 obtained their final result to Quiz 4 by treating the dz term of U's total differential as equivalent to the desired partial derivative. The third line of work shows the student solving what should be labeled as $\left(\frac{\partial U}{\partial z}\right)_{x,y}$. The student found and substituted in an expression for dz, so this method bears some resemblance to Diff RE. However, in their final line of work, the student Misidentified the dz term of the total differential of U as the desired partial derivative. The student then substituted in their result from line 3, the prompt's expression for z, and line 4's expression for dz. Student 4 had some reason to not include a differential in their solution, as the dy from the expression for dz is not present in the student's answer. The student may have recognized that they should not set a differential expression equal to a partial derivative. However, it is also possible that the student dropped the dy from their solution, believing that this was equivalent to saying $dy = 0$.

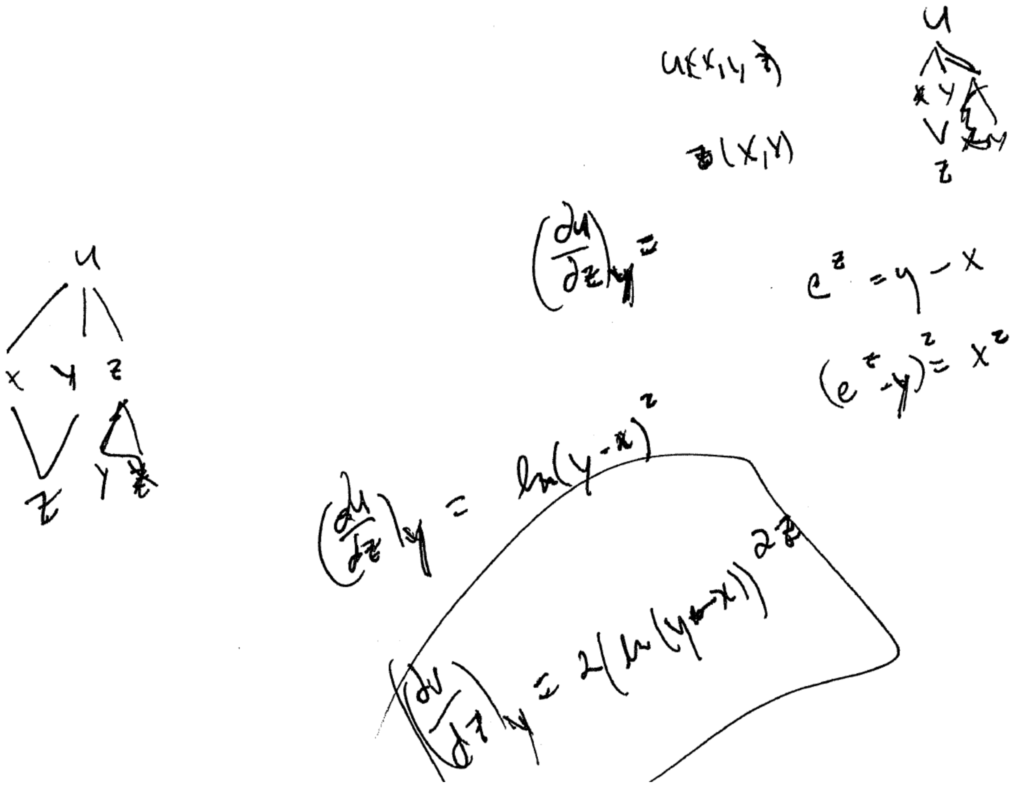
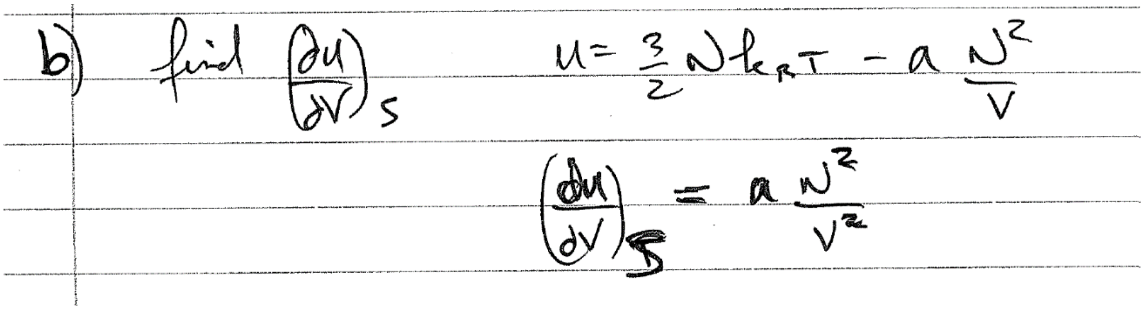


Figure 4.3 - Student 4's response to the Quiz 14 prompt.

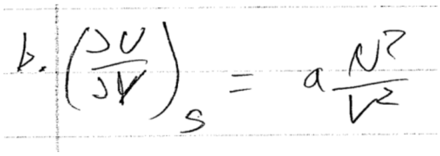
It is difficult to decipher the method used in the response seen in **Figure 4.3**. The student clearly did not intend to convey that $\ln(y-x)^2$, which is equivalent to z^2, is equal to the desired partial derivative. Although the student set the desired partial derivative equal to the z^2 term, they evaluated the derivative in their next step. It is not a reasonable assumption that the student intended to implicitly equate an equation with its own derivative. Based on their action in the next step, this line was meant to say that z^2 is the only term that need be considered to calculate the desired partial derivative. In their last step, student 4 calculated the desired partial derivative from their identified term in the previous step. This error is unique in the dataset, and is technically not Misidentification. A Misidentification error is where a student identified a term as equal to a partial derivative. The student also made a Differential error in their final, boxed step by multiplying the partial derivative's result by dz.



b) find $\left(\frac{\partial u}{\partial V}\right)_S$ $\quad u = \frac{3}{2}Nk_BT - a\frac{N^2}{V}$

$\left(\frac{du}{dV}\right)_S = a\frac{N^2}{V^2}$

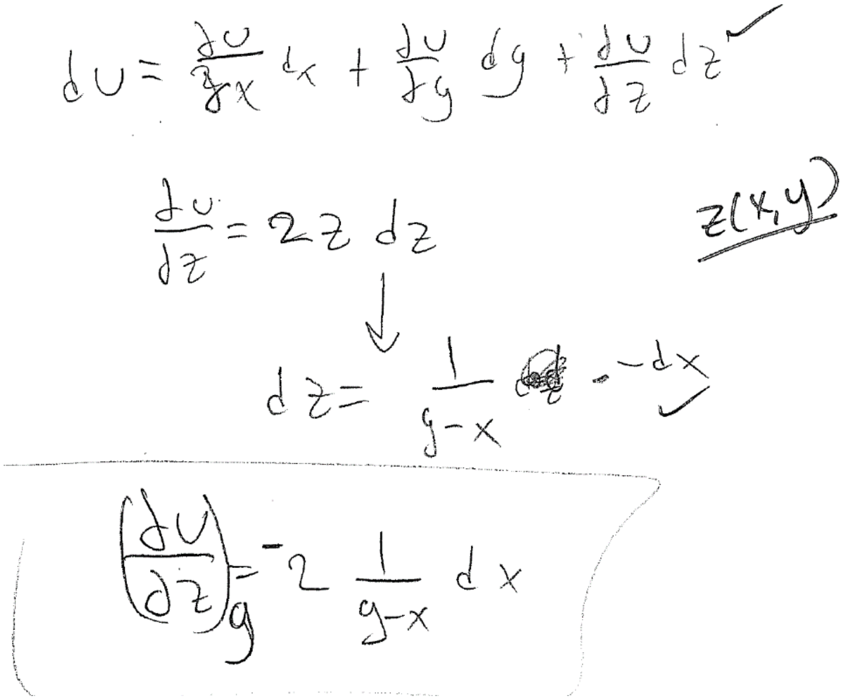
Figure 4.4 - Student 4's response to the Final 4.b prompt.



$$b.\ \left(\frac{\partial U}{\partial V}\right)_S = a\frac{N^2}{V^2}$$

Figure 4.5 - Student 28's response to the Final 4.b prompt.

Student 4's response to the Final 4.b prompt can be seen in **Figure 4.4**. As in both Quiz 4 and Quiz 14, student 4 pays no heed to variable dependencies not explicitly noted in the partial derivative. In this case, the student does not consider how the desired partial derivative depends on the other variable in the expression, temperature T. However, student 4 did not multiply their final result by a differential, as they did in their response to Quiz 14. This Equating error turned the prompt into a one-step problem. **Figure 4.5** shows student 28's work on Final 4.b. This student's method is identical to that used by student 4 on Final 4.b. In Quiz 14, student 28 made Incomplete, Differential, and Misidentification errors while using Imp Diff.



$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \checkmark$$

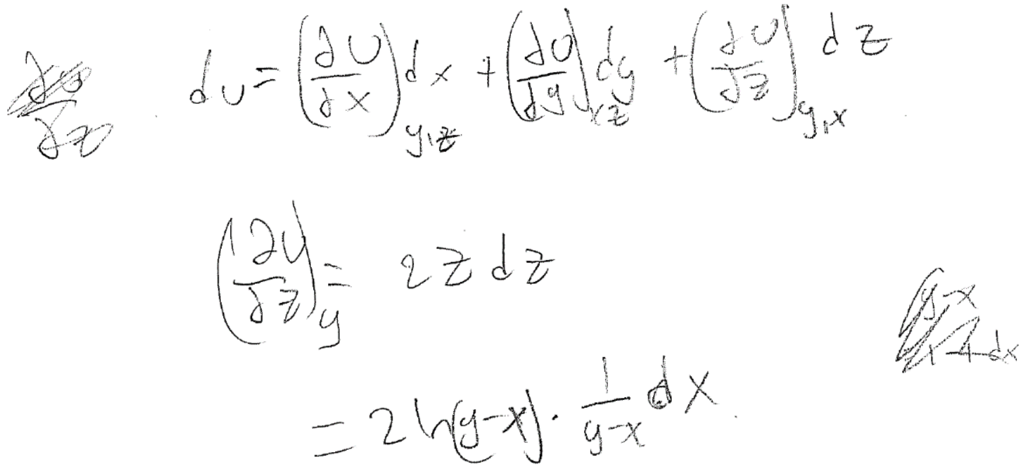
$$\frac{\partial U}{\partial z} = 2z\,dz \qquad z(x,y)$$

$$\downarrow$$

$$dz = \frac{1}{y-x} - dx \checkmark$$

$$\left(\frac{\partial U}{\partial z}\right)_y = 2\frac{1}{y-x}dx$$

Figure 4.6 - Student 11's response to the Quiz 4 prompt.



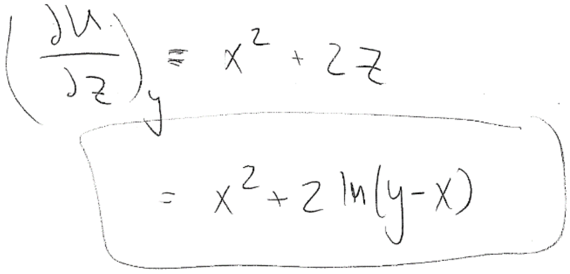
$$du = \left(\frac{\partial U}{\partial x}\right)_{y,z} dx + \left(\frac{\partial U}{\partial y}\right)_{x,z} dy + \left(\frac{\partial U}{\partial z}\right)_{y,x} dz$$

$$\left(\frac{\partial U}{\partial z}\right)_y = 2z\,dz$$

$$= 2\ln(y-x) \cdot \frac{1}{y-x} dx$$

Figure 4.7 - Student 11's response to the Quiz 14 prompt.

Figures 4.6 and **4.7** show student 11's work from Quiz 4 and Quiz 14 respectively. These are identical methods, setting aside the missing z in the last step of **Figure 4.6** and the sign error in the last step of **Figure 4.7**. This student used a flawed method similar to that seen in student 4's response to Quiz 4. In both cases, student 11 Misidentified the dz component of U's total differential as the desired partial derivative. The student found an expression for dz and substituted it into the misidentified term. In Quiz 14, the student also substituted in the prompt's equation for z. Both of student 11's solutions contain a differential. As described at the beginning of this chapter, this student's Differential error is likely a result of their Misidentification error. The student's final result has a differential because there was a differential in the term the student Misidentified as the desired partial derivative. Student 11 was one of the two students who used a dead-end method in Final 4.b. Student flailing will be discussed in **Chapter 5**.



$$\left(\frac{\partial u}{\partial z}\right)_y = x^2 + 2z$$

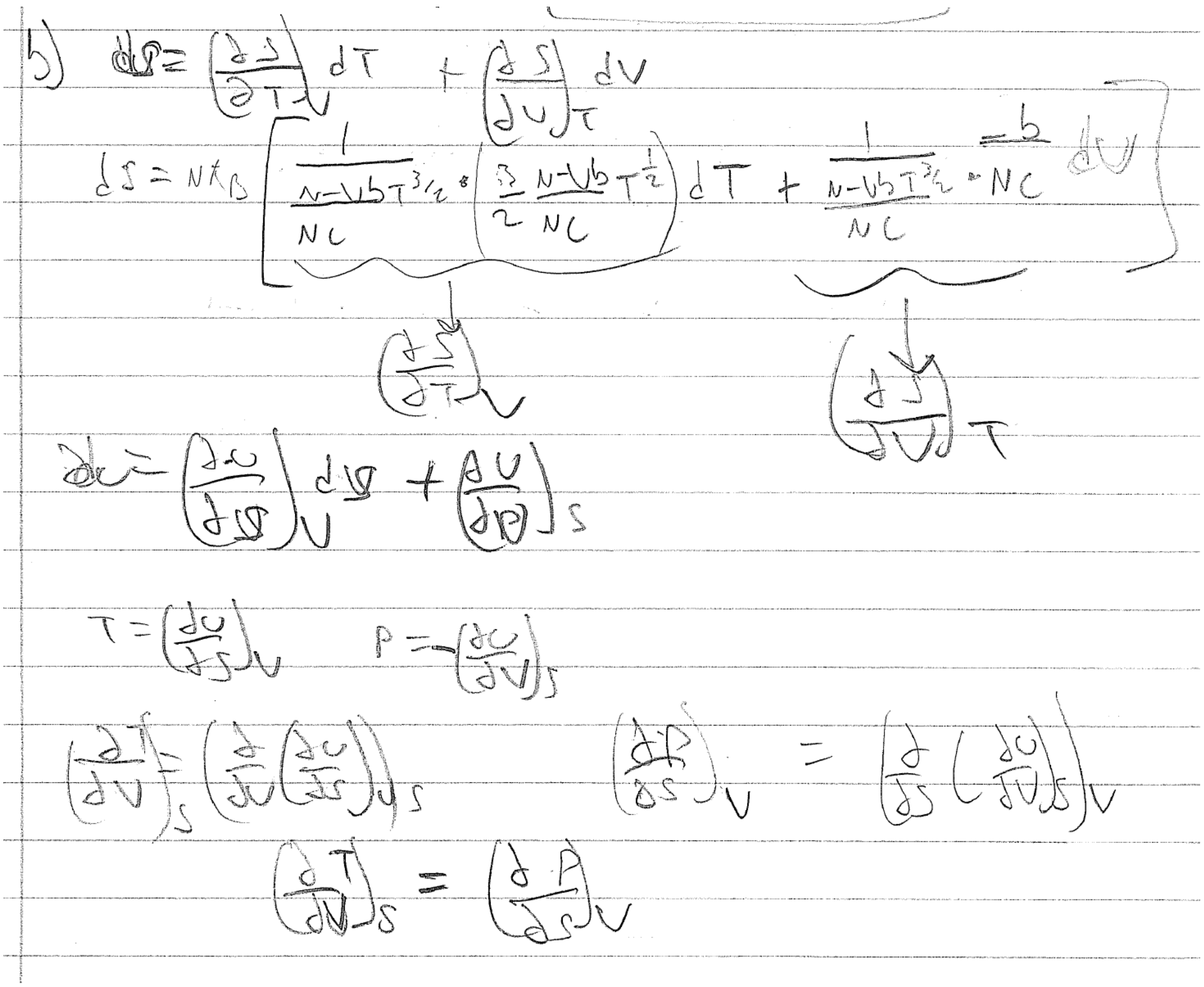
$$\boxed{= x^2 + 2\ln(y-x)}$$

Figure 4.8 - Student 9's response to the Quiz 14 prompt.

Figure 4.8 shows student 9's work from Quiz 14. Student 9 made a unique conceptual error in their response. The x^2 term was treated as if it were unaffected by the partial derivative. In their final answer, the student substituted in the prompt's expression for z. Although it bears resemblance to an Equating error, this student's error is unique in the dataset. Student 9 went on to use Var RE, without any conceptual error, in Final 4.b.

CHAPTER 5 – DEAD-END METHODS

Dead-end methods are methods that do not contain errors but do not yield acceptable solutions; while dead-end methods are methods, they are not *solution* methods. Acceptable solutions contain no indirect partial derivatives. Only one dead-end method was observed in the dataset. In this method, students used the thermodynamic identity to create an expression for the desired partial derivative. The students started by either explicitly writing or creating an expression from the thermodynamic identity. The students then identified the desired partial derivative from the expression and equated it with pressure p. This method will be referred to as ThermIdent. It is worth noting that part c of final exam question 4 asked students to use ThermIdent. Neither student who used ThermIdent on Final 4.b did their work out of order. Additionally, both students referred to their work in Final 4.b while responding to Final 4.c. This confirms that the below responses are indeed responses to Final 4.b. ThermIdent was described as an epistemic game variant in **Kustusch *et al.* [3]**. This method could be used to solve the prompt from **Kustusch *et al.* [3]**. ThermIdent does not apply to the Quiz 4/14 prompt since the prompt's equations have no explicit thermodynamic context. Although Final 4.b's prompt has thermodynamic context, the problem is not designed in a way that allows ThermIdent to yield an acceptable solution.



b) $dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$

$dS = Nk_B \left[\frac{1}{\frac{N-VbT^{3/2}}{NC}} * \left(\frac{3}{2}\frac{N-Vb}{NC}T^{\frac{1}{2}}\right) dT + \frac{1}{\frac{N-VbT^{3/2}}{NC}} * \frac{-b}{NC}\, dV \right]$

$\left(\frac{\partial S}{\partial T}\right)_V$ $\left(\frac{\partial S}{\partial V}\right)_T$

$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S$

$T = \left(\frac{\partial U}{\partial S}\right)_V$ $P = -\left(\frac{\partial U}{\partial V}\right)_S$

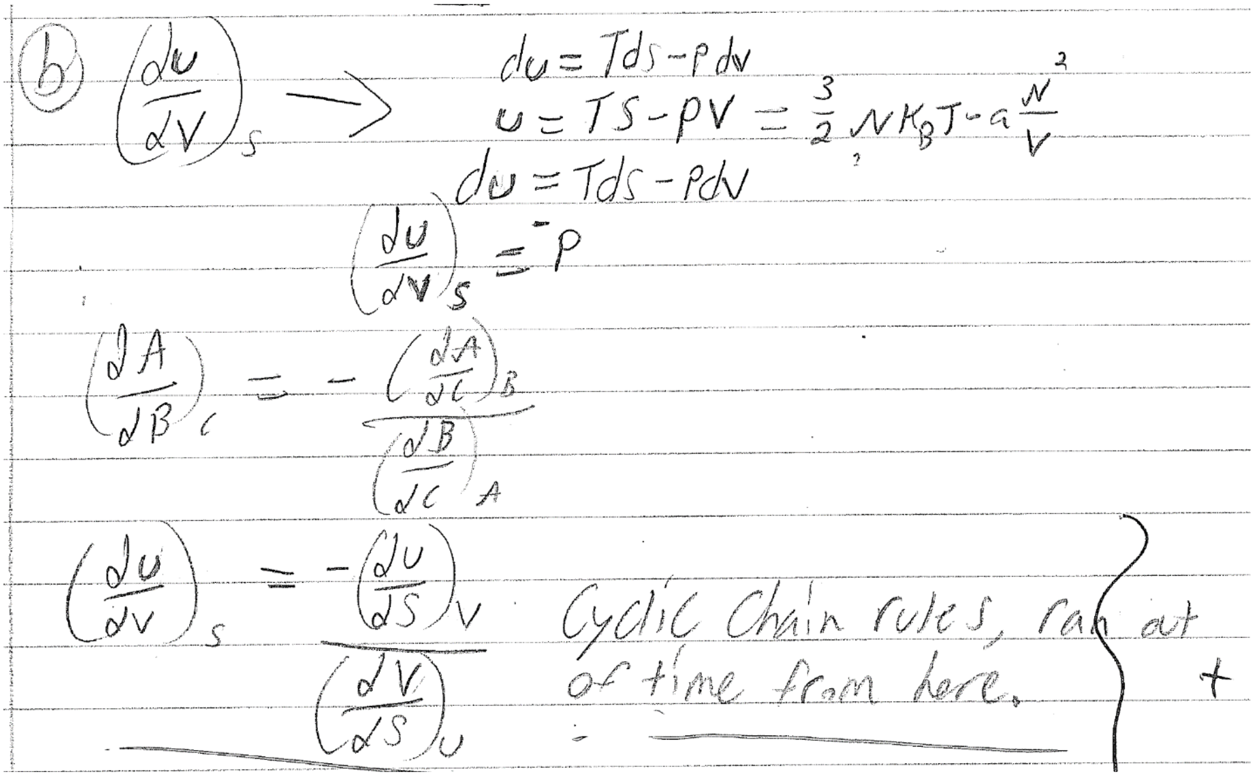
$\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{\partial}{\partial V}\left(\frac{\partial U}{\partial S}\right)\right)_S$ $\left(\frac{\partial P}{\partial S}\right)_V = \left(\frac{\partial}{\partial S}\left(\frac{\partial U}{\partial V}\right)_S\right)_V$

$\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{\partial P}{\partial S}\right)_V$

Figure 5.1 - Student 11's response to the Final 4.b prompt.

Student 11's approach to Final 4.b can be seen in **Figure 5.1**. After creating an expression for U's general total differential, the student correctly identified temperature T and the negative of pressure p from U's total differential.

It is clear that this student had the thermodynamic identity in mind. In this prompt, the desired partial derivative happens to be equivalent to the negative of pressure p. Student 11 then formed a Maxwell relation, presumably in an attempt to express pressure p in terms of the prompt's variables. Unfortunately, this process is not possible for Final 4.b's prompt. Final 4.b's prompt contains no information about p. Student 11 used a flawed method in their responses to Quiz 4 and Quiz 14.



ⓑ $\left(\frac{du}{dV}\right)_S \longrightarrow$

$du = Tds - pdv$

$u = TS - pV = \frac{3}{2} N k_B T - a \frac{N^2}{V}$

$du = Tds - PdV$

$\left(\frac{du}{dV}\right)_S = -p$

$\left(\frac{\partial A}{\partial B}\right)_C = -\frac{\left(\frac{\partial A}{\partial C}\right)_B}{\left(\frac{\partial B}{\partial C}\right)_A}$

$\left(\frac{du}{dv}\right)_S = -\frac{\left(\frac{dU}{dS}\right)_V}{\left(\frac{dV}{dS}\right)_U}$

Cyclic Chain rules, ran out of time from here.

+

Figure 5.2 - Student 19's response to the Final 4.b prompt.

Student 19's approach to Final 4.b can be seen in **Figure 5.2**. The student equated the thermodynamic identity to the prompt's expression for U, correctly identifying that the desired partial derivative was equivalent to the negative of pressure p. Student 19 then split the desired partial derivative using a cyclical chain rule. Even if the student had had time to continue their process, both partial derivatives produced by the cyclical chain rule are indirect partial derivatives; performing the cyclical chain rule moved the student further from finding a derivative-less expression for the desired partial derivative. On Quiz 14, student 19 had successfully used Var RE.

Both student 11 and student 19 used the dead-end method ThermIdent. Since this method is not applicable to the Quiz 4 and Quiz 14 prompts, it is impossible to discuss the ThermIdent's evolution over the course. It is even difficult to know when students became aware of this method. Would students have applied this method to Quiz 4 or Quiz 14 if these assignments had been designed to allow it? Students had been shown the thermodynamic identity, cyclical chain rule, and Maxwell equations before the third class week; even before Quiz 4 was assigned, students had the information necessary to produce the work seen in **Figures 5.1** and **5.2**. If these quizzes had been designed to permit ThermIdent's use, some students may have used this method. However, it is possible that students did not have enough time to process the material relevant to ThermIdent before Quiz 4 was assigned.

CHAPTER 6 – CONCLUSIONS

The goal of this project was to better understand how students approach thermodynamics-related chain rule problems. Two such problems were assigned to students in the Fall 2016 thermodynamics course at Oregon State University (OSU). Students' responses were anonymized and their solution methods and conceptual errors were studied. Students used many solution methods, all but one of which were taught during the course. At least one instance of every possible solution method was observed in responses to each of the two primary assignments. Solution method prominence varied greatly within each assignment and between assignments. Many student responses contained conceptual errors. There was some correlation between the student solution methods and student conceptual errors (or lack of conceptual errors). These results suggest adjustments to the course content of OSU's thermodynamics Paradigm, as well as method improvements in possible future work.

6.1. Summary of Results

Student Solution Methods

Two solution methods stood out due to their superior effectiveness. The first method involves changing the differentials of the initial function's total differential, via substitution, and then identifying the desired partial derivative. This method is referred to as Diff RE. Diff RE was the most successful student solution method. In the second method, one uses a chain rule diagram of the initial function to build a chain rule for the desired partial derivative. This method is referred to as CRD. CRD was the most troublesome solution method for students. Most students who tried to use CRD on Quiz 14 either mis-built or misread their diagram. CRD showed drastic improvement in the short time between Quiz 14 and Final 4.b; most students who made conceptual errors with CRD on Quiz 14 went on to use CRD, without conceptual error, on the final exam. These students clearly saw value in CRD and worked to improve their understanding of the method over the weekend between Quiz 14 and Final 4.b.

Student Successes and Difficulties

Despite the many conceptual errors made by students, some of the student successes observed in **Thompson *et al.* [1]** were also observed in this project. Student successes observed by **Thompson *et al.* [1]** include: knowing the difference between derivatives of systems with one or more independent variables, and understanding (either conceptually or functionally) that partial derivatives treat some variables as constants.

Most students in this study know that derivatives differ based on whether a system has one or many independent variables. Most of these students understand that partial derivatives involve holding one or more variables constant. In almost every response, students somehow applied information about variables being held constant. These successes, however, do not come without related difficulties.

Aside from student difficulties with chain rule diagrams, two student difficulties were observed in this study. The first student difficulty is holding variables constant while evaluating partial derivatives. In many cases, students made errors while operating based on their misconceptions of what it means to hold a variable constant. Many students do not know how to properly hold variables constant while evaluating partial derivatives. The other observed student difficulty is identifying partial derivatives. Students often misidentified partial derivatives.

6.2. Implications for Instruction

Most students who used Diff RE had a good understanding of its workings. Current teaching methods are clearly effective. However, the less successful solution methods could be removed from the curriculum, allowing more students to succeed with Diff RE. Diff RE can reliably solve any thermodynamics-related chain rule problem, making this solution method a preferable take-home message for students.

Students need more practice with chain rule diagrams earlier in the class to be more successful with CRD. Like Diff RE, CRD is a valuable solution method because it can be universally applied to thermodynamics-related chain rule problems. Classroom activities already give students hands-on experience with this method. These activities could be enhanced to counteract the prominence of the two common CRD-related errors. Such errors could also be shown to the class in an attempt to inoculate students from making the errors. More extensive hands-on practice with chain rule diagrams would help rectify the issue. This hands-on practice could take the form of assigned worksheets. Such worksheets could have multiple prompts similar to those in this study. However, these prompts would only ask the student to obtain an unevaluated chain rule. This design would maximize students' experience with building and reading chain rule diagrams. Worksheets could be assigned in class and partially completed as in-class group activities, requiring students to finish on their own time.

Finally, students need more experience evaluating and identifying partial derivatives. Both of these concepts are likely fundamental enough to be further conveyed by repetitious practice. In both cases, assigned worksheets should suffice. Partial derivative evaluation worksheets could contain multiple one-equation partial derivatives with varying numbers of variables. Since partial derivatives differentiate with respect to only one variable, students would have practice holding the other variables constant while evaluating each partial derivative. In the case of partial derivative identification, prompts could provide students with the specific total differentials of multiple functions. Specific total differentials contain evaluated partial derivatives. Students would be required to identify the evaluated partial derivatives in each specific total differential.

6.3. Future Work

This study's dataset was far from ideal. The lack of data on Quiz 4 restricted possible insight into student method and error evolution. Student method and error evolution was meaningfully observed between two assignments (Quiz 14 and Final 4.b) that were only two days apart. This study would have greatly benefited from a complete dataset.

Prompt synonymy was somewhat lacking. Final 4.b's prompt had explicit thermodynamic context, while Quiz 4's / Quiz 14's prompt did not. All prompts need to be designed with, or without, physical context. Giving all prompts parallel physical context will avoid isolated occurrences of unique methods caused by contextually different prompts. In this way, the observation of each student method's evolution will be ensured.

To further improve this study, the dataset's scope could be extended. Examination of a pre-test problem would be valuable. With this addition, a total of (ideally) four data points would be collected per student: beginning, middle, and end of the class, as well as the final exam.

Furthermore, a larger dataset would allow exploration of the correlation (or lack thereof) between student conceptual errors and student method choice. The larger dataset would also improve chances of repeated trials yielding similar distributions of methods and errors. Unfortunately, the dataset's size is limited to the observed class's population.

ACKNOWLEDGEMENTS

Dr. Corinne Manogue, for her oversight, insight, and foresight. Without Dr. Manogue’s advising, this project and the resulting document would have been nigh impossible to complete.

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Mike Vignal, for his work as the interrater reliability tester and primary data anonymizer.

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REFERENCES

[1] John R. Thompson, Brandon R. Bucy, Donald B. Mountcastle. *Assessing Student Understanding of Partial Derivatives in Thermodynamics*

[2] Jonathan Tuminaro and Edward F. Redish. *Elements of a cognitive model of physics problem solving: Epistemic games*, Phys. Rev. ST Phys. Educ. Res. 3, 020101 (2007)

[3] Mary Bridget Kustusch, David Roundy, Tevian Dray, Corinne A. Manogue. *Partial derivative games in thermodynamics: A cognitive task analysis*, Phys. Rev. ST Phys. Educ. Res. 10, 010101 (2014)

[4] David Roundy, Eric Weber, Tevian Dray, Rabindra R. Bajracharya, Allison Dorko, Emily M. Smith, Corinne A. Manogue. *Experts' understanding of partial derivatives using the partial derivative machine*, Phys. Rev. ST Phys. Educ. Res. 11, 020126 (2015)

[5] Aaron Wangberg and Ben Johnson. *Discovering Calculus on the Surface*, Primus 23(7): 627-639 (2013)

APPENDIX A – CALCULUS IN OTHER PARADIGMS COURSES

The following is a list of the Paradigms in Physics classes at Oregon State University that contain calculus-based content and precede the thermodynamics Paradigm. After each course title is a list of the calculus-based concepts taught in the respective course.

Symmetries and Idealizations

- Power Series
- Scalar Line Integrals
- Surface and Volume Elements
- Surface and Volume Integrals
- Partial Derivatives
- Vector Differential
- Gradient
- Potentials due to Continuous Charge Distributions
- Electric Field as a Gradient
- Electric Field due to a Ring of Charge
- Finding Potentials from Fields
- Conservative Fields

Static Vector Fields

- Scalar Surface Integrals
- Flux
- Volume Integrals
- Partial Derivatives
- Gauss's Law
- Limit Definition of the Derivative
- Divergence and the Divergence Theorem
- Current
- Scalar and Vector Line Integrals
- Magnetic Vector Potential
- The Biot-Savart Law
- Ampere's Law
- Curl
- Stoke's Theorem
- Differential Form of Ampere's Law
- Continuity of Fields over a Boundary
- Path Independence
- Conservative Vector Fields
- The Laplacian
- Product Rules

Oscillations

- Ordinary Differential Equations
- Derivatives of Position
- Damped and Undamped Harmonic Oscillations
- RLC Oscillators
- Fourier Transforms
- Fourier Coefficients
- Fourier Series
- Impulses

Quantum Measurements and Spin

- Probability Calculus
- Eigenvalues and Eigenfunctions
- The Schrodinger Equation
- Quantum Operators
- Projections
- Time-Dependent Schrodinger Equation

One-Dimensional Waves

- Phase and Group Velocity
- Ordinary Differential Equations
- Initial Conditions
- Non-Dispersive Wave Equation
- Dispersion Relations
- Newton's Law and Transverse Motion
- Reflection, Transition, and Impedance
- Energy Density
- Fourier Series
- Fourier Superposition
- Wave Packets
- The Schrodinger Equation
- Eigenvalues and Eigenfunctions
- Quantum Operators
- Probability and Probability Density
- Time-Dependent Schrodinger Equation
- Barriers and Tunneling
- Time Evolution of Wave Packets

Periodic Systems

- Ordinary Differential Equations
- RLC Oscillators
- Linearly Coupled Oscillators
- N-Chain Oscillators
- Infinite Chains of Atoms and Diatomic Molecules
- The Equipartition Theorem
- Heat Capacity
- Specific Heat
- Chained Potential Wells
- Hamiltonian for the N-Well System
- Effective Mass

Reference Frames

- 2-D Relative Time Derivatives
- Coriolis Acceleration and Force
- Centrifugal Acceleration and Force
- Linear Motion in a Rotating Frame
- Coupled Second-Order Differential Equations
- 3-D Relative Time Derivatives
- The Foucault Pendulum
- Lorentz Transformations

APPENDIX B – TIMELINE OF RELEVANT COURSE MATERIALS

Table B.1 contains a timeline of relevant assignments and discussions from the thermodynamics class of the Paradigms in Physics program at Oregon State University. Material directly relevant to this study is shown bolded. See **Section 2.2** for the abbreviated solution codes contained within the "SOLUTIONS DISCUSSED" column below.

Table B.1 – The Spring 2016 thermodynamics paradigm's instructional timeline.

TIMELINE OF RELEVANT COURSE MATERIALS

DATE	ASSIGNMENTS	SOLUTIONS DISCUSSED	RELEVANT TOPICS DISCUSSED
Week 1			
4/18/16			Differentials Dependent and Independent Variables Partial Derivative Machine [4]
4/19/16			Isowidth and Isoforce Stretchability Total Differentials Partial Derivatives Multivariable Chain Rule (Differentials Version)
4/20/16	Homework 1 due	**Imp Diff**	Onion Method Chain Rule Multivariable Chain Rule (Differentials Version) Interpreting Partial Derivatives Thermodynamic Identity
4/21/16		**Diff RE (general total differential)** **CRD**	Multivariable Chain Rule (Differentials Version) Upside-down Derivatives Cyclic Chain Rule
4/22/16	Homework 2 due		Legendre Transforms Equality of Mixed Second Partial Derivatives Contoured Surfaces [5]
Week 2			
4/27/16	Homework 3 due		
4/29/16	**Quiz 4 assigned**		
Week 3			
5/2/16		**Quiz 4 Solutions:** **>CRD** **>Imp Diff** **>Diff Div** **>Diff RE (specific total differential)**	
5/4/16			Equality of Mixed Second Partial Derivatives Maxwell Relations
Week 4			
5/11/16	Homework 7 due		
5/13/16	**Quiz 14 assigned**		
Week 5			
5/16/16	**Final exam**		

APPENDIX C – CODING SYNTAX

Upon initial review of the Quiz 4 and Quiz 14 data from the Spring 2016 thermodynamics paradigm, I broke each student's responses down into steps and made codes for each unique step. I tried to make a flow chart that contained every possible correct student solution to the prompt; I hoped to represent each unique student response as a unique path through the flow chart. This proved too complex, so I instead developed a coding language. Even initially, my coding language allowed me to represent common sequences of steps in very little space.

I performed the initial coding with my scheme, inputting the codes into Excel in a tabular format with three columns: Step, Err, and VC. "Step" is the column where student moves are listed sequentially. The "Err" column contains a code representing the nature of any error made by the student in the corresponding step. Similarly, the "VC" column contains a code representing how, or if, the student applied information about variables being held constant in the corresponding step.

After the first round of IRT, I greatly simplified my codes by eliminating overlaps between coding definitions and better defining my codes for variable information application. After the second stage of IRT, I revised the coding syntax to disallow step nesting (when what is shown as one step actually contains multiple individually representable steps), which was used to represent common sequences of steps; I was unsatisfied by step nesting due to the loss of information about students' errors and variable information application within sequence's individual steps. The third round of IRT yielded a step-by-step agreement (all cells in a table's row must be the same to count as an agreement) of ~60% as opposed to cell-by-cell agreement (each individual cell counts for or against agreement), which was >80%.

After yet another large rework, which dramatically improved readability, I realized that I had obtained everything I needed from the coding scheme; common errors and patterns of steps were clear to me. I had little justification to continue dragging the coding scheme along. To clarify, the coding scheme was no longer necessary for me to observe student work at the detail required for my analysis. Additionally, every time I observed a new student move or solution, or came to realize that I had been classifying two distinguishably different moves as the same (even if such minute details were insignificant to my analysis), I had to rethink my coding scheme and all previous codings.

Although I ultimately scrapped the coding scheme, the following is as it was in its final state, included for posterity's sake. This scheme is far from perfect (there are several prominent student moves / steps that it fails to account for), and I would have made many additional changes had I continued to update it as my project progressed.

C.1. Piece Names

General Form of the Total Differential

Ex: Given a function $?(a, b, c) = a^2 + b^2 + c^2$, the general form of the total differential of U, represented as "D?G" is as follows:

$$d? = \left(\frac{\partial ?}{\partial a}\right)_{b,c} da + \left(\frac{\partial ?}{\partial b}\right)_{a,c} db + \left(\frac{\partial ?}{\partial c}\right)_{a,b} dc \tag{12}$$

Specific form of the Total Differential

Ex: Given a function $?(a, b, c) = a^2 + b^2 + c^2$, the specific form of the total differential of U, represented as "D?S" is as follows:

$$d? = 2ada + 2bdb + 2cdc \tag{13}$$

Partial Derivatives

Syntax: ABcd ; A is the numerator variable, B is the denominator variable, and all lowercase letters are the variables that are being held constant.

Ex: Given functions $?(a, b, c)$ and $a\ (\ b\ , c\ )$, a few of the possible partial derivatives are:

- ABc represents:

$$\left(\frac{\partial a}{\partial b}\right)_c \tag{14}$$

- CAb represents:

$$\left(\frac{\partial c}{\partial a}\right)_b \tag{15}$$

- ?Cab represents:

$$\left(\frac{\partial ?}{\partial c}\right)_{a,b} \tag{16}$$

C.2. Step Syntax

The step syntax appears in these forms, translating directly to the corresponding statements:

- A for B into C ; Used piece A to solve for piece B and substituted piece B into piece C
- B into C ; Substituted piece B into piece C
- A for B ; Used piece A to solve for piece B
- A in B ; Identified partial derivative "A" in piece B
- A by B ; divided piece A by differential B
- or ; It is unclear which of the listed moves / steps the student made
- and ; used only after "or," when one of the listed moves contains multiple steps

Other step syntax codes, which represent moves and steps to which the previous syntax does not apply:

Created (NOT solved for) **the General Form of a Total Differential**

Syntax: Given a function ? (a, b, c), D?G

Created (NOT solved for) **the Specific form of a Total Differential**

Syntax: Given a function ? (a, b, c), D?S

Performed Extensive Algebraic Manipulation

Syntax: ALG { piece on which the algebra was performed }

Used a Cyclical Chain Rule

Syntax: CCR { partial derivative that is expanded by the cyclical chain rule}

Made a Chain Rule Diagram

Syntax: CRD { primary branching differential "to" secondary branching differential }

Ex: Given functions ? (a, b, c) and $a\ (\ b\ , c\ )$, all possible chain rules with "D?" as the primary branching differential:

CRD { D? to DA } ; D? branching into DA, DB, and DC, DA branching into DB and DC. See **Figure C.1**.

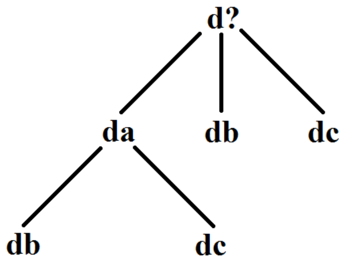


Figure C.1 – Chain rule diagram represented by CRD { D? to DA }

CRD { D? to DB } ; D? branching into DA, DB, and DC, DB branching into DA and DC. See **Figure C.2**.

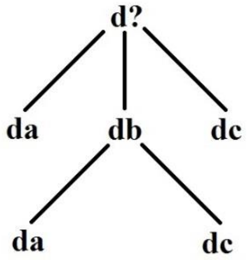


Figure C.2 – Chain rule diagram represented by CRD { D? to DB }

CRD { D? to DC } ; D? branching into DA, DB, and DC, DC branching into DA and DB. See **Figure C.3**.

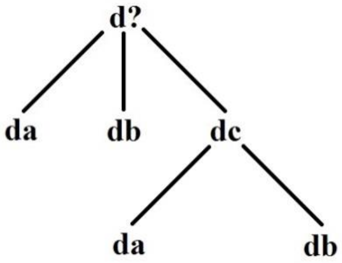


Figure C.3 – Chain rule diagram represented by CRD { D? to DC }

Built a Chain Rule (non-cyclical)

Syntax: BCR { partial derivative which is expanded by the chain rule }

Ex: Given functions ? (a, b, c) and $a\ (\ b\ , c\ )$, all possible chain rules with “?” as a common numerator:

- BCR { ?Ab }

$$\left(\frac{\partial ?}{da}\right)_b = \left(\frac{\partial ?}{\partial c}\right)_{a,b}\left(\frac{\partial c}{\partial a}\right)_b + \left(\frac{\partial ?}{\partial a}\right)_{b,c} \qquad (6)$$

- BCR { ?Ac }

$$\left(\frac{\partial ?}{da}\right)_c = \left(\frac{\partial ?}{\partial b}\right)_{a,c}\left(\frac{\partial b}{\partial a}\right)_c + \left(\frac{\partial ?}{\partial a}\right)_{b,c} \qquad (7)$$

- BCR { ?Ba }

$$\left(\frac{\partial ?}{db}\right)_a = \left(\frac{\partial ?}{\partial c}\right)_{a,b}\left(\frac{\partial c}{\partial b}\right)_a + \left(\frac{\partial ?}{\partial b}\right)_{a,c} \qquad (8)$$

- BCR { ?Bc}

$$\left(\frac{\partial ?}{db}\right)_a = \left(\frac{\partial ?}{\partial a}\right)_{b,c}\left(\frac{\partial a}{\partial b}\right)_c + \left(\frac{\partial ?}{\partial b}\right)_{a,c} \qquad (9)$$

- BCR { ?Ca }

$$\left(\frac{\partial ?}{\partial c}\right)_a = \left(\frac{\partial ?}{\partial b}\right)_{a,c}\left(\frac{\partial b}{\partial c}\right)_a + \left(\frac{\partial ?}{\partial c}\right)_{a,b} \qquad (10)$$

- BCR { ?Cb }

$$\left(\frac{\partial ?}{\partial c}\right)_b = \left(\frac{\partial ?}{\partial a}\right)_{b,c}\left(\frac{\partial a}{\partial c}\right)_b + \left(\frac{\partial ?}{\partial c}\right)_{a,b} \qquad (11)$$

In short, the syntax says: “the student built a chain rule for the partial derivative within the brackets.” Once a student equates a chain rule to a partial derivative, the partial derivative is considered to be a piece whose expression is the chain rule, regardless of whether the student matched the partial derivative with the correct chain rule (incorrectly matching a partial derivative and a chain rule is considered to be a conceptual error).

Obtained a Solution

Syntax:

- SOLN ; obtained a correct solution, disregarding sign errors
- SOLN# ; obtained a partially correct solution
- SOLN## ; did not obtain a correct or partially correct solution

C.3. Variable Information Codes (each step has a code from this section)

EXP ; Explicitly applied information about a variable being held constant
Ex: Stated that $dy = 0$ and crossed out the dy term.
Ex: Showed y as a subscript in a partial derivative

IMP ; Implicitly applied information about a variable being held constant
Ex: Left a term out of a calculation because it would have resulted in zero, but did not explicitly show and cross out the term

DNA ; Did not apply information about a variable being held constant

NA ; No information about a variable being held constant can be applied to this step

C.4. Error Codes (each step has a code from this section)

NER ; No mathematical or conceptual errors were made in the specified step

SIN ; A sign error was made in the specified step

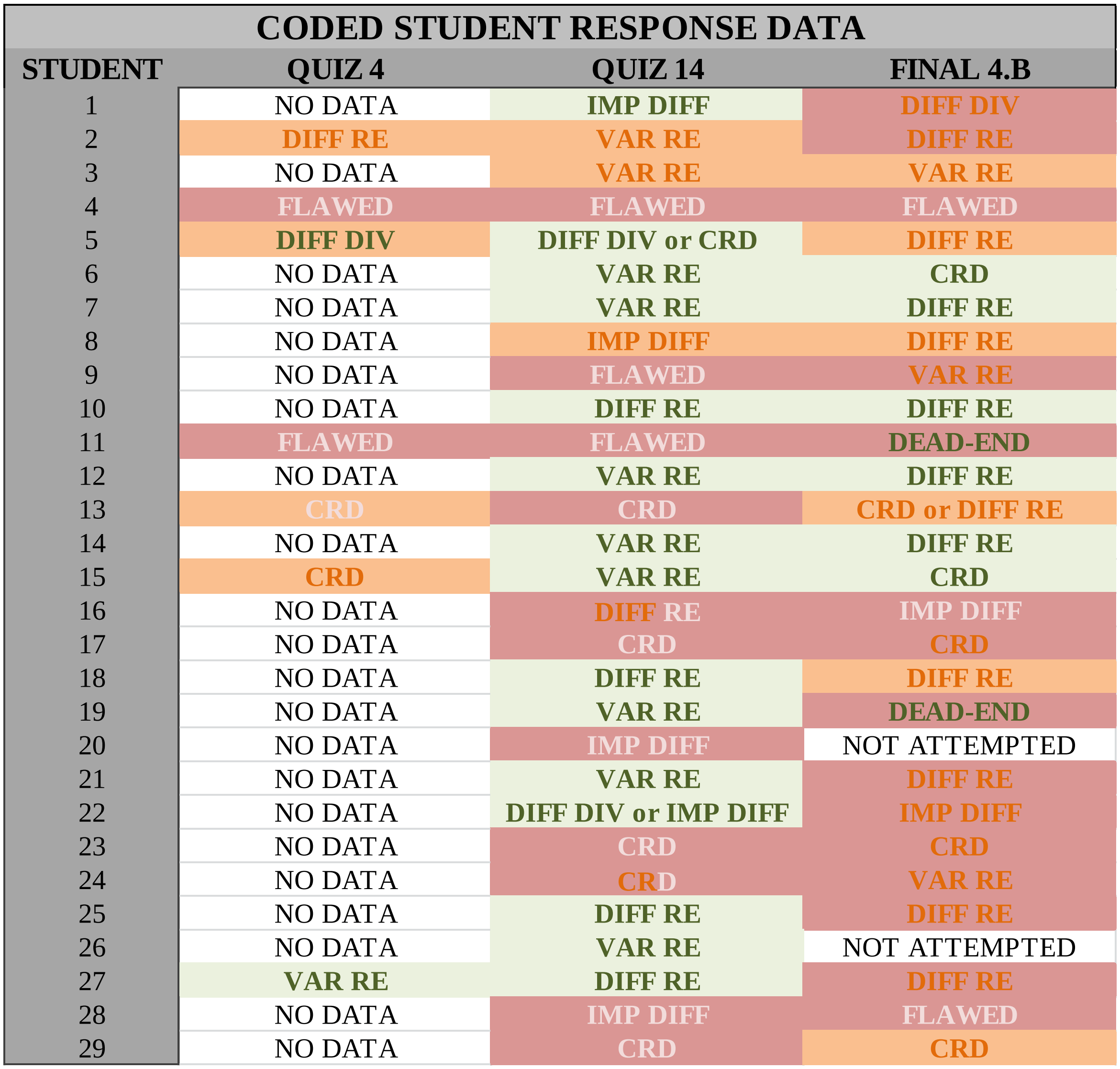
MTH ; A non-sign mathematical error was made in the specified step

CPL ; A conceptual error was made in the specified step

APPENDIX D – CODED STUDENT RESPONSES

Table D.1 contains the coding scheme's final output. A list of anonymized student responses is in **Appendix E**. The Analysis in **Sections 3.1**, **3.2**, and **4.1** can be derived from the **Table D.1**'s contents. The table's cells contain codes for each student's solution method. Some student responses had work reflecting two correct methods. These responses were therefore marked and counted as containing both methods. Green, orange, and red cell fill colors respectively represent a correct answer, partially correct answer, and incorrect answer. Although information about student answer correctness was recorded and shown, it is not this project's focus. Green, orange, and red font colors respectively signify that the student made no errors, a math error(s), and a conceptual error(s).

Table D.1 – Coded data, outputted by the final coding scheme. Green, orange, and red cell fill colors respectively represent a correct answer, partially correct answer, and incorrect answer. Green, orange, and red font colors respectively signify that the student made no errors, a math error(s), and a conceptual error(s).



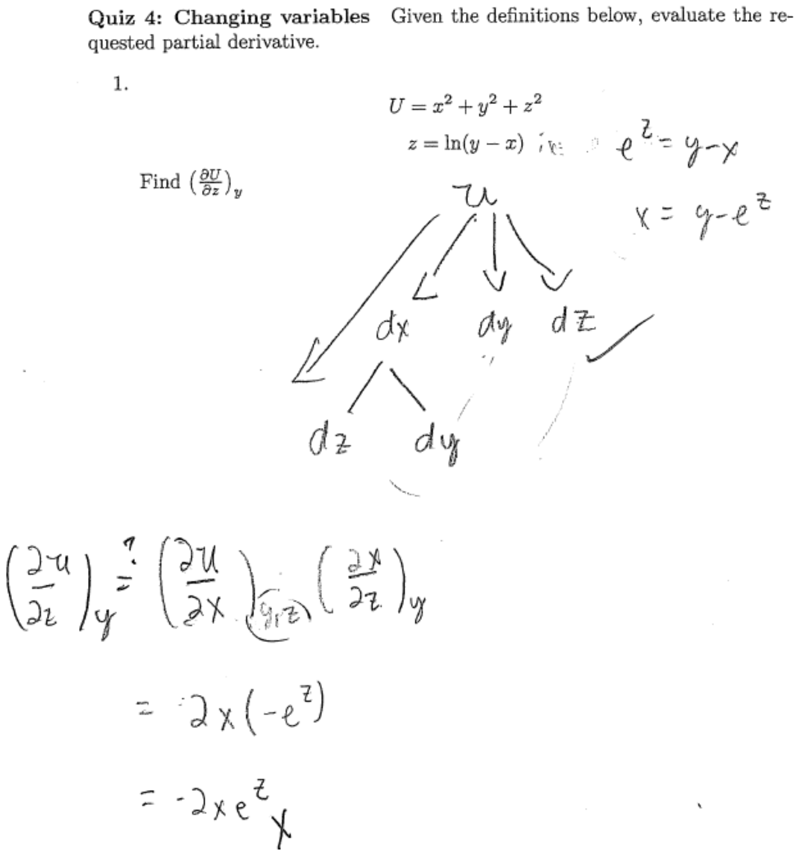
CODED STUDENT RESPONSE DATA

STUDENT	QUIZ 4	QUIZ 14	FINAL 4.B
1	NO DATA	**IMP DIFF**	**DIFF DIV**
2	**DIFF RE**	**VAR RE**	**DIFF RE**
3	NO DATA	**VAR RE**	**VAR RE**
4	**FLAWED**	**FLAWED**	**FLAWED**
5	**DIFF DIV**	**DIFF DIV or CRD**	**DIFF RE**
6	NO DATA	**VAR RE**	**CRD**
7	NO DATA	**VAR RE**	**DIFF RE**
8	NO DATA	**IMP DIFF**	**DIFF RE**
9	NO DATA	**FLAWED**	**VAR RE**
10	NO DATA	**DIFF RE**	**DIFF RE**
11	**FLAWED**	**FLAWED**	**DEAD-END**
12	NO DATA	**VAR RE**	**DIFF RE**
13	**CRD**	**CRD**	**CRD or DIFF RE**
14	NO DATA	**VAR RE**	**DIFF RE**
15	**CRD**	**VAR RE**	**CRD**
16	NO DATA	**DIFF RE**	**IMP DIFF**
17	NO DATA	**CRD**	**CRD**
18	NO DATA	**DIFF RE**	**DIFF RE**
19	NO DATA	**VAR RE**	**DEAD-END**
20	NO DATA	**IMP DIFF**	NOT ATTEMPTED
21	NO DATA	**VAR RE**	**DIFF RE**
22	NO DATA	**DIFF DIV or IMP DIFF**	**IMP DIFF**
23	NO DATA	**CRD**	**CRD**
24	NO DATA	**CRD**	**VAR RE**
25	NO DATA	**DIFF RE**	**DIFF RE**
26	NO DATA	**VAR RE**	NOT ATTEMPTED
27	**VAR RE**	**DIFF RE**	**DIFF RE**
28	NO DATA	**IMP DIFF**	**FLAWED**
29	NO DATA	**CRD**	**CRD**

APPENDIX E – ANONYMIZED STUDENT RESPONSES

E.1. Quiz 4

13



Quiz 4: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$U = x^2 + y^2 + z^2$

$z = \ln(y - x)$ ie $e^z = y - x$

$x = y - e^z$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

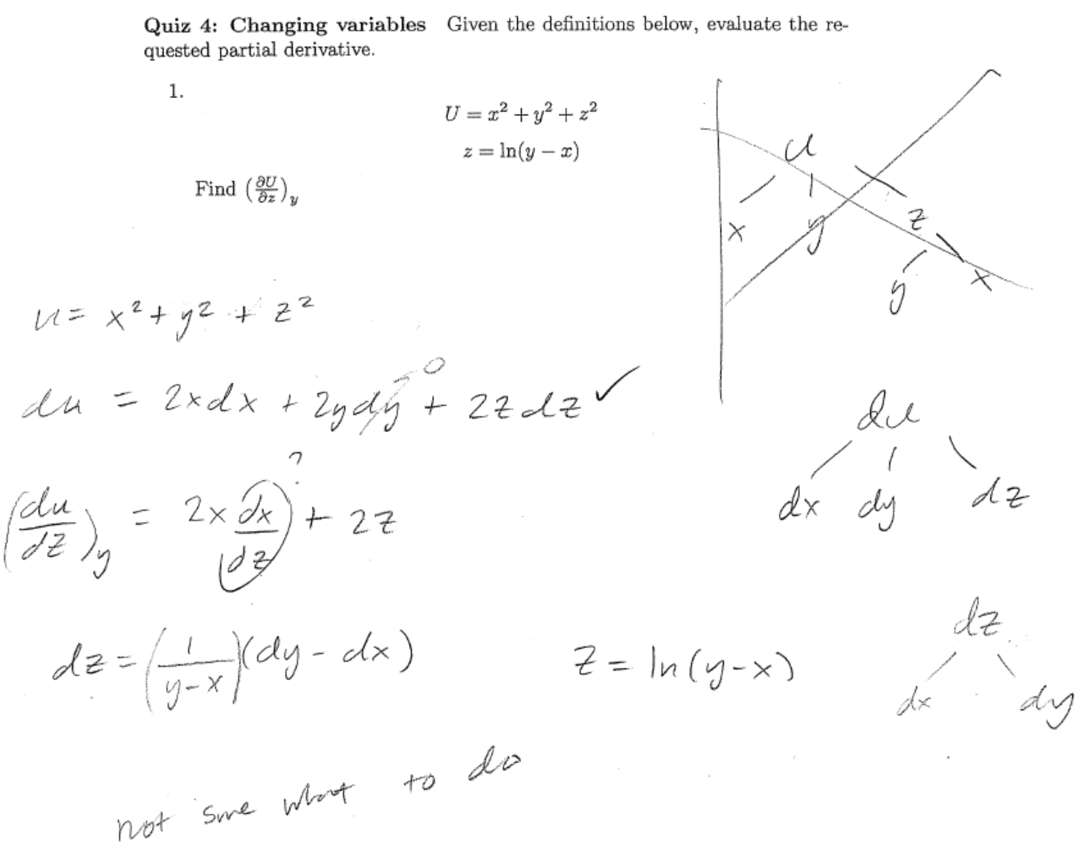
$$\left(\frac{\partial u}{\partial z}\right)_y \overset{?}{=} \left(\frac{\partial u}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial z}\right)_y$$

$= 2x(-e^z)$

$= -2xe^z$

✗

5



Quiz 4: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

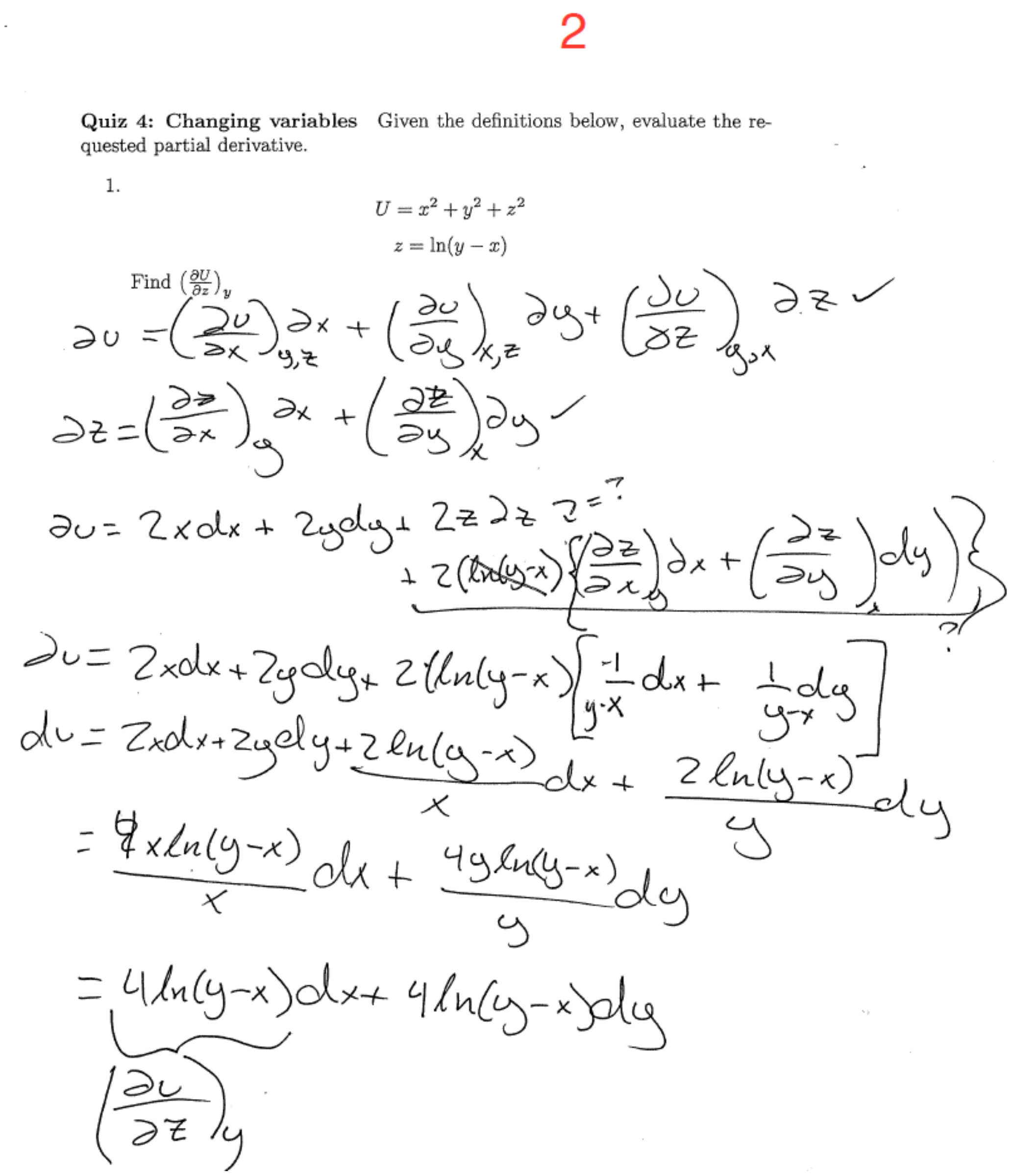
$u = x^2 + y^2 + z^2$

$du = 2x\,dx + 2y\,\overset{0}{\cancel{dy}} + 2z\,dz$ ✓

$\left(\frac{du}{dz}\right)_y = 2x \overset{?}{\left(\frac{dx}{dz}\right)} + 2z$

$dz = \left(\frac{1}{y-x}\right)(dy - dx)$ $z = \ln(y-x)$

not sure what to do



2

Quiz 4: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$\partial U = \left(\frac{\partial U}{\partial x}\right)_{y,z} \partial x + \left(\frac{\partial U}{\partial y}\right)_{x,z} \partial y + \left(\frac{\partial U}{\partial z}\right)_{y,x} \partial z \checkmark$$

$$\partial z = \left(\frac{\partial z}{\partial x}\right)_y \partial x + \left(\frac{\partial z}{\partial y}\right)_x \partial y \checkmark$$

$$\partial U = 2x\,dx + 2y\,dy + 2z\,\partial z \quad z = ?$$

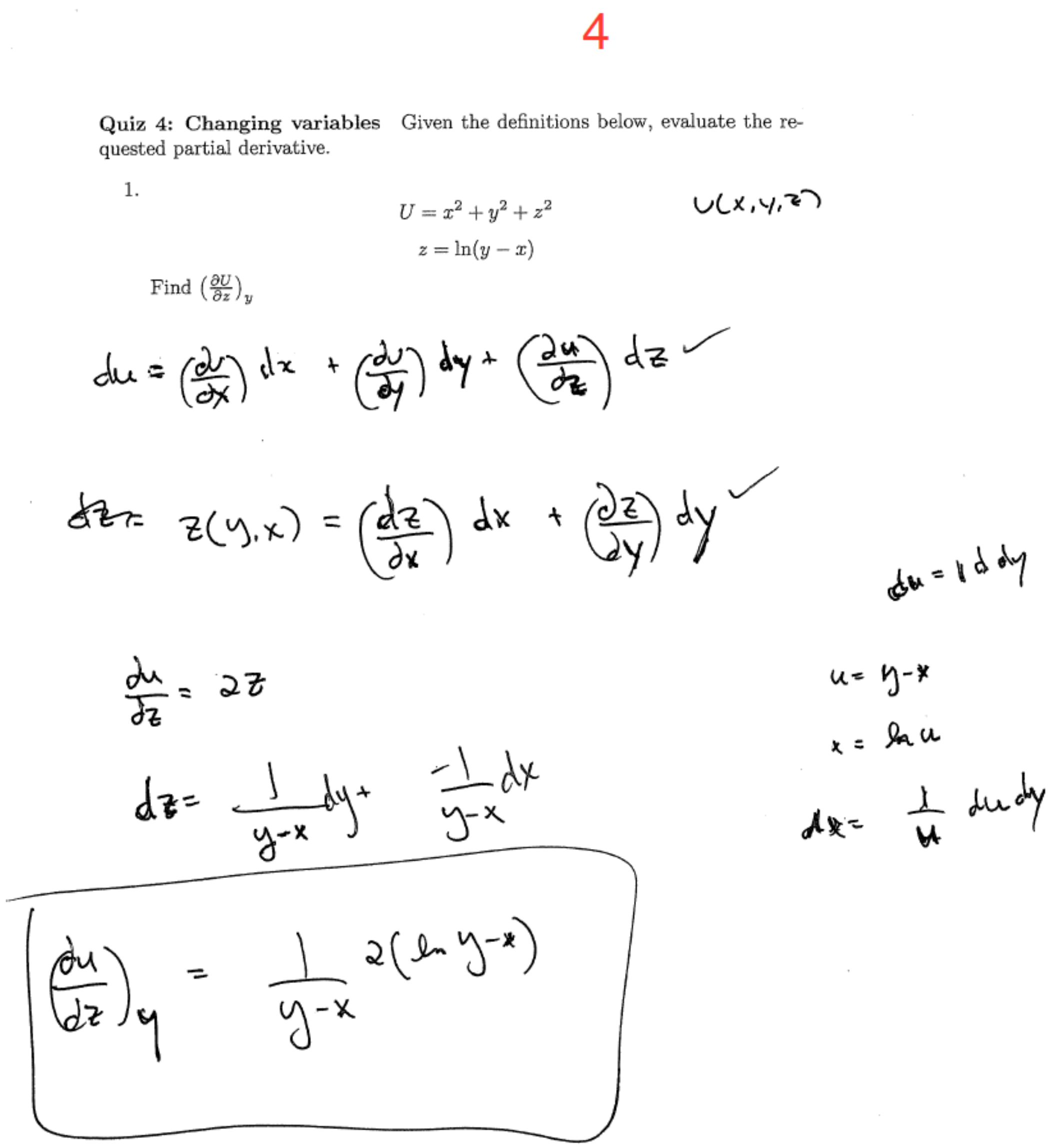
$$\underline{+\,2(\cancel{\ln(y-x)})\left\{\left(\frac{\partial z}{\partial x}\right)_y \partial x + \left(\frac{\partial z}{\partial y}\right)_x dy\right)\right\}} \; ?$$

$$\partial U = 2x\,dx + 2y\,dy + 2(\ln(y-x))\left[\frac{-1}{y-x}dx + \frac{1}{y-x}dy\right]$$

$$dU = 2x\,dx + 2y\,dy + \frac{2\ln(y-x)}{x}dx + \frac{2\ln(y-x)}{y}dy$$

$$= \frac{4x\ln(y-x)}{x}dx + \frac{4y\ln(y-x)}{y}dy$$

$$= \underbrace{4\ln(y-x)}_{\left(\frac{\partial U}{\partial z}\right)_y}dx + 4\ln(y-x)dy$$



4

Quiz 4: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

$U(x, y, z)$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$du = \left(\frac{\partial U}{\partial x}\right) dx + \left(\frac{\partial U}{\partial y}\right) dy + \left(\frac{\partial u}{\partial z}\right) dz \checkmark$$

$$dz = z(y, x) = \left(\frac{dz}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy \checkmark$$

$$du = 1\, d\, dy$$

$$\frac{\partial u}{\partial z} = 2z$$

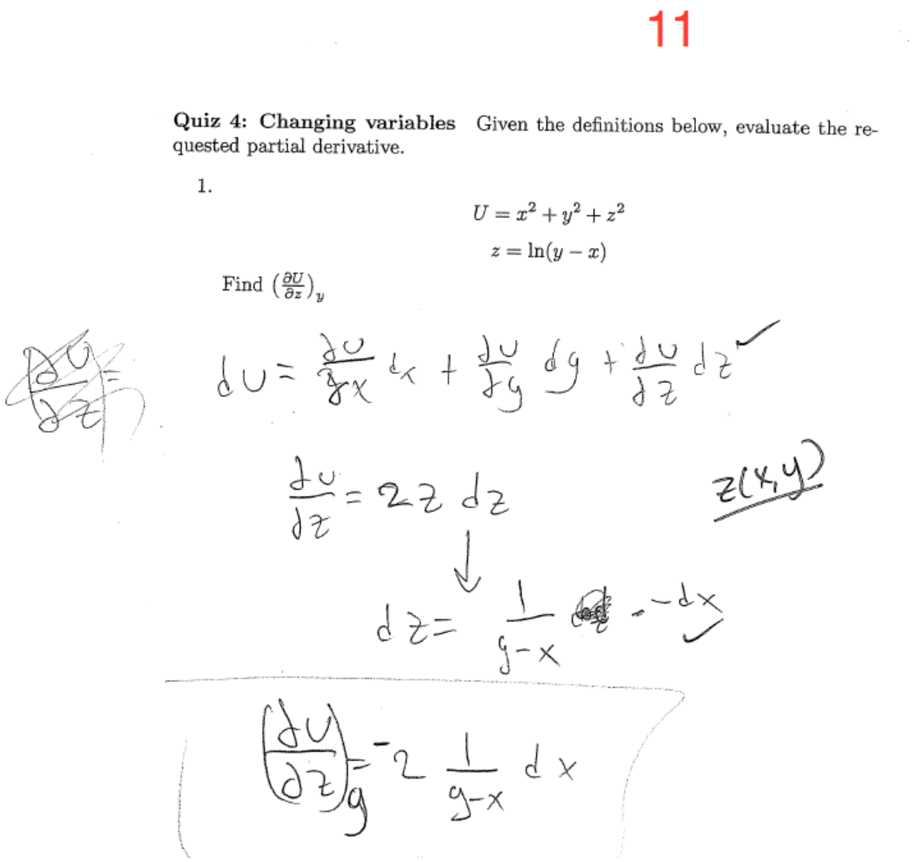
$$u = y - x$$

$$x = \ln u$$

$$dz = \frac{1}{y-x} dy + \frac{-1}{y-x} dx$$

$$dx = \frac{1}{u} du\, dy$$

$$\boxed{\left(\frac{\partial u}{dz}\right)_y = \frac{1}{y-x} 2(\ln y - x)}$$



11

Quiz 4: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

~~$\frac{\partial U}{\partial z} =$~~

$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$ ✓

$\frac{\partial U}{\partial z} = 2z\, dz$ $\quad z(x,y)$

$\downarrow$

$dz = \frac{1}{y-x} - dx$ ✓

$\left(\frac{\partial U}{\partial z}\right)_y = -2 \frac{1}{y-x} dx$

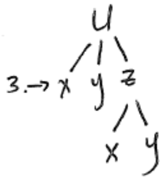
15

Quiz 4: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$



du
$|$
dx
$|$
dz

$\left(\frac{\partial u}{\partial x}\right)$

$\left(\frac{\partial x}{\partial z}\right)$

$$\left(\frac{\partial u}{\partial z}\right)_y = \left(\frac{\partial u}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\right)_{x,y}.$$

$$\left(\frac{\partial u}{\partial x}\right) = 2x$$

$$\left(\frac{\partial x}{\partial z}\right) = \left(\frac{\partial z}{\partial x}\right)^{-1} = \left(\frac{1}{y-x}\right)^{-1} = \quad y-x$$

$$\boxed{\left(\frac{\partial u}{\partial z}\right)_y = -2x(y-x) = 2xy - 2x^2} = 2(y-e^z)y - 2(y-e^z)^2$$

$z = \ln(y-x)$

$e^z = e^{\ln(y-x)}$

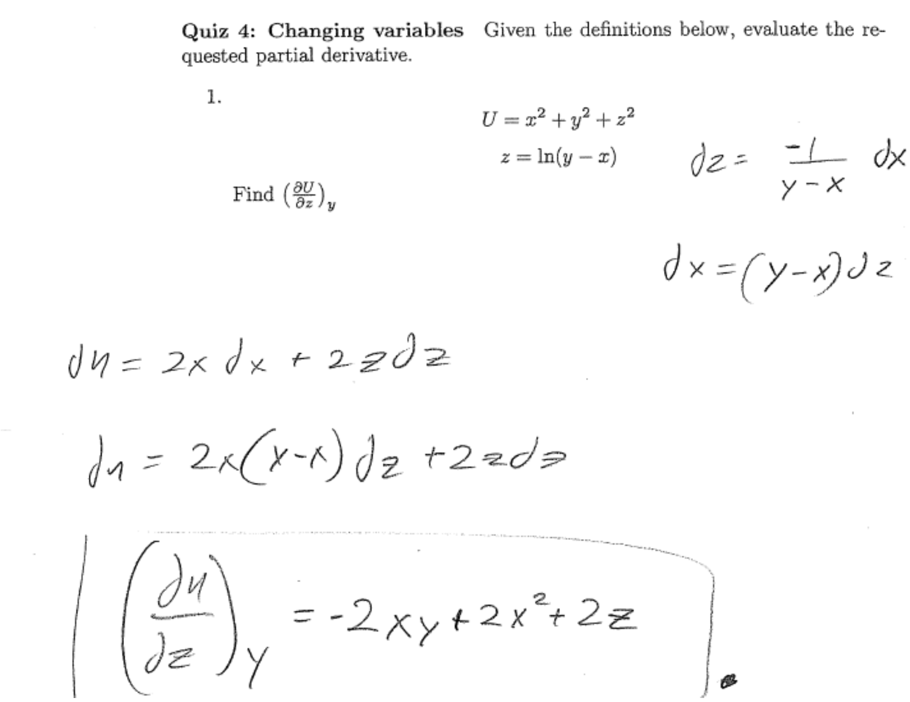
$e^z = y-x$

$x = y - e^z$

~~$u = z^2 + y^2 + (y-e^z)^2$~~

~~$\left(\frac{\partial u}{\partial z}\right)_y = 2z + 2(y-e^z)e^z$~~

27



Quiz 4: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$\partial z = \frac{-1}{y-x} \, \partial x$$

$$\partial x = (y-x) \partial z$$

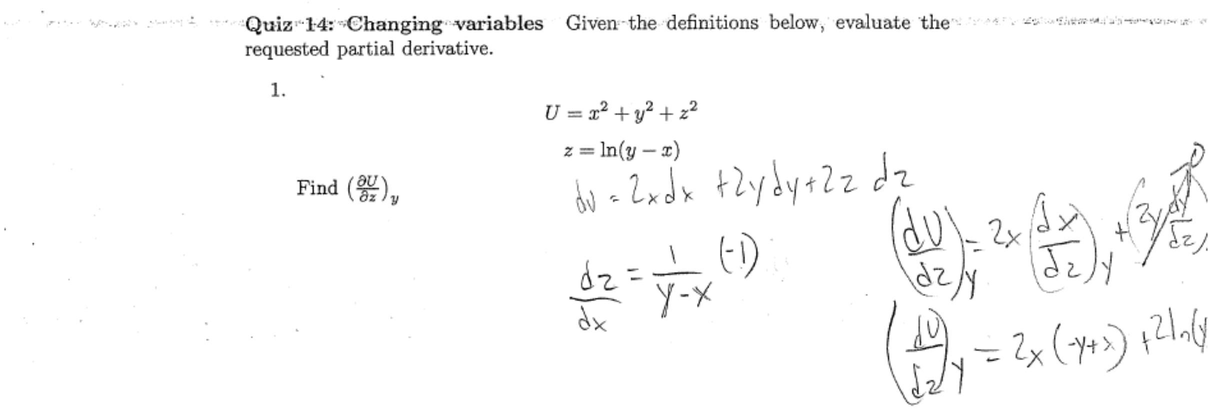
$$\partial u = 2x \, \partial x + 2z \partial z$$

$$\partial u = 2x(y-x) \partial z + 2z \partial z$$

$$\boxed{\left(\frac{\partial u}{\partial z}\right)_y = -2xy + 2x^2 + 2z}$$

E.2. Quiz 14

22



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

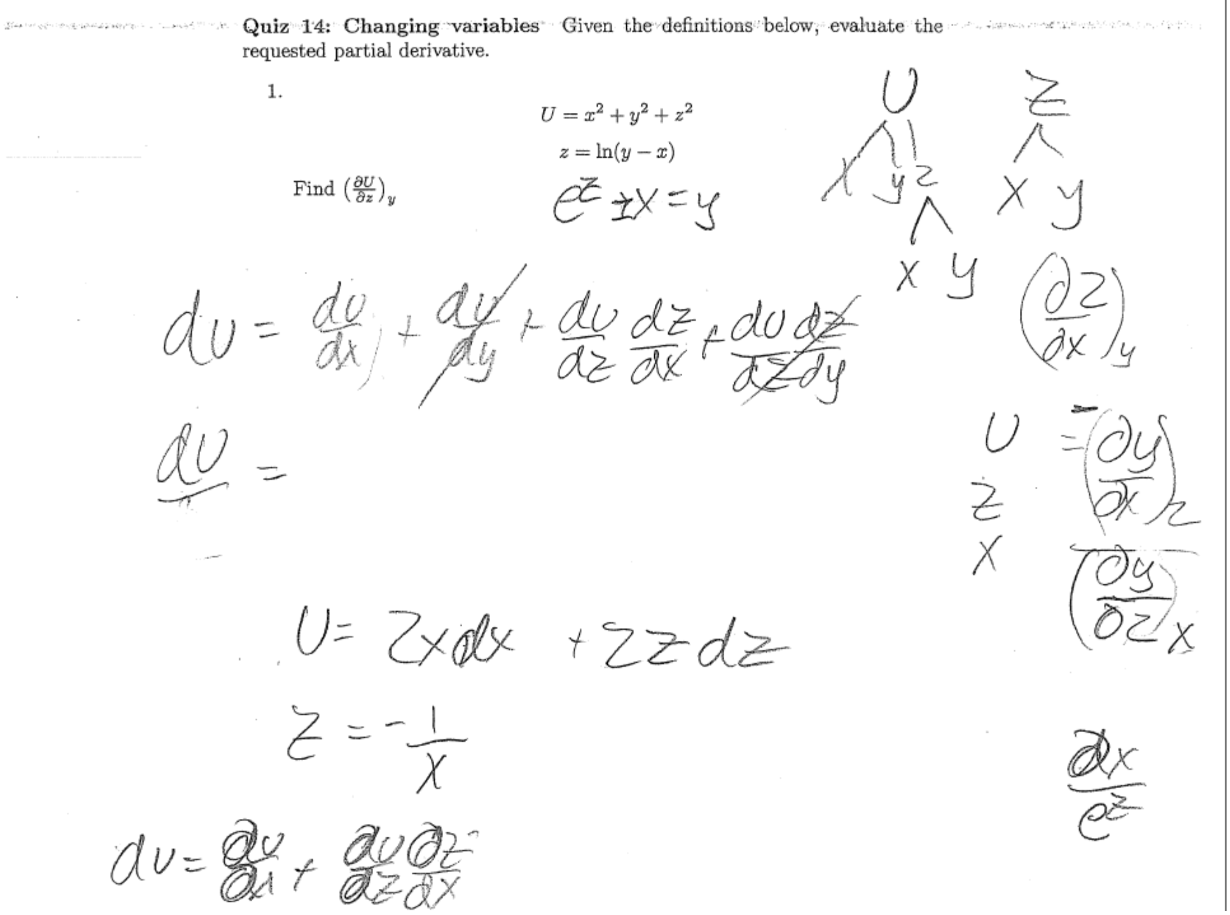
$dU = 2x\,dx + 2y\,dy + 2z\,dz$

$\frac{dz}{dx} = \frac{1}{y-x}(-1)$

$\left(\frac{dU}{dz}\right)_y = 2x\left(\frac{dx}{dz}\right)_y + \cancel{2y\frac{dy}{dz}}$

$\left(\frac{dU}{dz}\right)_y = 2x(-y+x) + 2\ln(y$

24



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$e^z \pm x = y$

$dU = \frac{dU}{dx} + \cancel{\frac{dU}{dy}} + \frac{dU}{dz}\frac{dz}{dx} + \frac{dU}{\cancel{dz}}\frac{\cancel{dz}}{dy}$

$\left(\frac{\partial z}{\partial x}\right)_y$

$\frac{dU}{} =$

U
z
x

$-\frac{\left(\frac{\partial y}{\partial x}\right)_z}{\left(\frac{\partial y}{\partial z}\right)_x}$

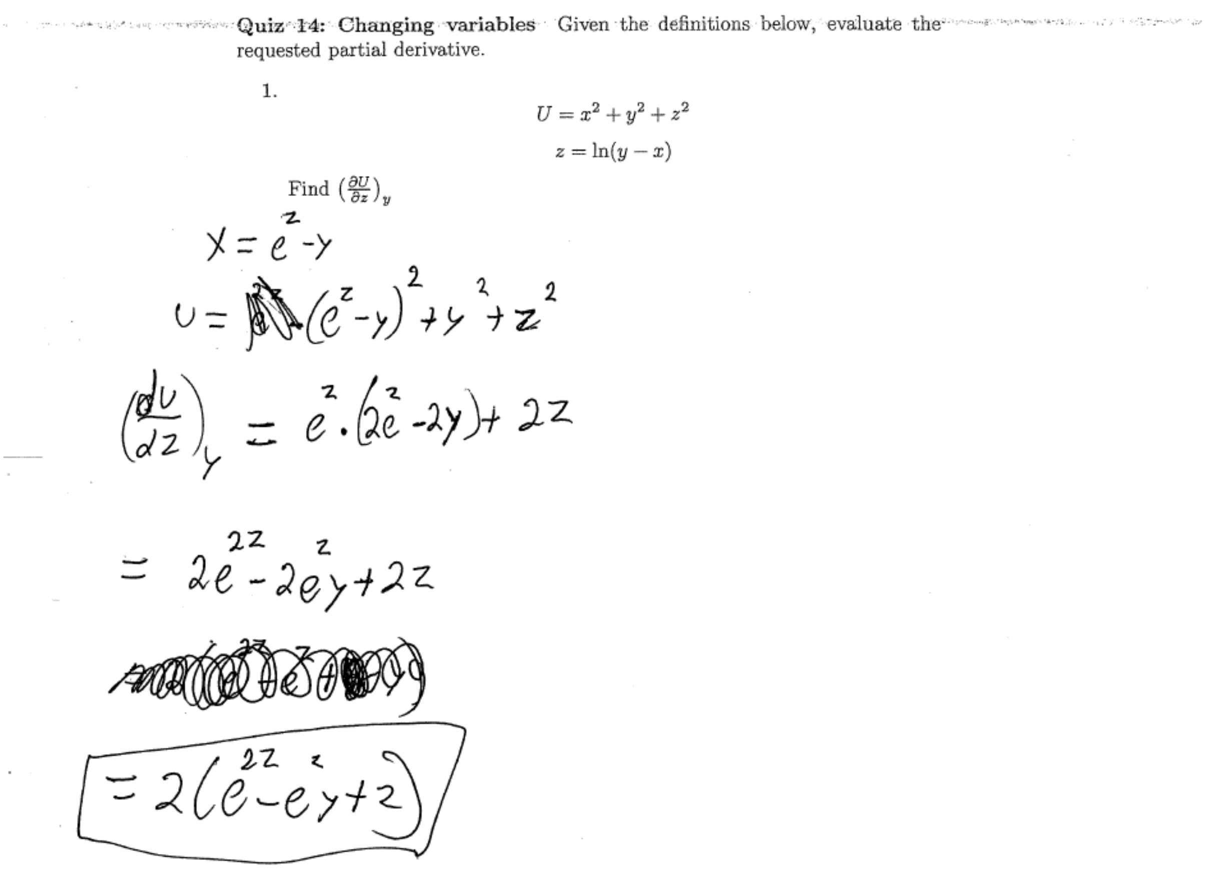
$U = 2x\,dx + 2z\,dz$

$z = -\frac{1}{x}$

$\frac{\partial x}{e^z}$

$dU = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial x}$

19



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$x = e^z - y$

$U = (e^z - y)^2 + y^2 + z^2$

$\left(\frac{dU}{dz}\right)_y = e^z \cdot (2e^z - 2y) + 2z$

$= 2e^{2z} - 2e^z y + 2z$

$\boxed{= 2(e^{2z} - e^z y + z)}$

21

Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

$e^z = y - x$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$x = y - e^z$

$u = (y-e^z)^2 + y^2 + \cancel{y}z^2$

$\cancel{x} \quad = y^2 - 2ye^z - e^{2z} + y^2 + z^2$

$$\left(\frac{\partial u}{\partial z}\right)_y = -2ye^z - 2e^{2z} + 2z$$

9

Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

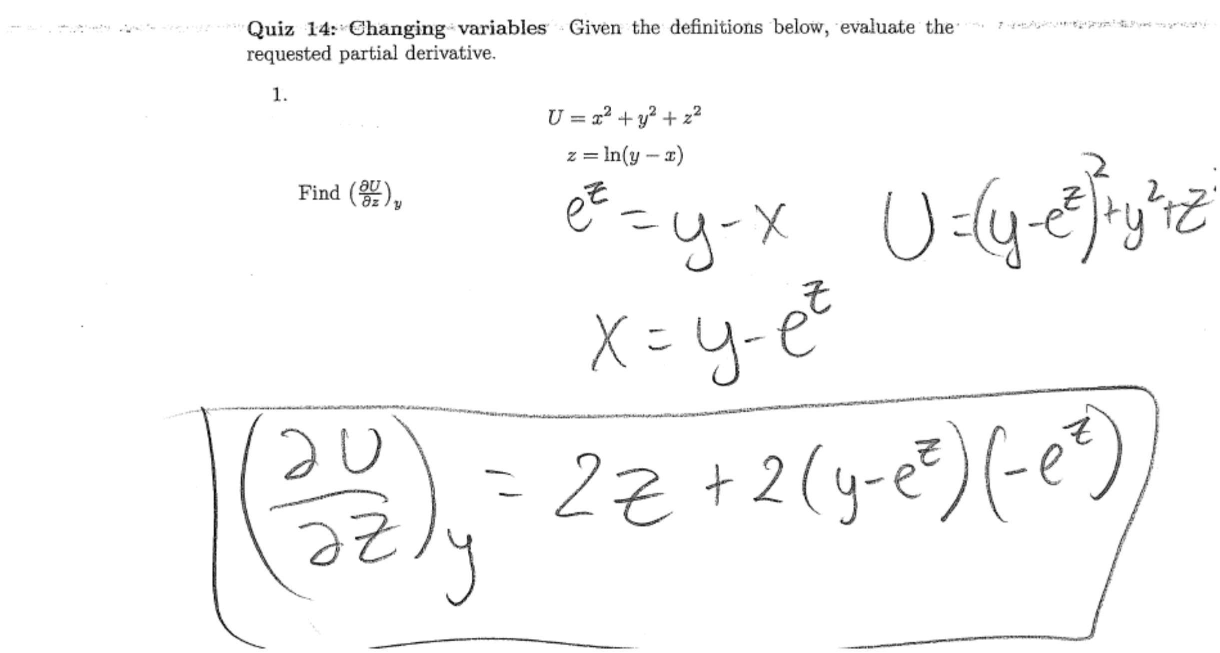
$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$\left(\frac{\partial u}{\partial z}\right)_y = x^2 + 2z$$

$$\boxed{= x^2 + 2\ln(y-x)}$$

6



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

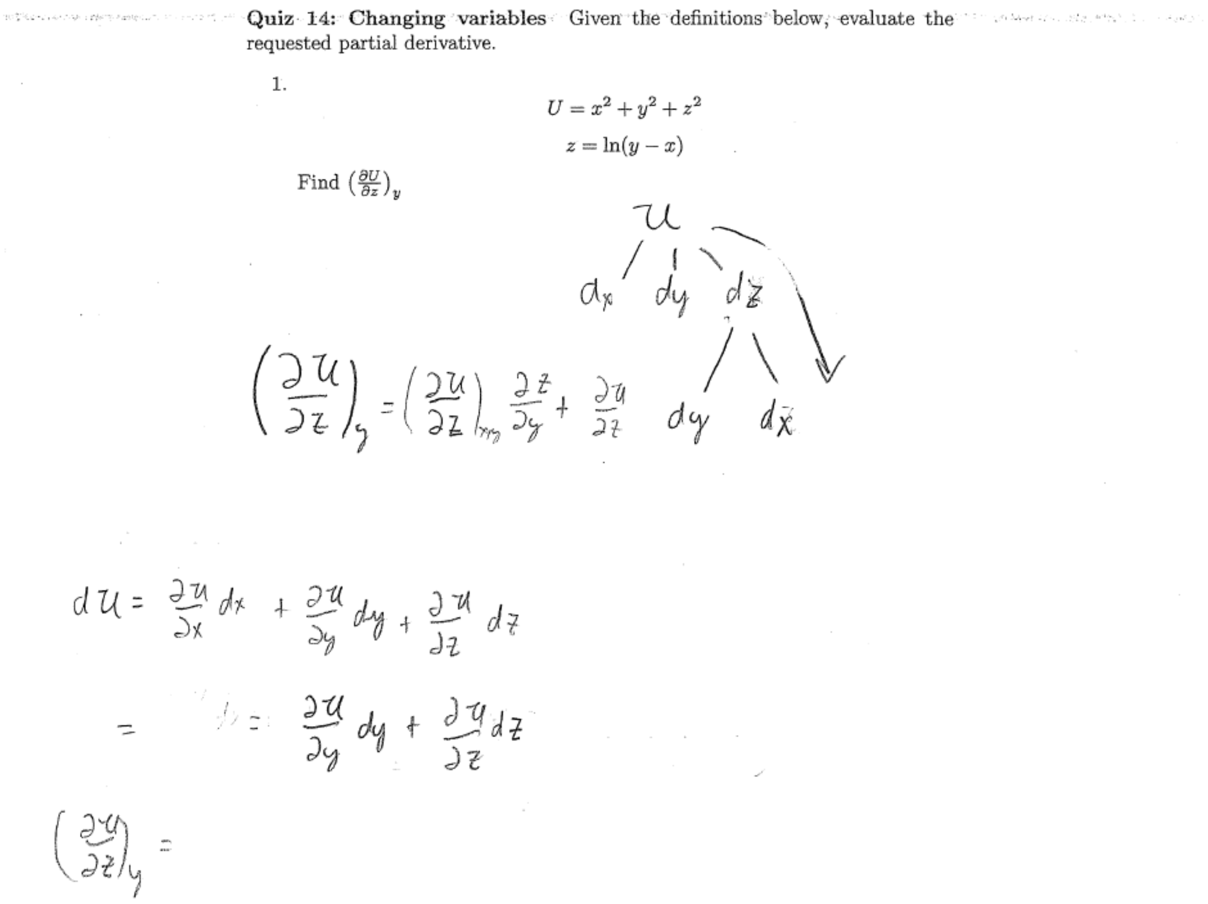
Find $\left(\frac{\partial U}{\partial z}\right)_y$

$e^z = y - x$ $U = (y - e^z)^2 + y^2 + z$

$x = y - e^z$

$$\boxed{\left(\frac{\partial U}{\partial z}\right)_y = 2z + 2(y - e^z)(-e^z)}$$

13



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

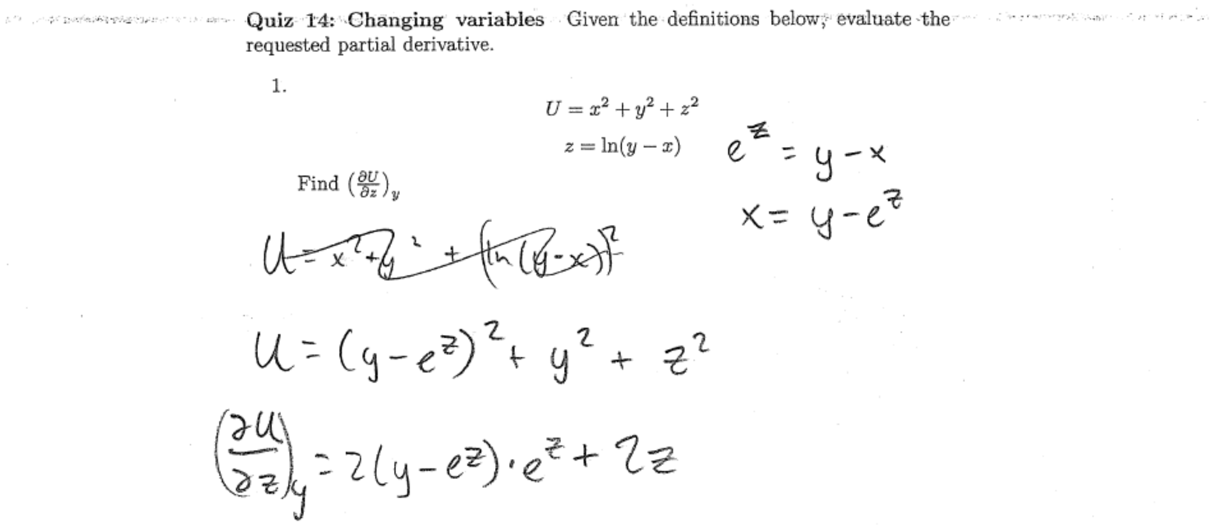
$$\left(\frac{\partial u}{\partial z}\right)_y = \left(\frac{\partial u}{\partial z}\right)_{xy} \frac{\partial z}{\partial y} + \frac{\partial u}{\partial z}$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$= \quad = \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$\left(\frac{\partial u}{\partial z}\right)_y =$$

7



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$e^z = y - x$

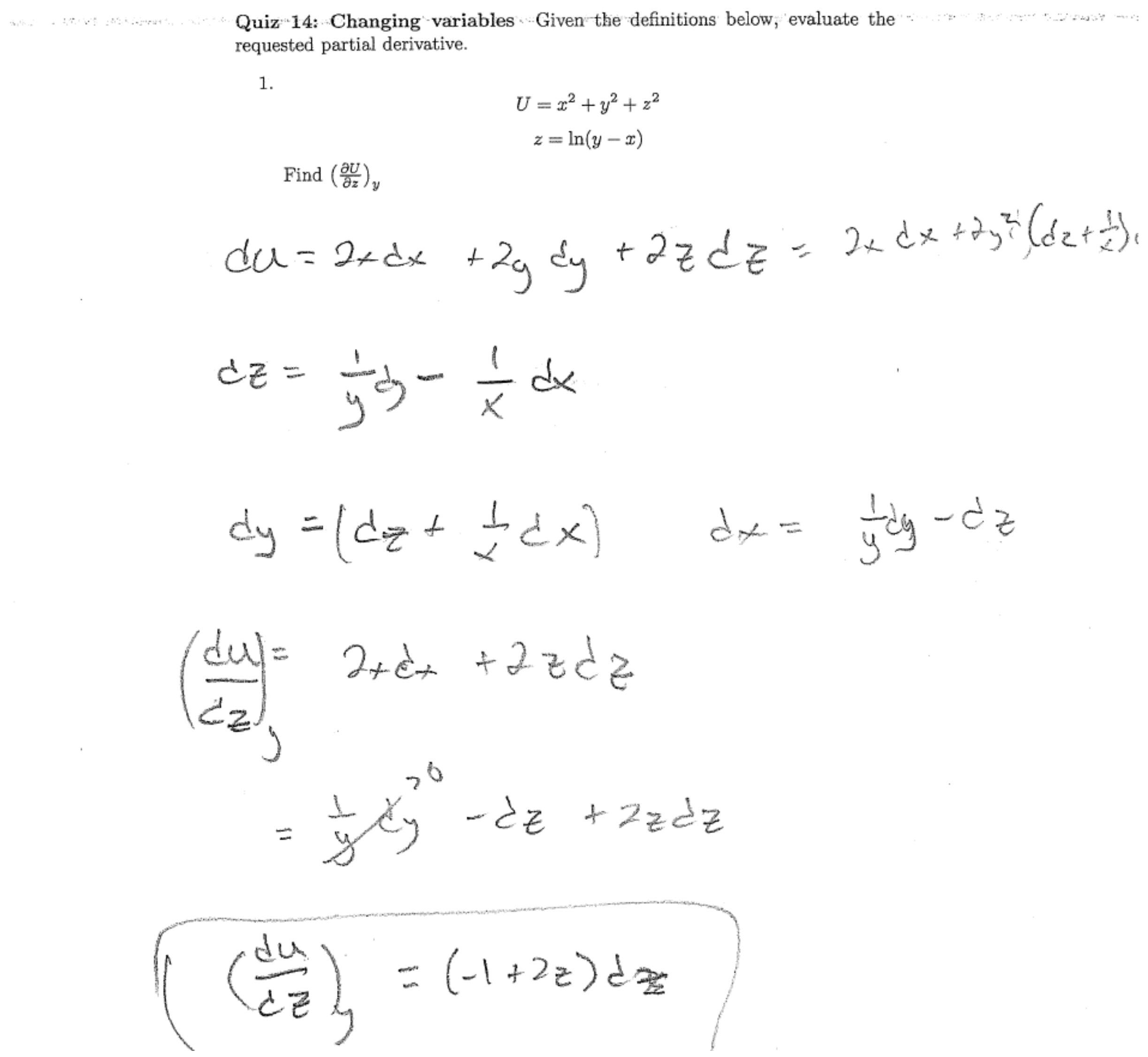
$x = y - e^z$

~~$U = x^2 + y^2 + (\ln(y-x))^2$~~

$U = (y - e^z)^2 + y^2 + z^2$

$\left(\frac{\partial U}{\partial z}\right)_y = 2(y - e^z) \cdot e^z + 2z$

16



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$du = 2x\,dx + 2y\,dy + 2z\,dz = 2x\,dx + 2y\,(dz + \frac{1}{x}$$

$$dz = \frac{1}{y}dy - \frac{1}{x}dx$$

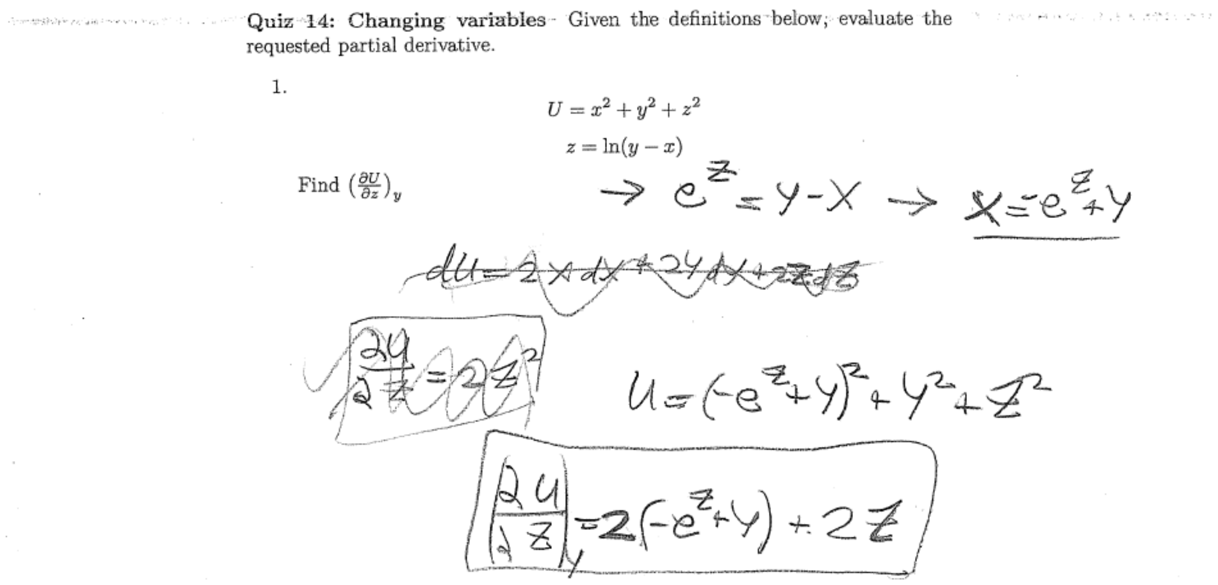
$$dy = (dz + \frac{1}{x}dx) \qquad dx = \frac{1}{y}dy - dz$$

$$\left(\frac{du}{dz}\right)_y = 2x\,dx + 2z\,dz$$

$$= \frac{1}{y}\cancel{dy}^{\to 0} - dz + 2z\,dz$$

$$\boxed{\left(\frac{du}{dz}\right)_y = (-1 + 2z)\,dz}$$

3



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$ $\rightarrow e^z = y - x \rightarrow \underline{x = -e^z + y}$

~~$dU = 2x\,dx + 2y\,dy + 2z\,dz$~~

~~$\boxed{\frac{\partial U}{\partial z} = 2z}$~~

$$U = (-e^z + y)^2 + y^2 + z^2$$

$$\boxed{\left(\frac{\partial U}{\partial z}\right)_y = 2(-e^z + y) + 2z}$$

29

Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

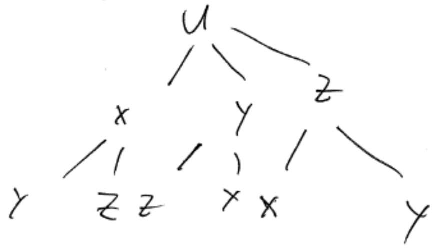
$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$du = 2x\,dx + 2y\,dy + 2z\,dz$

$dz =$

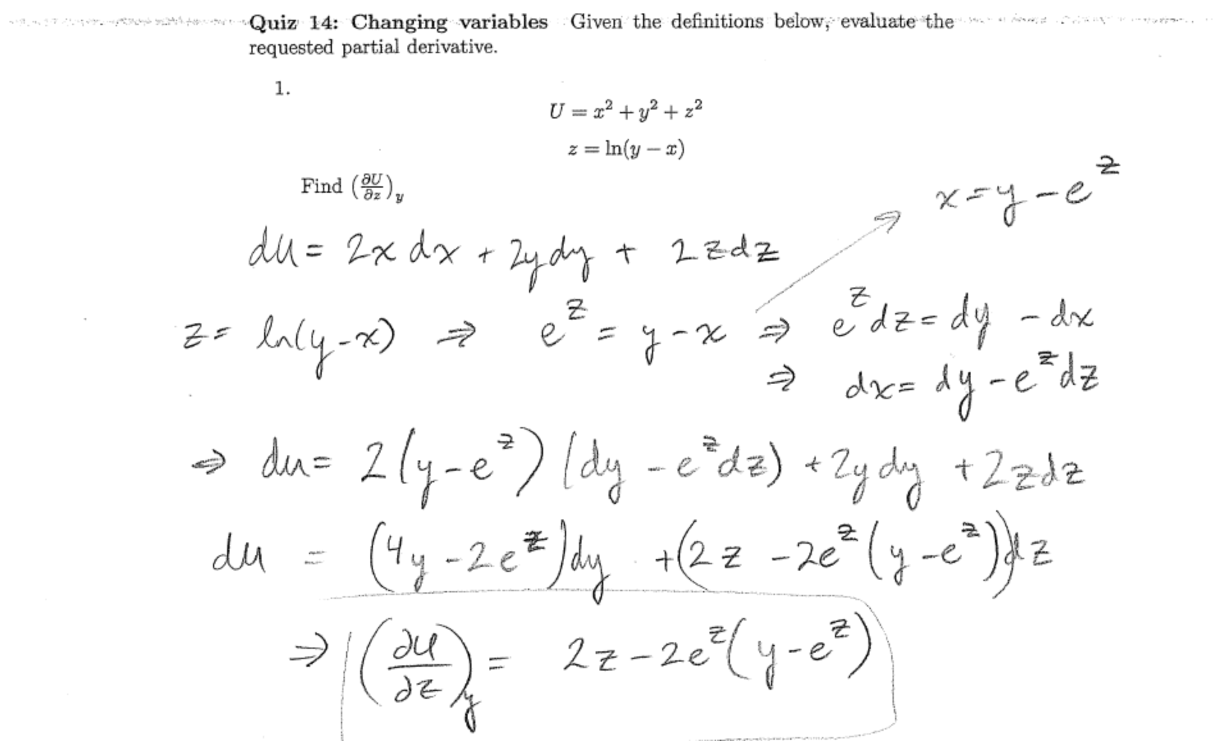


$$\left(\frac{\partial u}{\partial z}\right)_y = \cancel{\left(\frac{\partial u}{\partial z}\right)_{xy}\left(\frac{\partial z}{\partial x}\right)_y}$$

$$= \left(\frac{\partial u}{\partial z}\right)_{xy}\left(\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial z}\right)_y\right) + \cancel{\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial z}\right)}$$

$$\left(\frac{\partial u}{\partial x}\right)_{yz}\left(\frac{\partial x}{\partial z}\right)_y$$

10



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$du = 2x\,dx + 2y\,dy + 2z\,dz$

$x = y - e^z$

$z = \ln(y-x) \Rightarrow e^z = y - x \Rightarrow e^z dz = dy - dx$

$\Rightarrow dx = dy - e^z dz$

$\Rightarrow du = 2(y-e^z)(dy - e^z dz) + 2y\,dy + 2z\,dz$

$du = (4y - 2e^z)dy + (2z - 2e^z(y-e^z))dz$

$\Rightarrow \boxed{\left(\frac{\partial u}{\partial z}\right)_y = 2z - 2e^z(y-e^z)}$

17

Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

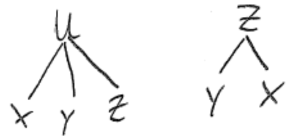
$$U = x^2 + y^2 + z^2$$

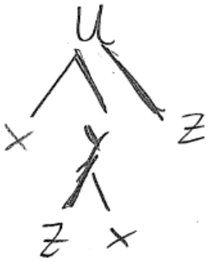
$$z = \ln(y - x)$$

$e^z = y - x$

$y = e^z + x$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

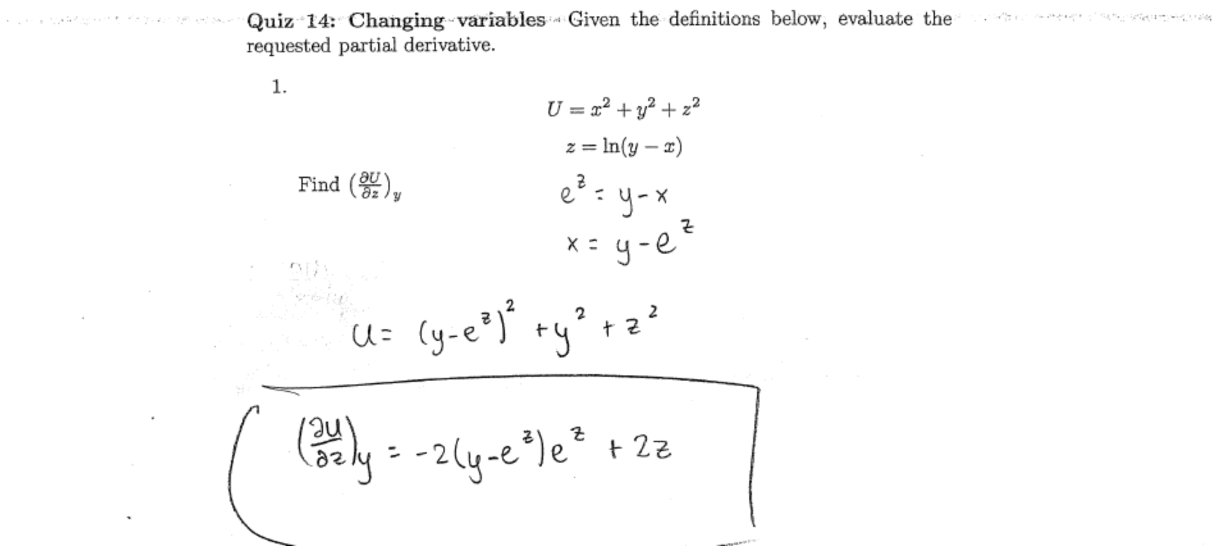




$$\left(\frac{\partial u}{\partial z}\right)_y = \left(\frac{\partial u}{\partial z}\right)_{xy} + \left(\frac{\partial y}{\partial z}\right)_x$$

$$= \boxed{2z + e^z}$$

14



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

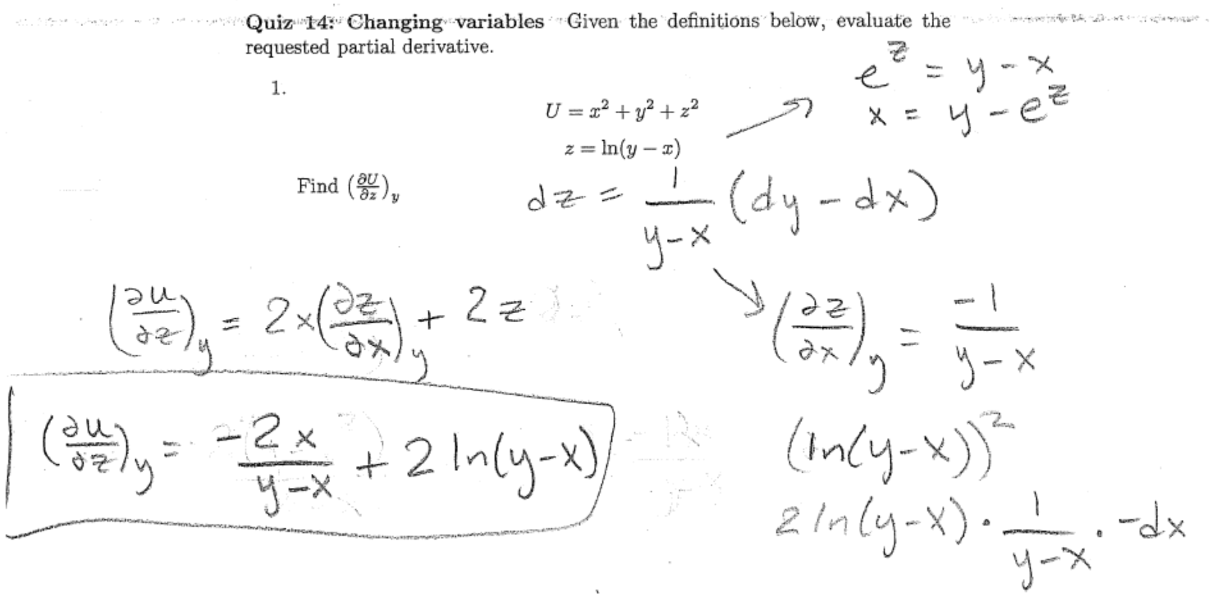
$e^z = y - x$

$x = y - e^z$

$U = (y - e^z)^2 + y^2 + z^2$

$\left(\frac{\partial U}{\partial z}\right)_y = -2(y - e^z)e^z + 2z$

8



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$e^z = y - x$

$x = y - e^z$

$dz = \frac{1}{y-x}(dy - dx)$

$\left(\frac{\partial z}{\partial x}\right)_y = \frac{-1}{y-x}$

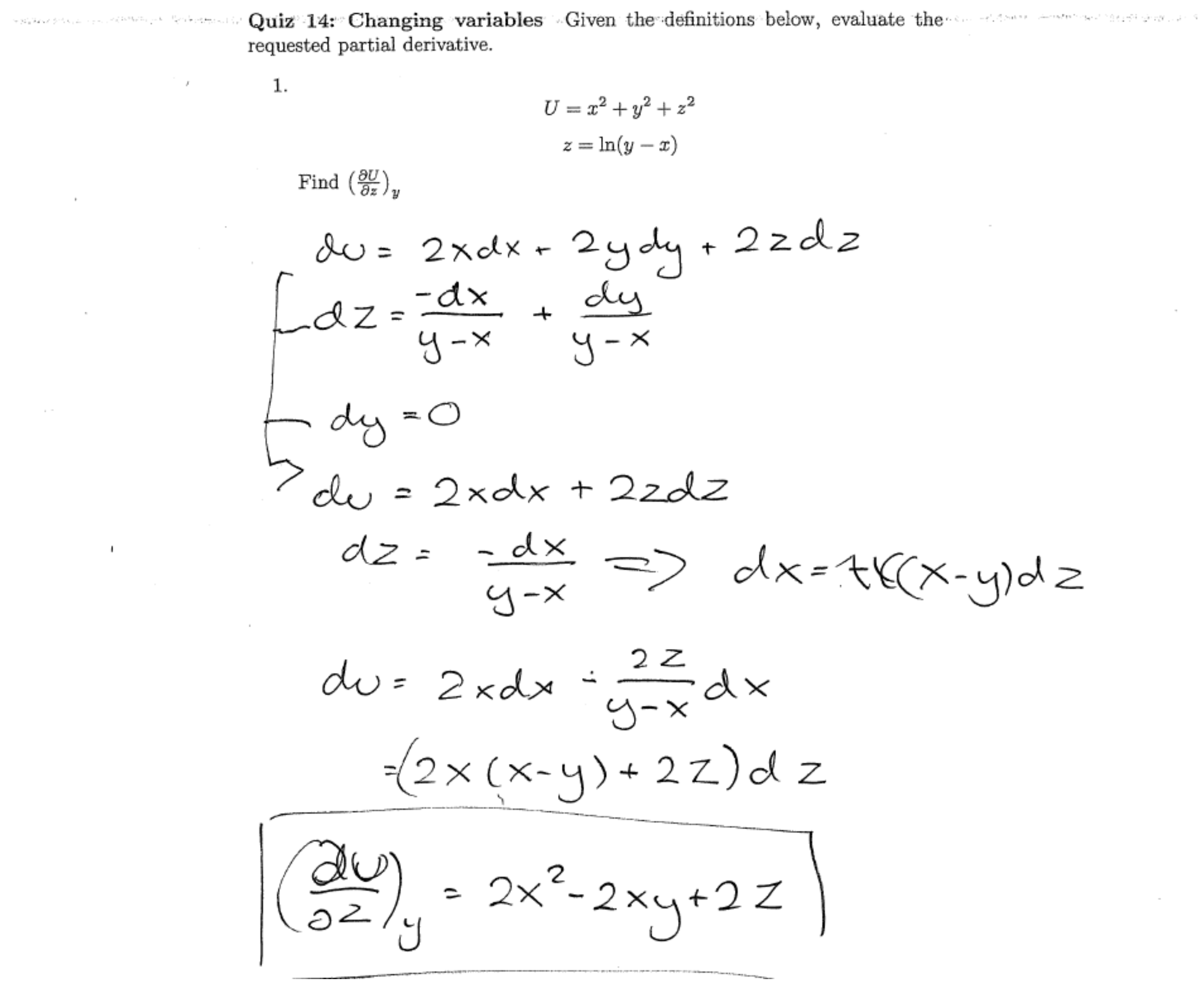
$\left(\frac{\partial u}{\partial z}\right)_y = 2x\left(\frac{\partial z}{\partial x}\right)_y + 2z$

$\left(\frac{\partial u}{\partial z}\right)_y = \frac{-2x}{y-x} + 2\ln(y-x)$

$(\ln(y-x))^2$

$2\ln(y-x) \cdot \frac{1}{y-x} \cdot -dx$

18



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$dU = 2x\,dx + 2y\,dy + 2z\,dz$

$dz = \frac{-dx}{y-x} + \frac{dy}{y-x}$

$dy = 0$

$dU = 2x\,dx + 2z\,dz$

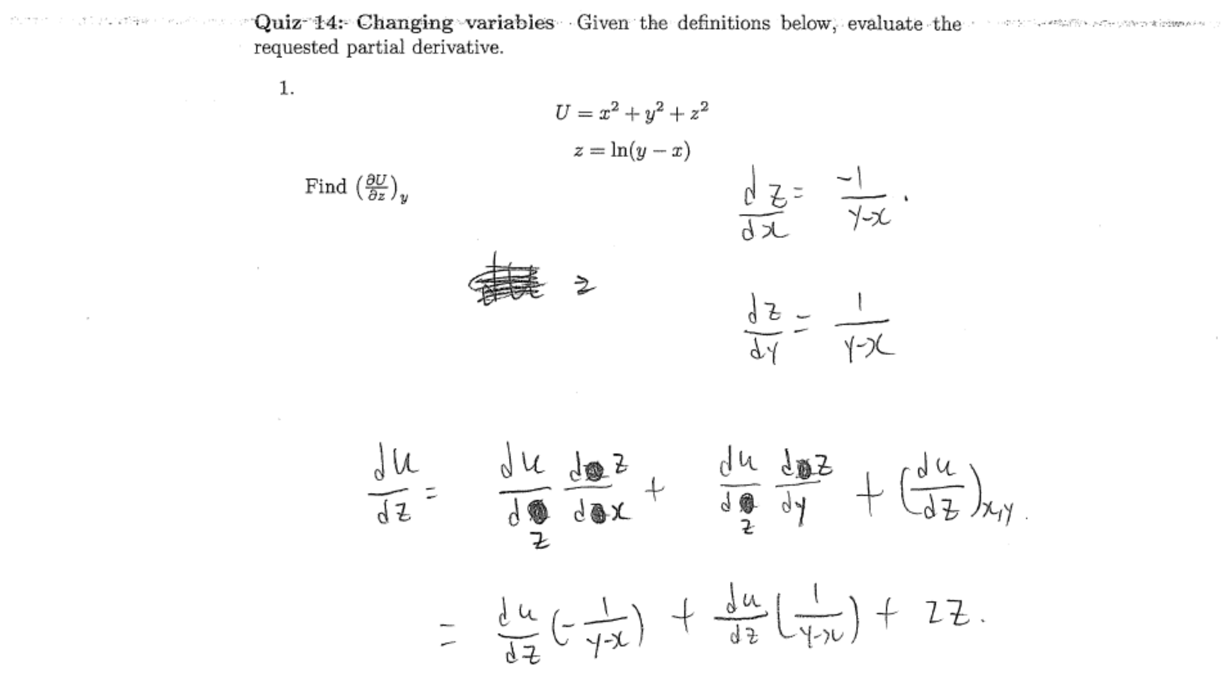
$dz = \frac{-dx}{y-x} \Rightarrow dx = +((x-y)dz$

$dU = 2x\,dx \div \frac{2z}{y-x} dx$

$= (2x(x-y) + 2z)\,dz$

$$\boxed{\left(\frac{\partial U}{\partial z}\right)_y = 2x^2 - 2xy + 2z}$$

20



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$\frac{dz}{dx} = \frac{-1}{y-x}.$$

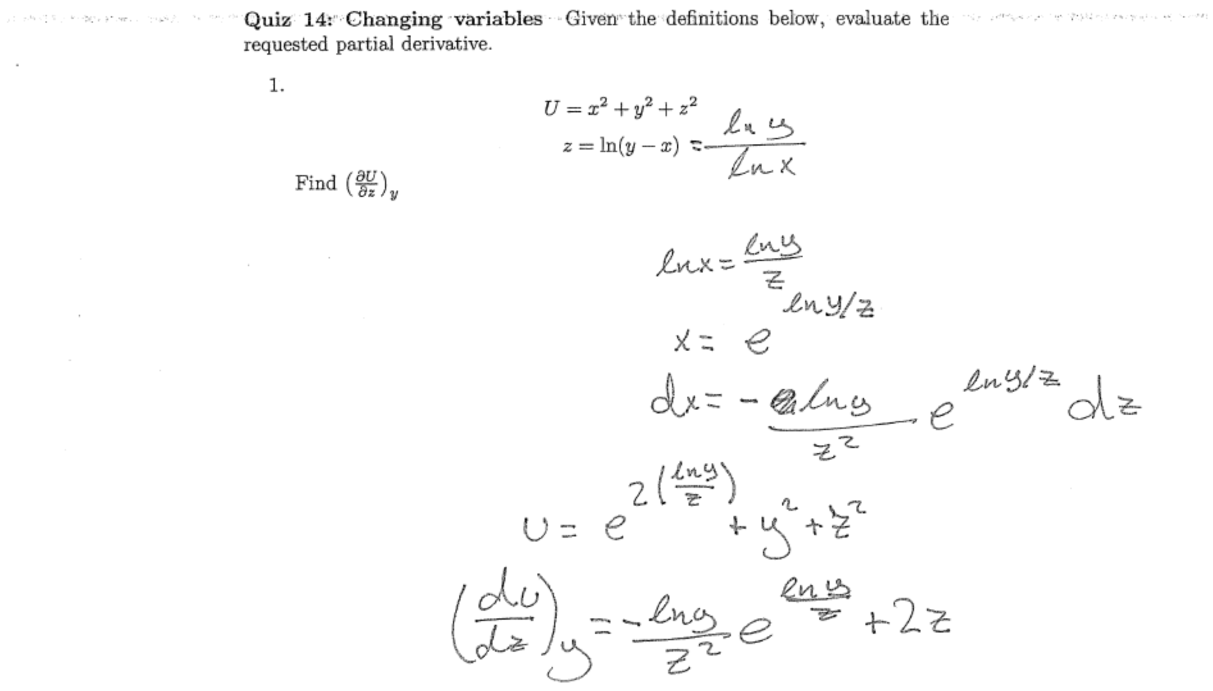
z

$$\frac{dz}{dy} = \frac{1}{y-x}$$

$$\frac{du}{dz} = \frac{du}{dz}\frac{dz}{dx} + \frac{du}{dz}\frac{dz}{dy} + \left(\frac{du}{dz}\right)_{x,y}.$$

$$= \frac{du}{dz}\left(-\frac{1}{y-x}\right) + \frac{du}{dz}\left(\frac{1}{y-x}\right) + 2z.$$

2



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x) = \frac{\ln y}{\ln x}$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$\ln x = \frac{\ln y}{z}$$

$$x = e^{\ln y/z}$$

$$dx = -\frac{\ln y}{z^2} e^{\ln y/z} dz$$

$$U = e^{2\left(\frac{\ln y}{z}\right)} + y^2 + z^2$$

$$\left(\frac{du}{dz}\right)_y = -\frac{\ln y}{z^2} e^{\frac{\ln y}{z}} + 2z$$

5

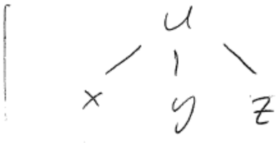
Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

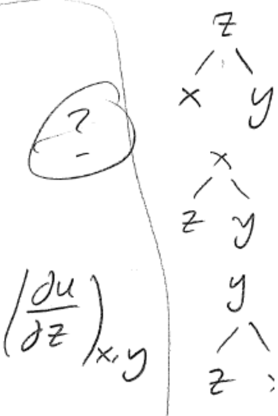
$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$



$$du = \left(\frac{\partial u}{\partial x}\right)_{y,z} dx + \left(\frac{\partial u}{\partial y}\right)_{x,z} dy + \left(\frac{\partial u}{\partial z}\right)_{x,y} \partial z$$



$$\left(\frac{\partial u}{\partial z}\right)_y = \left(\frac{\partial u}{\partial x}\right)_{yz}\left(\frac{\partial x}{\partial z}\right)_y + \overset{0}{\cancel{\left(\frac{\partial u}{\partial y}\right)_{x,z}\left(\frac{\partial y}{\partial z}\right)_x}} + \left(\frac{\partial u}{\partial z}\right)_{x,y}$$

$$\left(\frac{\partial u}{\partial z}\right)_y = 2x \underbrace{\left(\frac{dx}{\partial z}\right)_y}_{} + 2z$$

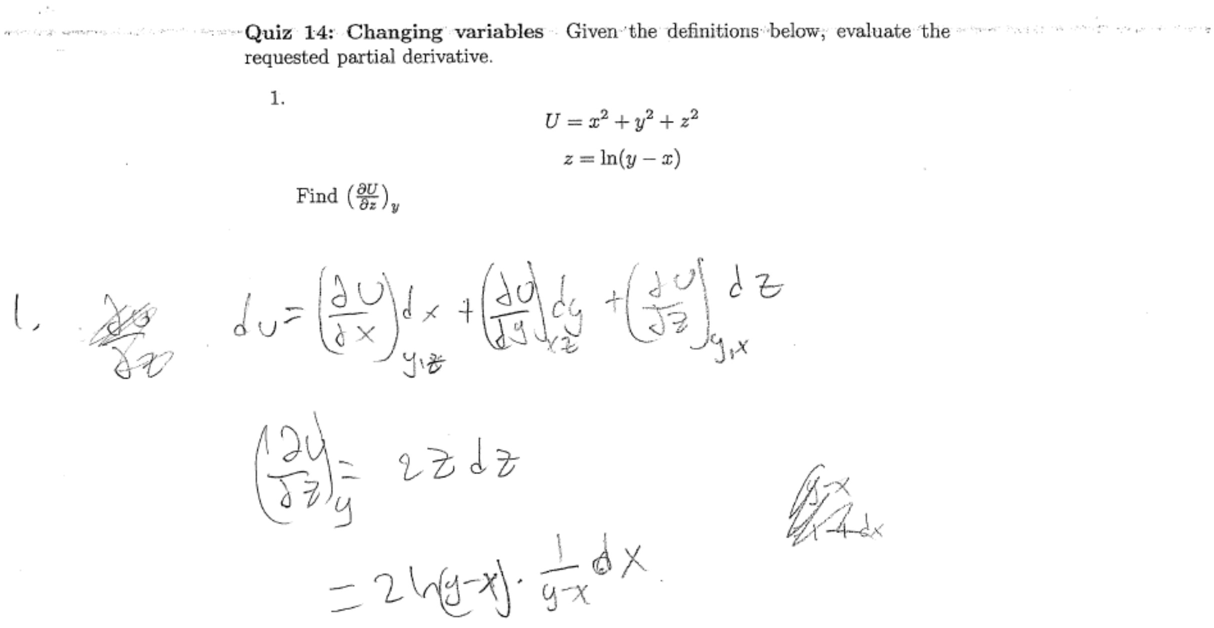
$$e^z = y - x$$

$$x = y - e^z$$

$$\left(\frac{\partial x}{\partial z}\right)_y = 0 - e^z$$

$$\left(\frac{\partial u}{\partial z}\right)_y = 2x(-e^z) + 2z \; {}_1 \quad = \quad 2z - 2xe^z$$

11



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

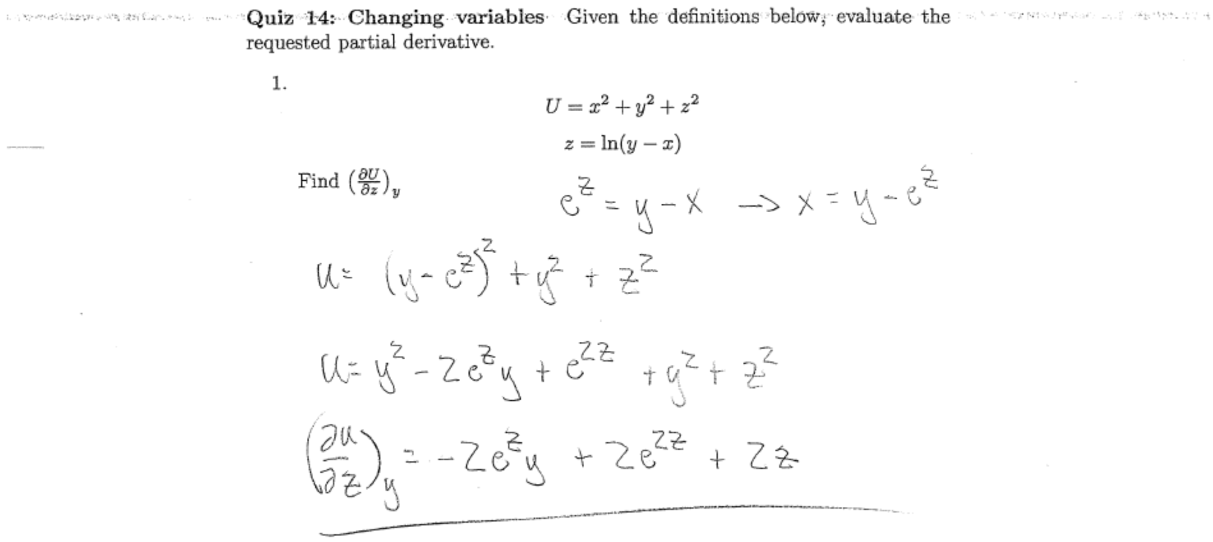
1.

$$dU = \left(\frac{\partial U}{\partial x}\right)_{y,z} dx + \left(\frac{\partial U}{\partial y}\right)_{x,z} dy + \left(\frac{\partial U}{\partial z}\right)_{y,x} dz$$

$$\left(\frac{\partial U}{\partial z}\right)_y = 2z\,dz$$

$$= 2\ln(y-x) \cdot \frac{1}{y-x} dx$$

26



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$e^z = y - x \longrightarrow x = y - e^z$$

$$U = (y - e^z)^2 + y^2 + z^2$$

$$U = y^2 - 2e^z y + e^{2z} + y^2 + z^2$$

$$\left(\frac{\partial u}{\partial z}\right)_y = -2e^z y + 2e^{2z} + 2z$$

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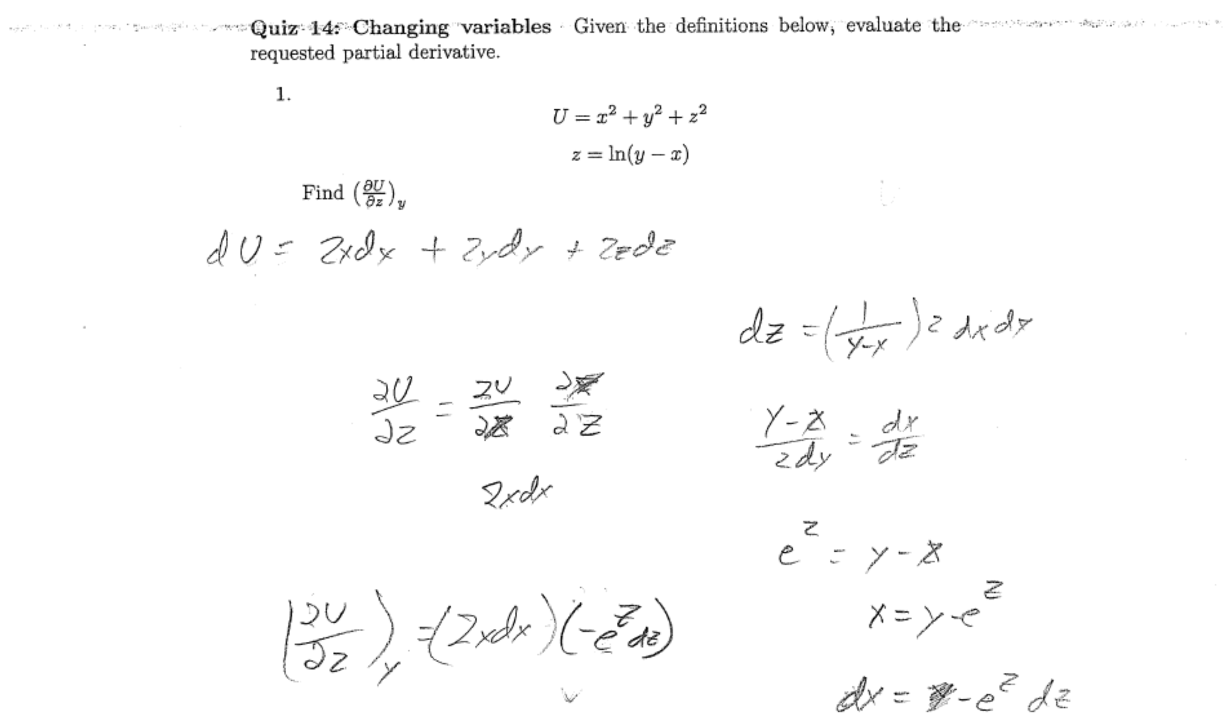

Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$dU = 2x\,dx + 2y\,dy + 2z\,dz$

$dz = \left(\frac{1}{y-x}\right) 2\, dx\, dx$

$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial \cancel{x}} \frac{\partial \cancel{x}}{\partial z}$

$\frac{y - \cancel{x}}{2dy} = \frac{dx}{dz}$

$2x\,dx$

$e^z = y - \cancel{x}$

$x = y - e^z$

$\left(\frac{\partial U}{\partial z}\right)_y = (2x\,dx)(-e^z dz)$

$dx = \cancel{y} - e^z\, dz$

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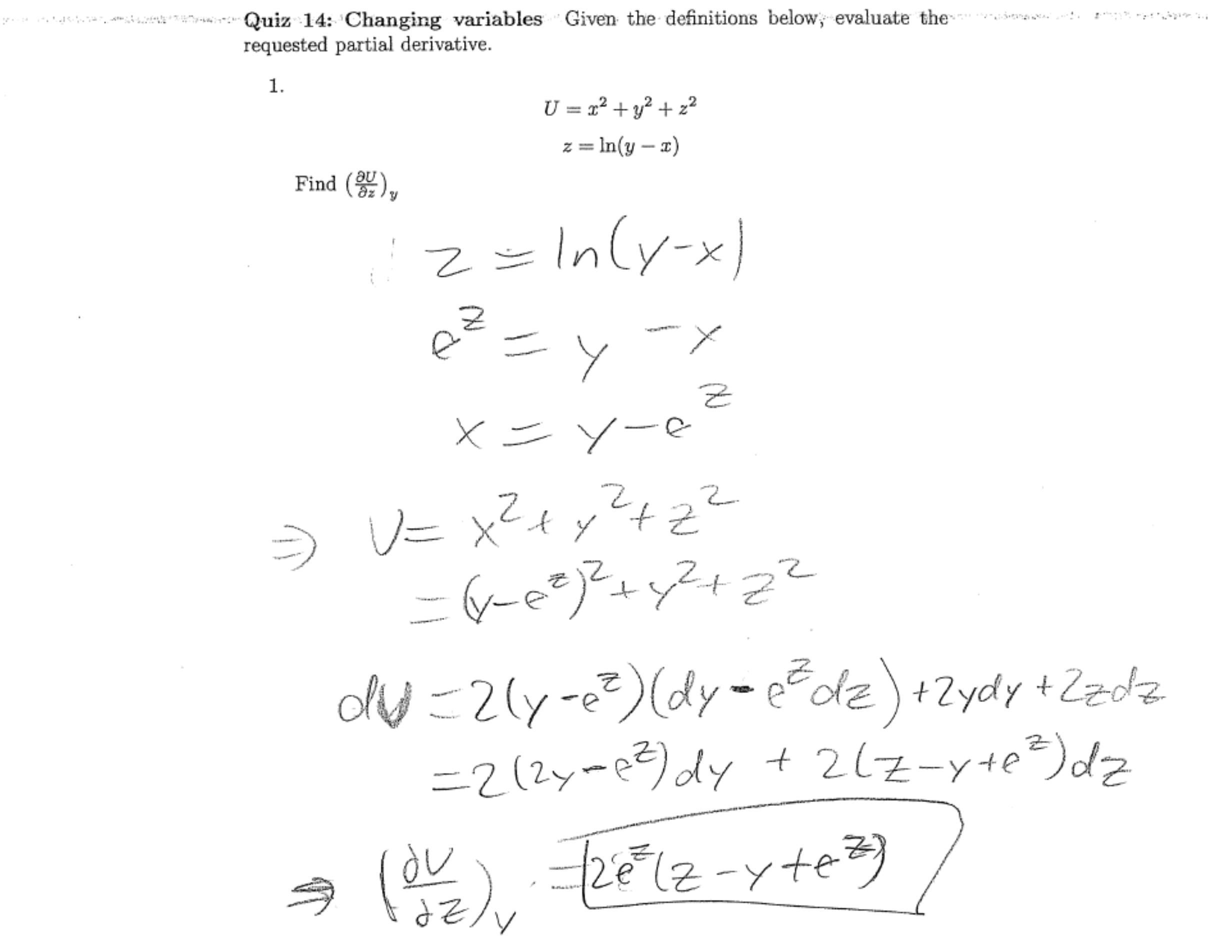


Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$$z = \ln(y-x)$$

$$e^z = y - x$$

$$x = y - e^z$$

$$\Rightarrow V = x^2 + y^2 + z^2$$

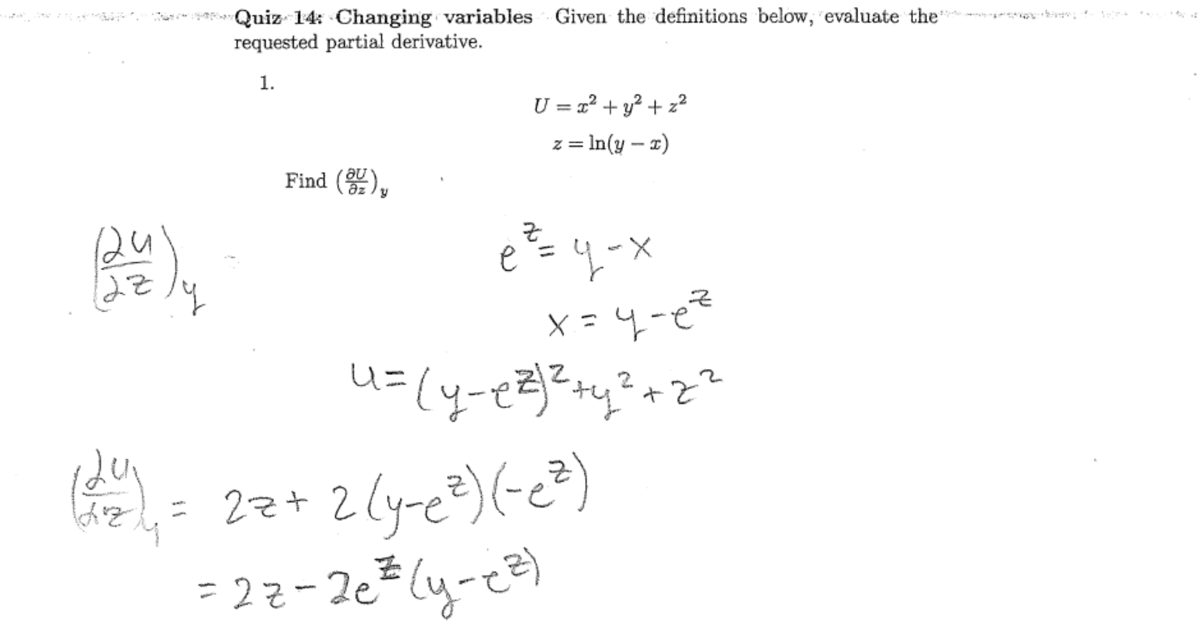
$$= (y-e^z)^2 + y^2 + z^2$$

$$dV = 2(y-e^z)(dy - e^z dz) + 2y\,dy + 2z\,dz$$

$$= 2(2y-e^z)\,dy + 2(z-y+e^z)\,dz$$

$$\Rightarrow \left(\frac{\partial V}{\partial z}\right)_y = \boxed{2e^z(z-y+e^z)}$$

15



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$\left(\frac{\partial u}{\partial z}\right)_y$

$e^z = y - x$

$x = y - e^z$

$u = (y - e^z)^2 + y^2 + z^2$

$\left(\frac{\partial u}{\partial z}\right)_y = 2z + 2(y - e^z)(-e^z)$

$= 2z - 2e^z(y - e^z)$

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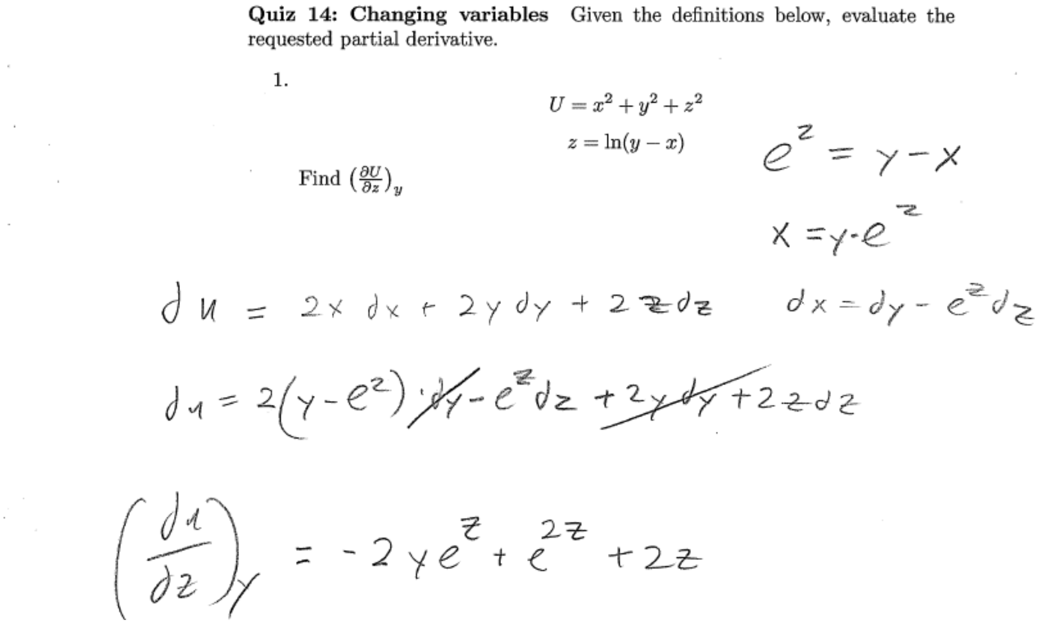


Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$e^z = y - x$

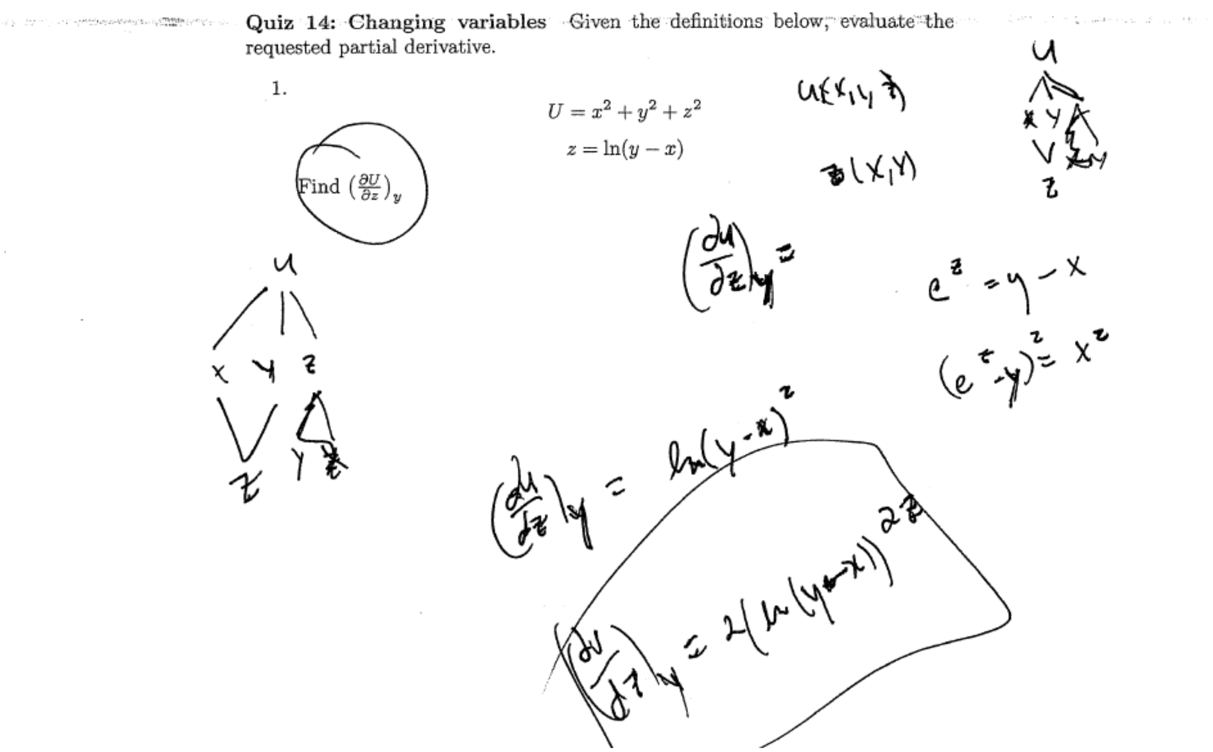
$x = y - e^z$

$du = 2x\,dx + 2y\,dy + 2z\,dz$ $\quad dx = dy - e^z dz$

$du = 2(y - e^z)\cdot \cancel{dy} - e^z dz + \cancel{2y\,dy} + 2z\,dz$

$\left(\frac{du}{dz}\right)_y = -2ye^z + e^{2z} + 2z$

4



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$U(x, y, z)$

$z(x, y)$

$\left(\frac{\partial u}{\partial z}\right)_y =$

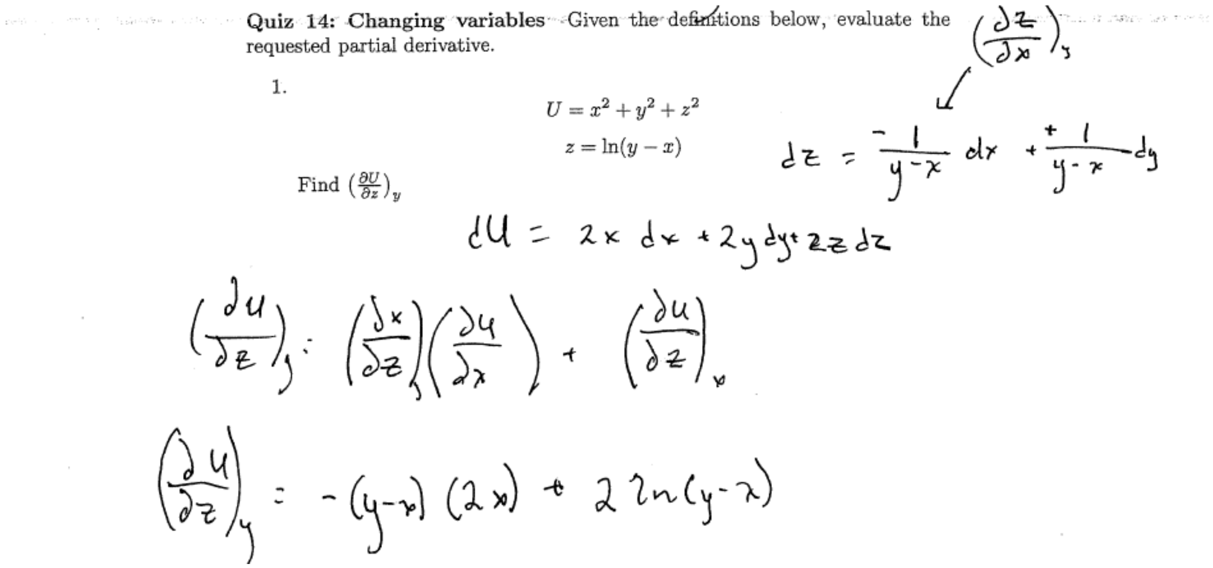
$e^z = y - x$

$(e^z - y)^2 = x^2$

$\left(\frac{\partial u}{\partial z}\right)_y = \ln(y-x)^2$

$\left(\frac{\partial u}{\partial z}\right)_y = 2(\ln(y-x))^{2z}$

1



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$\left(\frac{\partial z}{\partial x}\right)_y$ ✓

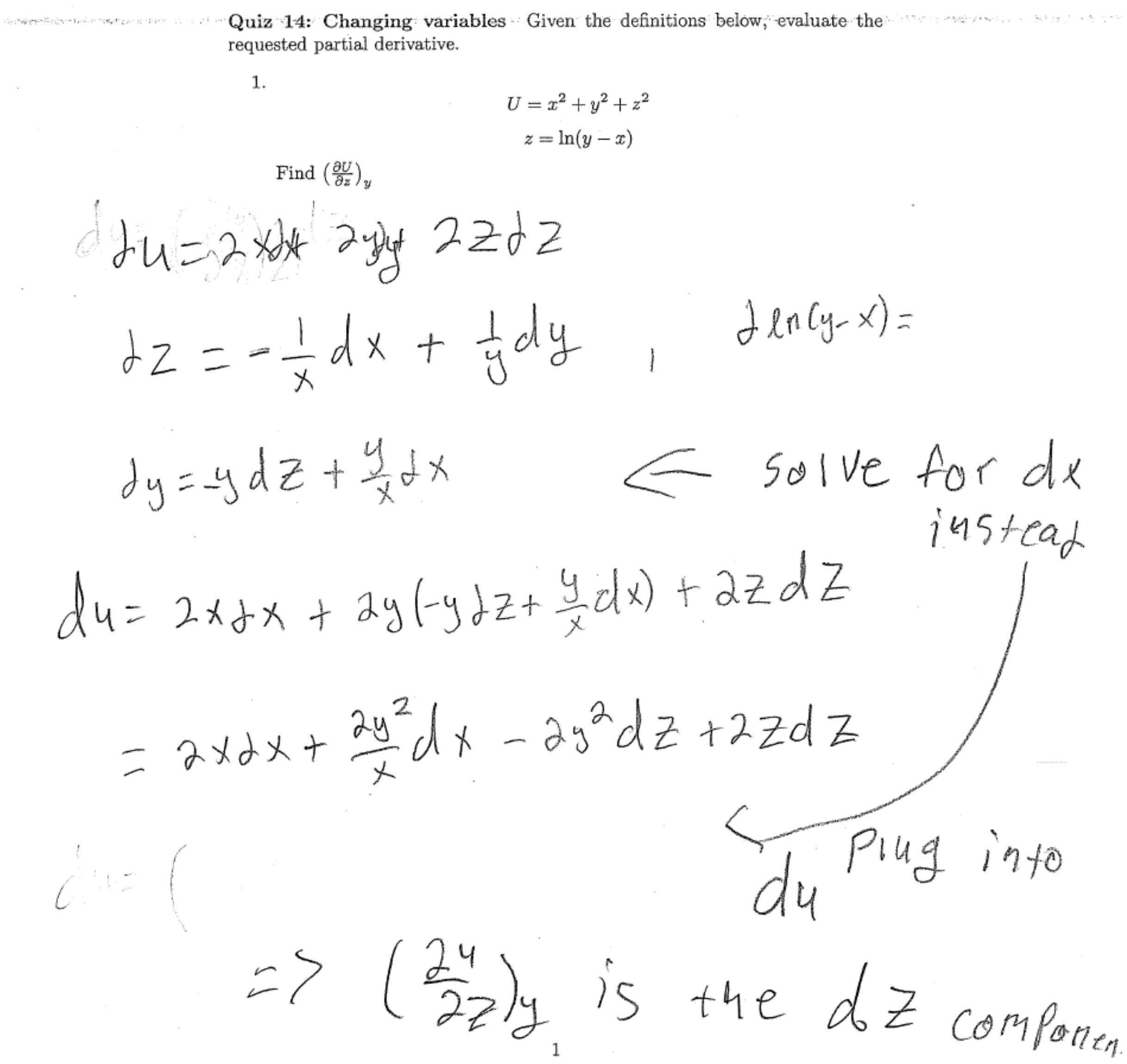
$dz = \frac{-1}{y-x}\,dx + \frac{+1}{y-x}\,dy$

$dU = 2x\,dx + 2y\,dy + 2z\,dz$

$\left(\frac{\partial u}{\partial z}\right)_y = \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial z}\right)_x$

$\left(\frac{\partial u}{\partial z}\right)_y = -(y-x)(2x) + 2\ln(y-x)$

25



Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

1.

$$U = x^2 + y^2 + z^2$$
$$z = \ln(y - x)$$

Find $\left(\frac{\partial U}{\partial z}\right)_y$

$du = 2xdx \quad 2ydy \quad 2zdz$

$dz = -\frac{1}{x}dx + \frac{1}{y}dy$, $d\ln(y-x) =$

$dy = -ydz + \frac{y}{x}dx$ ← solve for dx instead

$du = 2xdx + 2y(-ydz + \frac{y}{x}dx) + 2zdz$

$= 2xdx + \frac{2y^2}{x}dx - 2y^2dz + 2zdz$

du Plug into

$\Rightarrow \left(\frac{\partial u}{\partial z}\right)_y$ is the dz componen

1

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Quiz 14: Changing variables Given the definitions below, evaluate the requested partial derivative.

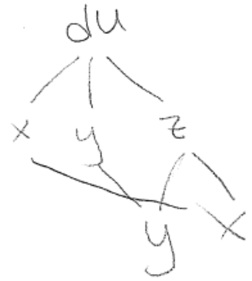
1.

Uhh oh, this wasn't the practice quiz I looked at! :(

$$U = x^2 + y^2 + z^2$$

$$z = \ln(y - x)$$

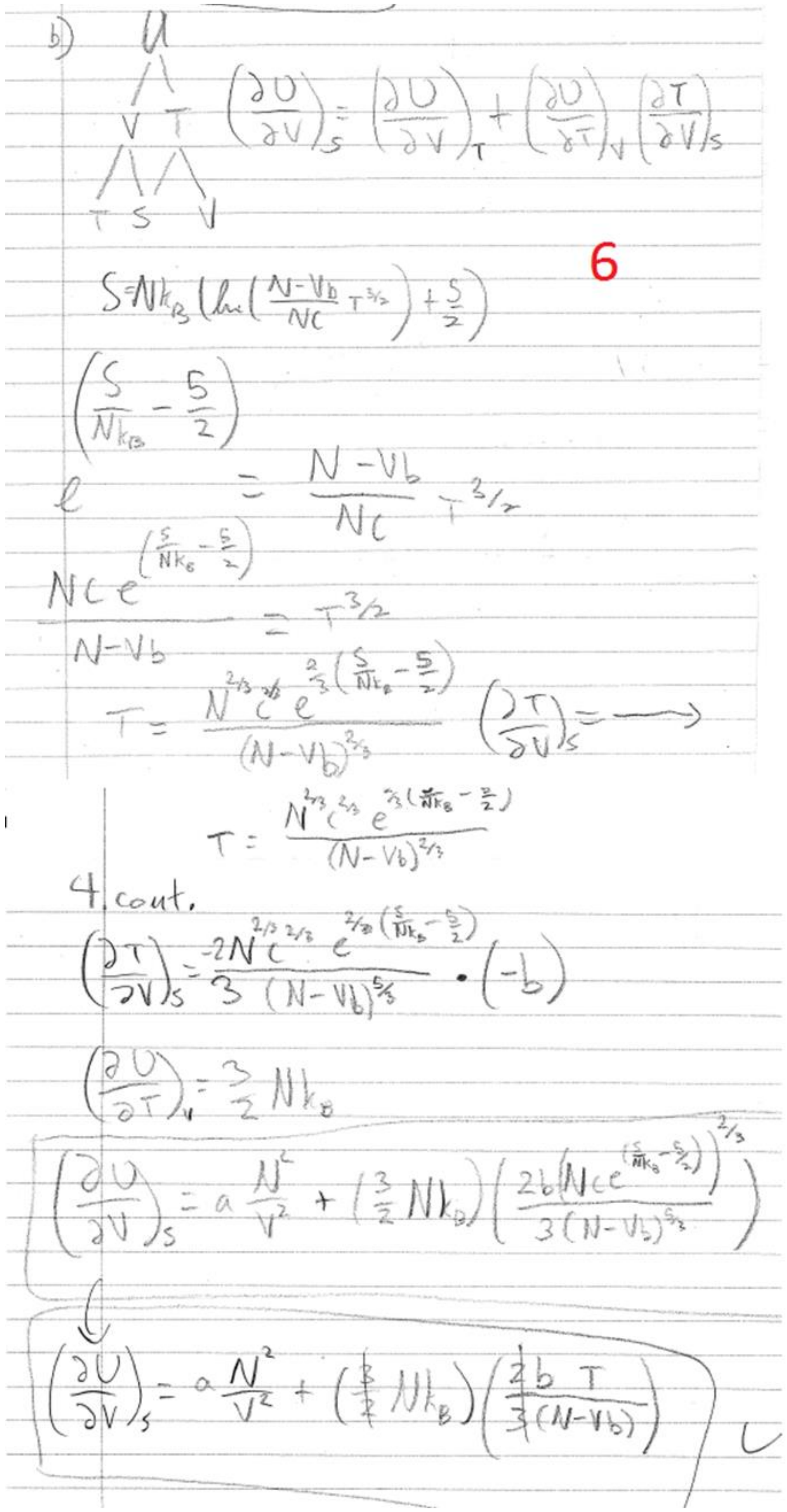
Find $\left(\frac{\partial U}{\partial z}\right)_y$



$$\left(\frac{\partial u}{\partial z}\right)_y = \left(\frac{\partial u}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_y$$

$$\left(\frac{\partial u}{\partial z}\right)_y = (2z)\left(\frac{-1}{y-x}\right) = \boxed{\frac{-2\ln(y-x)}{y-x}}$$

E.3. Final exam, question 4.b



b) U

V T

T S V

$$\left(\frac{\partial U}{\partial V}\right)_S = \left(\frac{\partial U}{\partial V}\right)_T + \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S$$

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$$S = Nk_B\left(\ln\left(\frac{N-Vb}{Nc}T^{3/2}\right)+\frac{5}{2}\right)$$

$$\left(\frac{S}{Nk_B}-\frac{5}{2}\right)$$

$$e \quad = \frac{N-Vb}{Nc}T^{3/2}$$

$$\frac{Nce^{\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}}{N-Vb} = T^{3/2}$$

$$T = \frac{N^{2/3\,\cancel{2/3}}c^{2/3}e^{\frac{2}{3}\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}}{(N-Vb)^{2/3}} \qquad \left(\frac{\partial T}{\partial V}\right)_S = \longrightarrow$$

$$T = \frac{N^{2/3}c^{2/3}e^{\frac{2}{3}\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}}{(N-Vb)^{2/3}}$$

4. cont.

$$\left(\frac{\partial T}{\partial V}\right)_S = \frac{-2N^{2/3}c^{2/3}e^{2/3\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}}{3\,(N-Vb)^{5/3}}\cdot(-b)$$

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}Nk_B$$

$$\boxed{\left(\frac{\partial U}{\partial V}\right)_S = a\frac{N^2}{V^2} + \left(\frac{3}{2}Nk_B\right)\left(\frac{2b\left(Nce^{\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}\right)^{2/3}}{3(N-Vb)^{5/3}}\right)}$$

$$\boxed{\left(\frac{\partial U}{\partial V}\right)_S = a\frac{N^2}{V^2} + \left(\frac{\cancel{3}}{\cancel{2}}Nk_B\right)\left(\frac{\cancel{2}b\;T}{\cancel{3}(N-Vb)}\right)}$$

✓

4. a) First, zap S and U:

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$$dS = Nk_B\left[\frac{b}{N-Vb}dV + \frac{3}{2}\frac{1}{T}dT\right]$$

$$dU = \frac{3}{2}Nk_B dT + aN^2V^{-2}dV$$

Then, from dU we have

$$\boxed{\left(\frac{\partial U}{\partial V}\right)_T = aN^2V^{-2}}$$

b) Rearranging dS above,

$$\frac{1}{Nk_B}dS = \frac{b}{N-Vb}dV + \frac{3}{2T}dT$$

$$\frac{1}{Nk_B}dS - \frac{b}{N-Vb}dV = \frac{3}{2T}dT$$

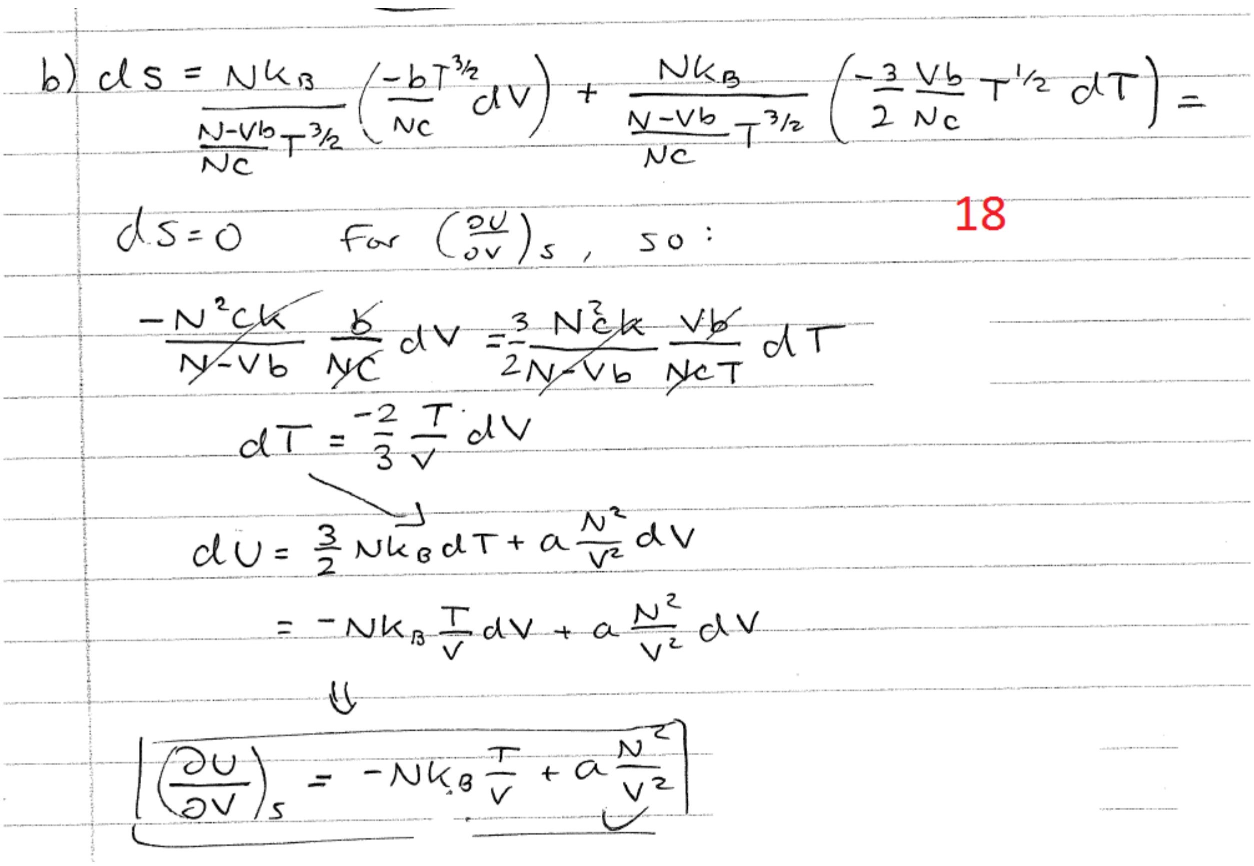
$$dT = \left[\frac{1}{Nk_B}dS - \frac{b}{N-Vb}dV\right]\frac{2T}{3}$$

Substituting into dU, we have

$$dU = \frac{3}{2}Nk_B\left[\frac{2T}{3}\left(\frac{1}{Nk_B}dS - \frac{b}{N-Vb}dV\right)\right] + aN^2V^{-2}dV$$

$$= TdS + \left(aN^2V^{-2} + \frac{TNk_Bb}{N-Vb}\right)dV. *$$

Thus, $\boxed{\left(\frac{\partial U}{\partial V}\right)_S = aN^2V^{-2} + \frac{TNk_Bb}{N-Vb}}$



b) $dS = \frac{Nk_B}{\frac{N-Vb}{Nc}T^{3/2}}\left(\frac{-bT^{3/2}}{Nc}dV\right) + \frac{Nk_B}{\frac{N-Vb}{Nc}T^{3/2}}\left(-\frac{3}{2}\frac{Vb}{Nc}T^{1/2}dT\right) =$

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$dS = 0$ for $\left(\frac{\partial U}{\partial V}\right)_S$, so:

$$-\cancel{\frac{N^2ck}{N-Vb}}\frac{\cancel{b}}{\cancel{N}c}dV = -\frac{3}{2}\cancel{\frac{N^2ck}{N-Vb}}\frac{V\cancel{b}}{\cancel{N}cT}dT$$

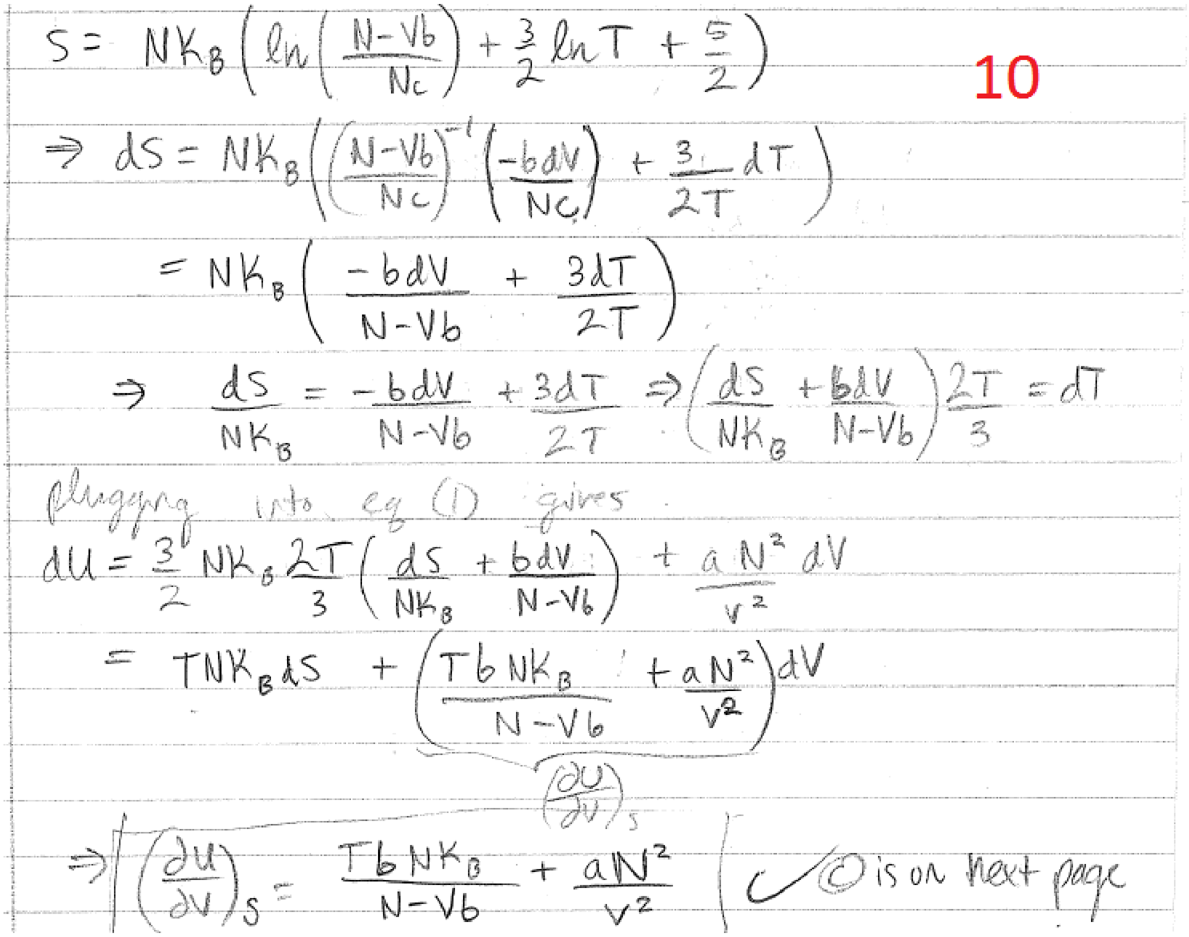
$$dT = \frac{-2}{3}\frac{T}{V}dV$$

$$dU = \frac{3}{2}Nk_B dT + a\frac{N^2}{V^2}dV$$

$$= -Nk_B\frac{T}{V}dV + a\frac{N^2}{V^2}dV$$

$$\Downarrow$$

$$\boxed{\left(\frac{\partial U}{\partial V}\right)_S = -Nk_B\frac{T}{V} + a\frac{N^2}{V^2}}$$



10

$$S = Nk_B\left(\ln\left(\frac{N-Vb}{Nc}\right) + \frac{3}{2}\ln T + \frac{5}{2}\right)$$

$$\Rightarrow dS = Nk_B\left(\left(\frac{N-Vb}{Nc}\right)^{-1}\left(\frac{-b\,dV}{Nc}\right) + \frac{3}{2T}dT\right)$$

$$= Nk_B\left(\frac{-b\,dV}{N-Vb} + \frac{3dT}{2T}\right)$$

$$\Rightarrow \frac{dS}{Nk_B} = \frac{-b\,dV}{N-Vb} + \frac{3dT}{2T} \Rightarrow \left(\frac{dS}{Nk_B} + \frac{b\,dV}{N-Vb}\right)\frac{2T}{3} = dT$$

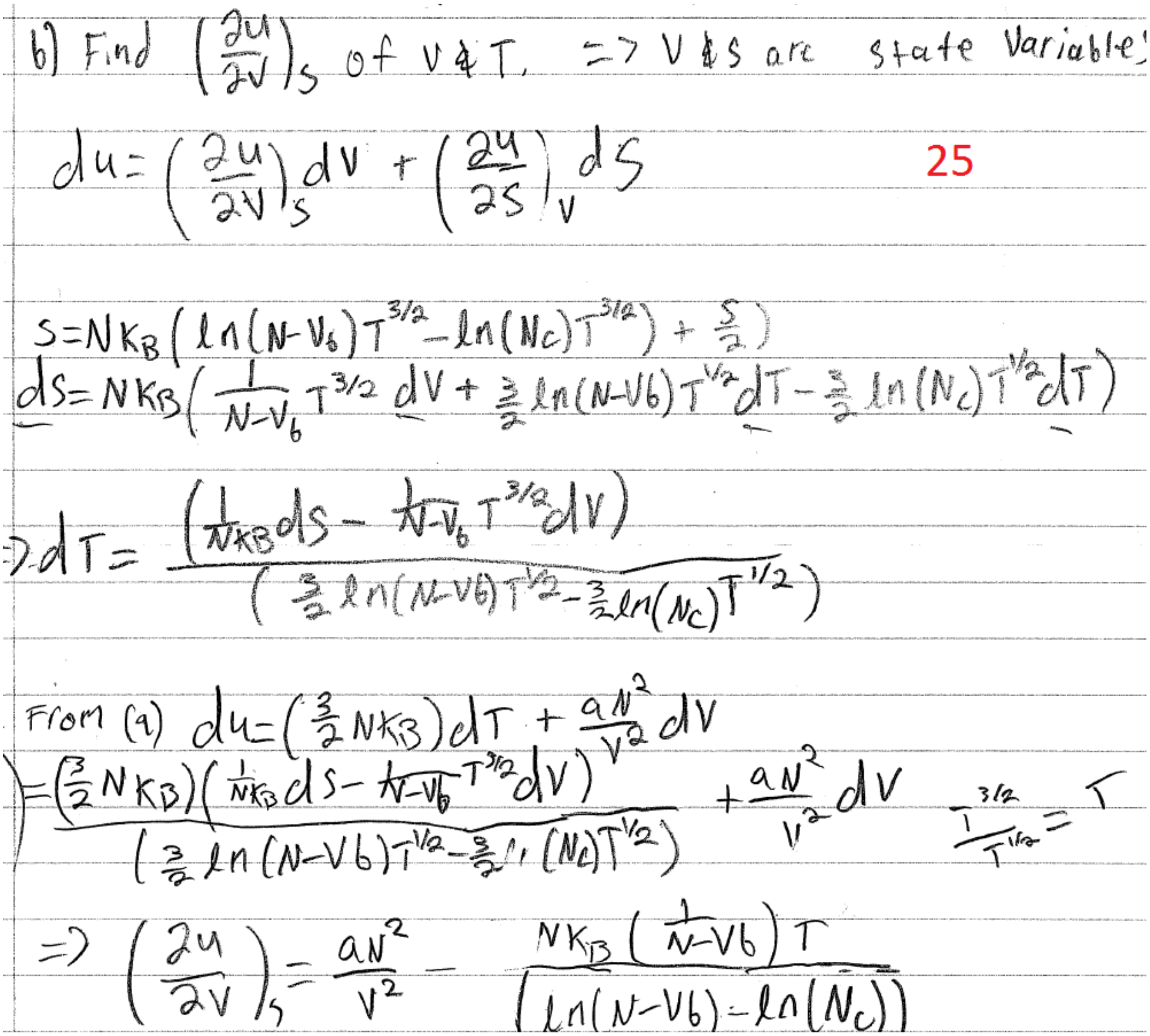
plugging into eq (1) gives

$$dU = \frac{3}{2}Nk_B\frac{2T}{3}\left(\frac{dS}{Nk_B} + \frac{b\,dV}{N-Vb}\right) + \frac{aN^2}{V^2}dV$$

$$= TNk_B\,dS + \underbrace{\left(\frac{TbNk_B}{N-Vb} + \frac{aN^2}{V^2}\right)}_{\left(\frac{\partial U}{\partial V}\right)_S}dV$$

$$\Rightarrow \boxed{\left(\frac{\partial U}{\partial V}\right)_S = \frac{TbNk_B}{N-Vb} + \frac{aN^2}{V^2}}$$

✓ ⓒ is on next page



b) Find $\left(\frac{\partial u}{\partial V}\right)_S$ of V & T. => V & S are state variables

$du = \left(\frac{\partial u}{\partial V}\right)_S dV + \left(\frac{\partial u}{\partial S}\right)_V dS$

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$S = Nk_B\left(\ln(N-V_b)T^{3/2} - \ln(N_c)T^{3/2}\right) + \frac{5}{2})$

$dS = Nk_B\left(\frac{1}{N-V_b}T^{3/2}\,dV + \frac{3}{2}\ln(N-Vb)T^{1/2}\,dT - \frac{3}{2}\ln(N_c)T^{1/2}\,dT\right)$

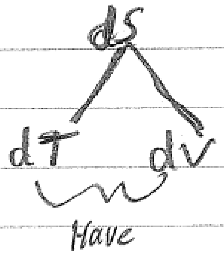
$$\Rightarrow dT = \frac{\left(\frac{1}{Nk_B}dS - \frac{1}{N-V_b}T^{3/2}dV\right)}{\left(\frac{3}{2}\ln(N-Vb)T^{1/2} - \frac{3}{2}\ln(N_c)T^{1/2}\right)}$$

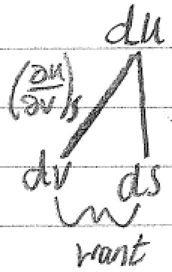
From (a) $du = \left(\frac{3}{2}Nk_B\right)dT + \frac{aN^2}{V^2}dV$

$$= \frac{\left(\frac{3}{2}Nk_B\right)\left(\frac{1}{Nk_B}dS - \frac{1}{N-Vb}T^{3/2}dV\right)}{\left(\frac{3}{2}\ln(N-Vb)T^{1/2} - \frac{3}{2}\ln(N_c)T^{1/2}\right)} + \frac{aN^2}{V^2}dV \qquad \frac{T^{3/2}}{T^{1/2}} = T$$

$$\Rightarrow \left(\frac{\partial u}{\partial V}\right)_S = \frac{aN^2}{V^2} - \frac{Nk_B\left(\frac{1}{N-Vb}\right)T}{\left(\ln(N-Vb) - \ln(N_c)\right)}$$

b)





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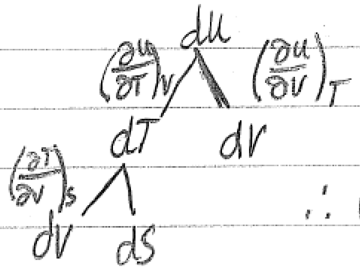


$$\therefore \left(\frac{\partial u}{\partial V}\right)_S = \left(\frac{\partial u}{\partial V}\right)_T dV + \left(\frac{\partial u}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S dV$$

We know $\left(\frac{\partial u}{\partial V}\right)_T = a\frac{N^2}{V^2}$ and $\left(\frac{\partial u}{\partial T}\right)_V = \frac{3}{2}Nk_B$, $\left(\frac{\partial T}{\partial V}\right)_S$ is what we need.

To get $\left(\frac{\partial T}{\partial V}\right)_S$ we use the cyclic chain rule:

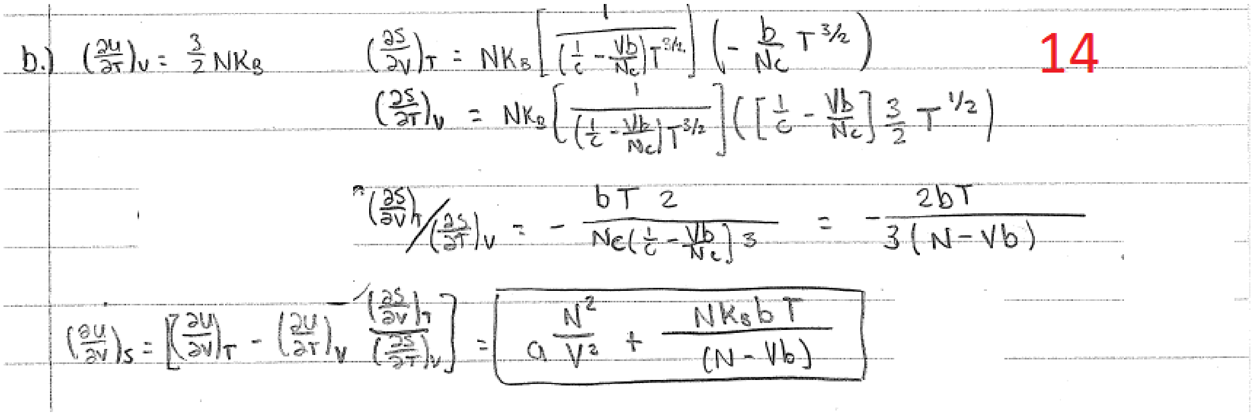
$$\cdot \left(\frac{\partial T}{\partial V}\right)_S = -\frac{\left(\frac{\partial T}{\partial S}\right)_V}{\left(\frac{\partial V}{\partial S}\right)_T} = -\underline{\frac{\left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V}}$$

$$\cdot \left(\frac{\partial S}{\partial V}\right)_T = Nk_B\left(\frac{N-Vb}{Nc}\right)\frac{1}{T^{3/2}} = Nk_B\left(\frac{N-Vb}{Nc}\right)T^{-3/2}.$$

$$\cdot \left(\frac{\partial S}{\partial T}\right)_V = Nk_B\frac{bT_{3/2}}{Nc}\frac{1}{V} = \frac{Nk_B bT_{3/2}}{NcV} = \frac{k_B b\,T_{3/2}}{cV}$$

$$\therefore \left(\frac{\partial u}{\partial V}\right)_S = a\frac{N^2}{V^2} + \left(\frac{3}{2}Nk_B\right)\left(\frac{N\cancel{k_B}\left(\frac{N-Nb}{Nc}T^{-3/2}\right)}{\frac{\cancel{k_B}\,b\,T^{3/2}}{cV}}\right)$$

$$\therefore \boxed{\left(\frac{\partial u}{\partial V}\right)_S = a\frac{N^2}{V^2} + \frac{3}{2}Nk_B\left(\frac{NcVT^{9/4}\left(\frac{N-Nb}{Nc}\right)}{b}\right)}$$



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b.) $\left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}NK_B$

$$\left(\frac{\partial S}{\partial V}\right)_T = NK_B\left[\frac{1}{\left(\frac{1}{c}-\frac{Vb}{Nc}\right)T^{3/2}}\right]\left(-\frac{b}{Nc}T^{3/2}\right)$$

$$\left(\frac{\partial S}{\partial T}\right)_V = NK_B\left[\frac{1}{\left(\frac{1}{c}-\frac{Vb}{Nc}\right)T^{3/2}}\right]\left(\left[\frac{1}{c}-\frac{Vb}{Nc}\right]\frac{3}{2}T^{1/2}\right)$$

$$\left(\frac{\partial S}{\partial V}\right)_T \Big/ \left(\frac{\partial S}{\partial T}\right)_V = -\frac{bT\ 2}{Nc\left(\frac{1}{c}-\frac{Vb}{Nc}\right)3} = -\frac{2bT}{3(N-Vb)}$$

$$\left(\frac{\partial U}{\partial V}\right)_S = \left[\left(\frac{\partial U}{\partial V}\right)_T - \left(\frac{\partial U}{\partial T}\right)_V \frac{\left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V}\right] = \boxed{a\frac{N^2}{V^2} + \frac{NK_B bT}{(N-Vb)}}$$

b)

$$dS = 0 = Nk_B\left(\frac{1}{\frac{N-V\cdot b}{N\cdot c}T^{3/2}}\cdot\left(-\frac{b}{N\cdot c}dV\,T^{3/2} + -\frac{V\cdot b}{N\cdot C}\frac{3}{2}T^{1/2}\cdot dT\right)\right)$$

Then $$0 = Nk_B\left(\frac{\cancel{N}c}{(N-Vb)\cancel{T^{3/2}}}\left(-\frac{b\cancel{T^{3/2}}dV}{\cancel{N}c} - \frac{3}{2}\frac{Vb\cancel{T^{1/2}}}{\cancel{N}C\,T}dT\right)\right)$$

$$0 = -Nk_B\left(\frac{b\,dV + \frac{3}{2}\frac{Vb}{T}dT}{N-Vb}\right)$$

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$$0 = \frac{-Nk_B b}{N-Vb}dV - \frac{3}{2}\frac{Nk_B Vb}{T(N-Vb)}dT$$

$\circledast$ $$\frac{Nk_B b}{N-Vb}dV = \frac{-3}{2}\frac{Nk_B Vb}{T(N-Vb)}dT \Bigg\}$$ These differentials are really partials w.r.t. S, since dS = 0

Now $$dU = \frac{3}{2}Nk_B dT + \frac{aN^2}{V^2}dV \quad (\circledast\circledast)$$

From $\circledast$ equation, we find

$$-\frac{2}{3}\frac{T}{V}dV = dT. \iff -\frac{2}{3}\frac{T}{V}(\partial V)_S = (\partial T)_S$$

Thus, for dS = 0, plugging into $(\circledast\circledast)$,

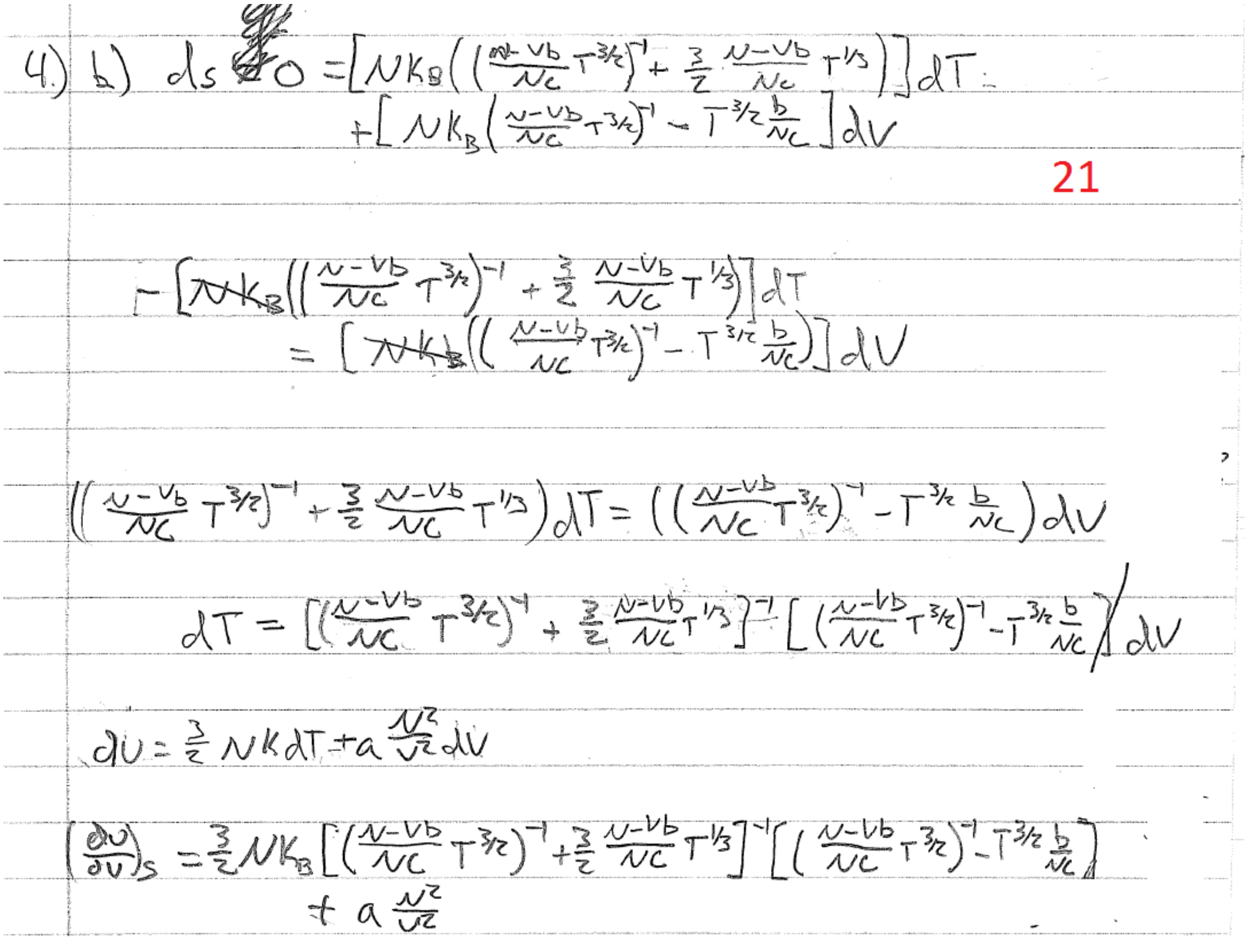
$$(\partial U)_S = \frac{3}{2}Nk_B\left(-\frac{2}{3}\frac{T}{V}(\partial V)_S\right) + \frac{aN^2}{V^2}(\partial V)_S$$

Then

$$(\partial U)_S = \frac{-Nk_BT}{V}(\partial V)_S + \frac{aN^2}{V^2}(\partial V)_S$$

nd we arrive at

$$\left(\frac{\partial U}{\partial V}\right)_S = \frac{-Nk_BT}{V} + \frac{aN^2}{V^2}.$$



4.) b)

$$ds = 0 = \left[Nk_B\left(\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} + \frac{3}{2}\frac{N-Vb}{Nc}T^{1/3}\right)\right]dT$$
$$+\left[Nk_B\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} - T^{3/2}\frac{b}{Nc}\right]dV$$

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$$-\left[\cancel{Nk_B}\left(\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} + \frac{3}{2}\frac{N-Vb}{Nc}T^{1/3}\right)\right]dT$$
$$= \left[\cancel{Nk_B}\left(\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} - T^{3/2}\frac{b}{Nc}\right)\right]dV$$

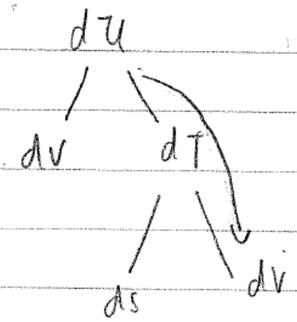
$$\left(\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} + \frac{3}{2}\frac{N-Vb}{Nc}T^{1/3}\right)dT = \left(\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} - T^{3/2}\frac{b}{Nc}\right)dV$$

$$dT = \left[\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} + \frac{3}{2}\frac{N-Vb}{Nc}T^{1/3}\right]^{-1}\left[\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} - T^{3/2}\frac{b}{Nc}\right]dV$$

$$dU = \frac{3}{2}NkdT + a\frac{N^2}{V^2}dV$$

$$\left(\frac{\partial U}{\partial V}\right)_S = \frac{3}{2}Nk_B\left[\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} + \frac{3}{2}\frac{N-Vb}{Nc}T^{1/3}\right]^{-1}\left[\left(\frac{N-Vb}{Nc}T^{3/2}\right)^{-1} - T^{3/2}\frac{b}{Nc}\right]$$
$$+ a\frac{N^2}{V^2}$$

b) $\left(\frac{\partial U}{\partial V}\right)_S$



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$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

$$= \left(\frac{\partial U}{\partial T}\right)_V \left[\left(\frac{\partial T}{\partial V}\right)_S dV + \left(\frac{\partial T}{\partial S}\right)_V dS\right] + \left(\frac{\partial U}{\partial V}\right)_T dV$$

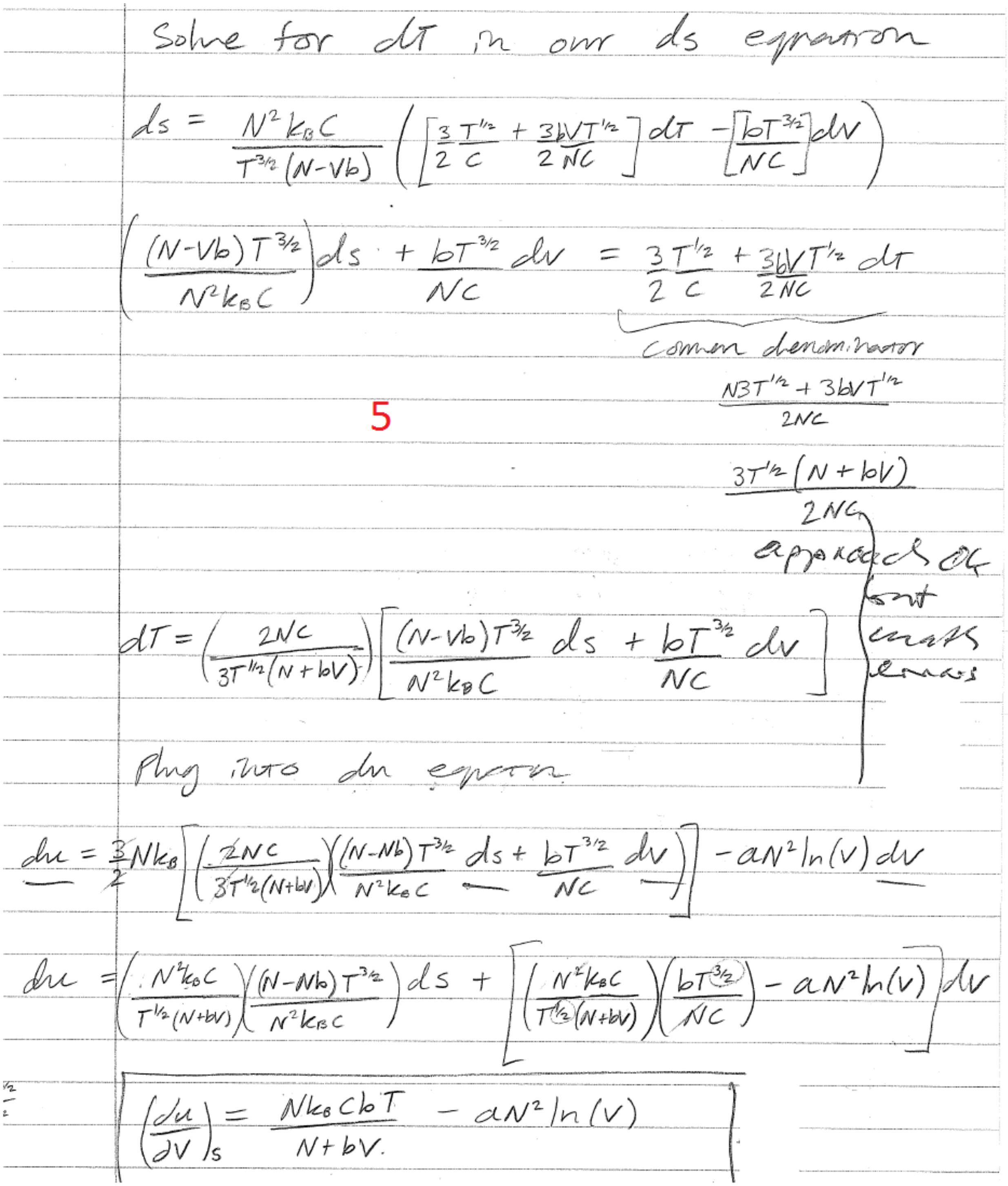
$$= \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S dV + \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_T dV$$

$$\left(\frac{\partial U}{\partial V}\right)_S = \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S + \left(\frac{\partial U}{\partial V}\right)_T$$

There is work on the algebra page to find all of these derivatives

$$= \left(\frac{\cancel{3}}{\cancel{2}} \frac{\cancel{\;}}{Nk_B}\right)\left(e^{\left(\frac{S}{Nk_B} - \frac{5}{2}\right)} \frac{bNc}{(N-Vb)^2} \frac{\cancel{2}}{\cancel{3}} T^{1/2}\right) + \frac{aN^2}{V^2}$$

$$= \frac{1}{Nk_B T^{1/2}}) \frac{(bNc)}{(N-Vb)^2} e^{\left(\frac{S}{Nk_B} - \frac{5}{2}\right)} + \frac{aN^2}{V^2}$$



Solve for dT in our ds equation

$$ds = \frac{N^2 k_B C}{T^{3/2}(N-Vb)}\left(\left[\frac{3}{2}\frac{T^{1/2}}{C} + \frac{3bVT^{1/2}}{2NC}\right]dT - \left[\frac{bT^{3/2}}{NC}\right]dV\right)$$

$$\left(\frac{(N-Vb)T^{3/2}}{N^2 k_B C}\right)ds + \frac{bT^{3/2}}{NC}dV = \underbrace{\frac{3}{2}\frac{T^{1/2}}{C} + \frac{3bVT^{1/2}}{2NC}}_{\text{common denominator}} dT$$

$$\frac{N3T^{1/2} + 3bVT^{1/2}}{2NC}$$

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$$\frac{3T^{1/2}(N+bV)}{2NC}$$

$$dT = \left(\frac{2NC}{3T^{1/2}(N+bV)}\right)\left[\frac{(N-Vb)T^{3/2}}{N^2 k_B C}ds + \frac{bT^{3/2}}{NC}dV\right]$$

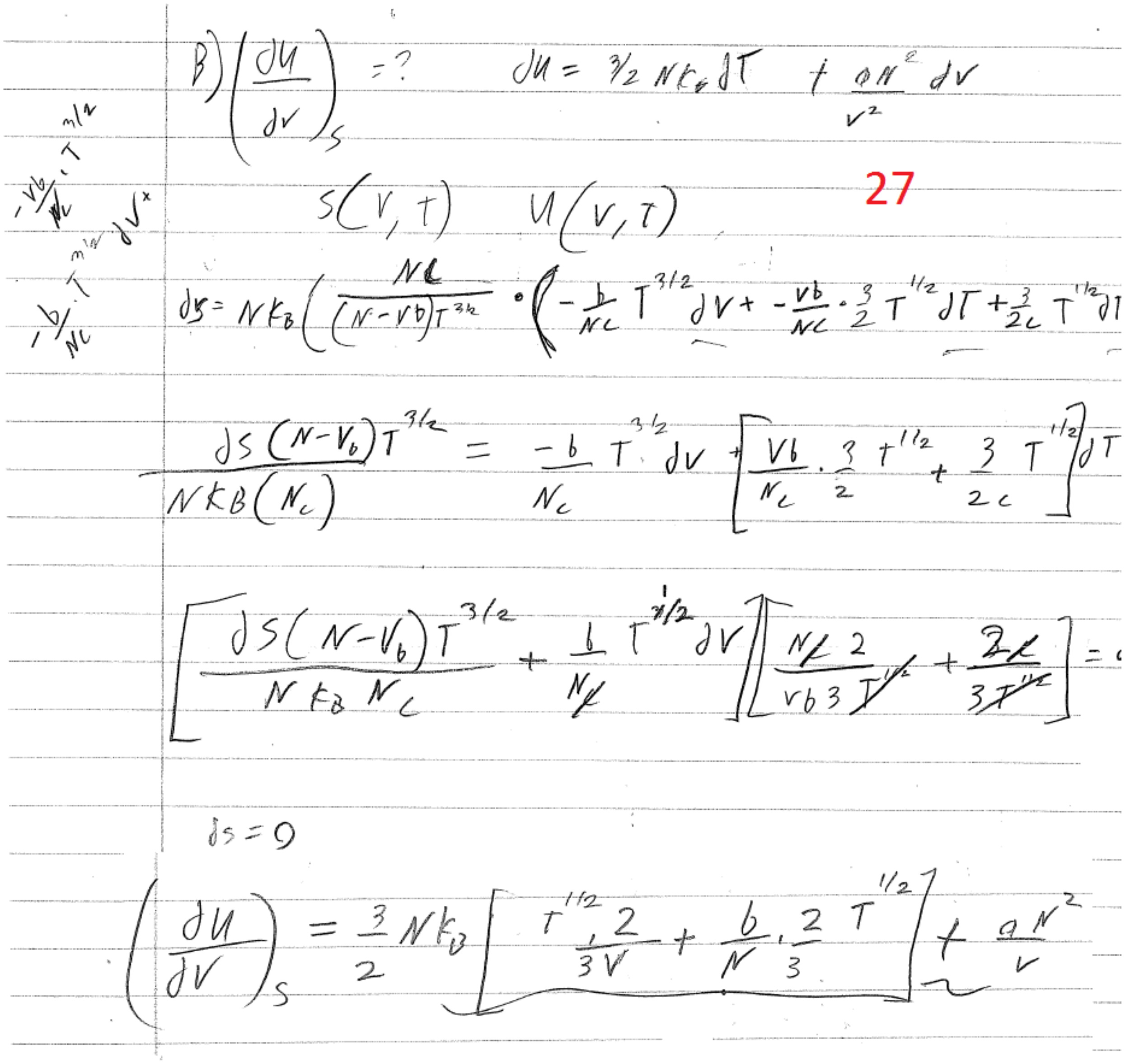
approach OK but makes errors

Plug into du equation

$$du = \frac{\cancel{3}}{\cancel{2}}Nk_B\left[\left(\frac{\cancel{2}NC}{\cancel{3}T^{1/2}(N+bV)}\right)\left(\frac{(N-Nb)T^{3/2}}{N^2 k_B C}ds + \frac{bT^{3/2}}{NC}dV\right)\right] - aN^2\ln(V)dV$$

$$du = \left(\frac{N^2 k_B C}{T^{1/2}(N+bV)}\right)\left(\frac{(N-Nb)T^{3/2}}{N^2 k_B C}\right)ds + \left[\left(\frac{N^2 k_B C}{T^{1/2}(N+bV)}\right)\left(\frac{bT^{3/2}}{NC}\right) - aN^2\ln(V)\right]dV$$

$$\boxed{\left(\frac{du}{\partial V}\right)_S = \frac{Nk_B CbT}{N+bV.} - aN^2\ln(V)}$$



$$B)\left(\frac{\partial U}{\partial V}\right)_S = ? \qquad \partial U = 3/2\, Nk_B \partial T + \frac{aN^2}{V^2}\partial V$$

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$$-\frac{Vb}{\cancel{Nc}} \cdot T^{1/2} \qquad \partial V^x \qquad \frac{b}{Nc}\, T^{1/2}$$

$$S(V,T) \qquad U(V,T)$$

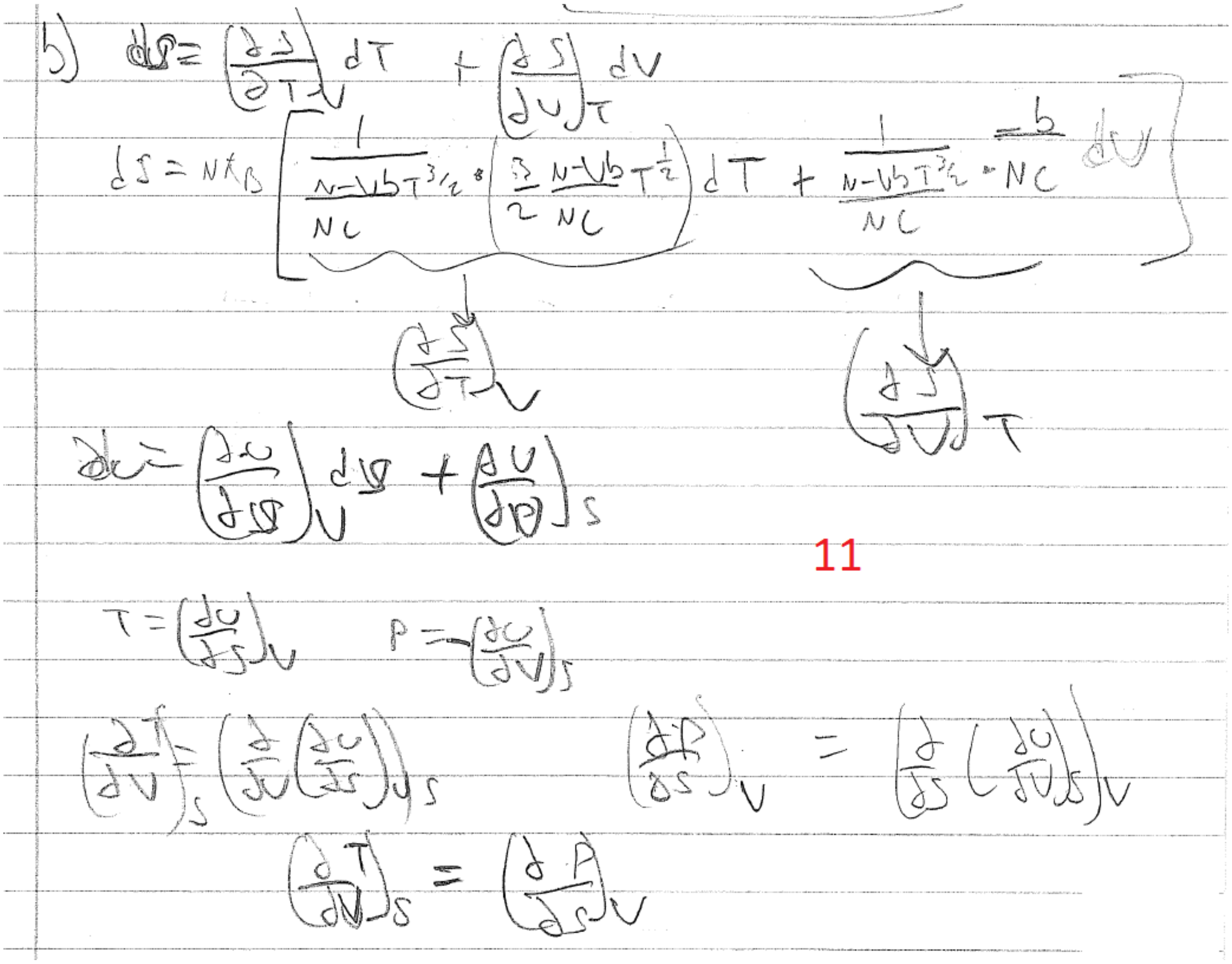
$$\partial S = NK_B\left(\frac{Nc}{(N-Vb)T^{3/2}} \cdot \left(-\frac{b}{Nc}T^{3/2}\partial V + -\frac{Vb}{Nc}\cdot\frac{3}{2}T^{1/2}\partial T + \frac{3}{2c}T^{1/2}\partial T\right.\right.$$

$$\frac{\partial S (N-V_b)T^{3/2}}{NKB(N_c)} = \frac{-b}{N_c}T^{3/2}\partial V + \left[\frac{Vb}{N_c}\cdot\frac{3}{2}T^{1/2} + \frac{3}{2c}T^{1/2}\right]\partial T$$

$$\left[\frac{\partial S(N-V_b)T^{3/2}}{N k_B N_c} + \frac{b}{N\cancel{c}}T^{3/2}\partial V\right]\left[\frac{N\cancel{c}\,2}{Vb\,3\,\cancel{T^{1/2}}} + \frac{2\cancel{c}}{3\cancel{T^{1/2}}}\right] =$$

$$\partial S = 0$$

$$\left(\frac{\partial U}{\partial V}\right)_S = \frac{3}{2}Nk_B\left[T^{1/2}\frac{2}{3V} + \frac{b}{N}\cdot\frac{2}{3}T^{1/2}\right] + \frac{aN^2}{V}$$



b) $dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$

$$dS = Nk_B \left[\underbrace{\frac{1}{\frac{N-Vb\,T^{3/2}}{Nc}} \cdot \left(\frac{3}{2}\frac{N-Vb}{Nc}T^{\frac{1}{2}}\right)}_{\left(\frac{\partial S}{\partial T}\right)_V} dT + \underbrace{\frac{1}{\frac{N-Vb\,T^{3/2}}{Nc}} \cdot \frac{-b}{Nc}}_{\left(\frac{\partial S}{\partial V}\right)_T} dV \right]$$

$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S$

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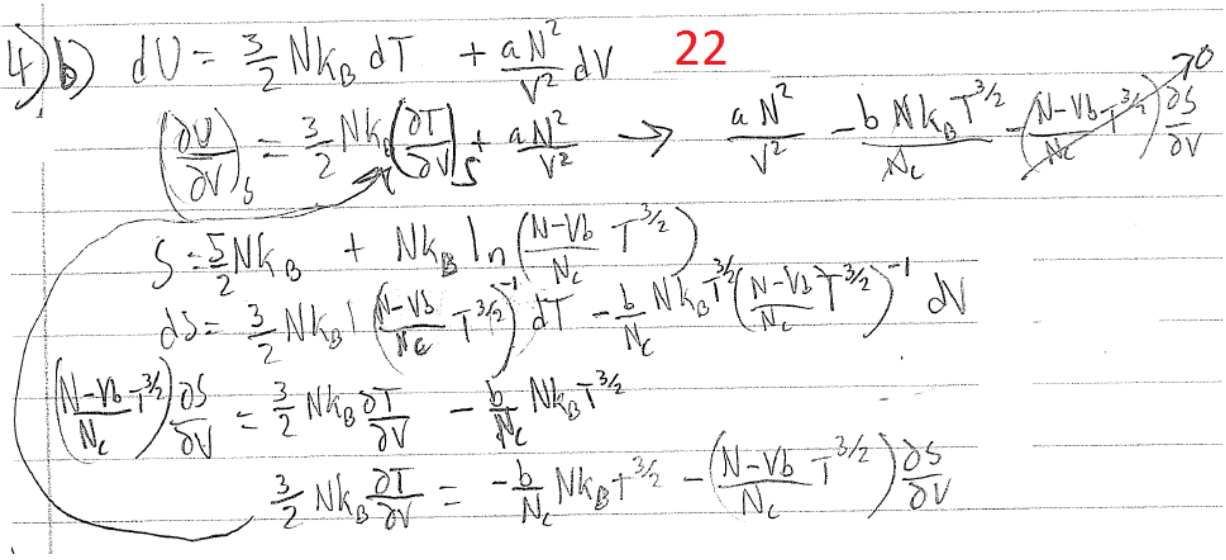
$T = \left(\frac{\partial U}{\partial S}\right)_V \qquad P = -\left(\frac{\partial U}{\partial V}\right)_S$

$\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{\partial}{\partial V}\left(\frac{\partial U}{\partial S}\right)_V\right)_S \qquad \left(\frac{\partial P}{\partial S}\right)_V = \left(\frac{\partial}{\partial S}\left(-\frac{\partial U}{\partial V}\right)_S\right)_V$

$\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{\partial P}{\partial S}\right)_V$



4) b) $dU = \frac{3}{2}Nk_B\,dT + \frac{aN^2}{V^2}dV$ 22

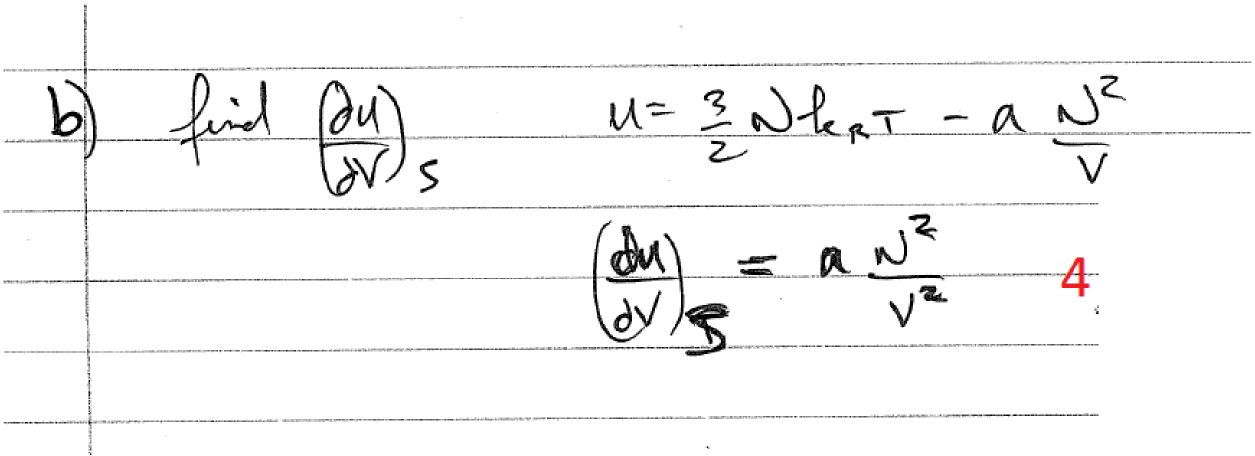
$\left(\frac{\partial U}{\partial V}\right)_S = \frac{3}{2}Nk_B\left(\frac{\partial T}{\partial V}\right)_S + \frac{aN^2}{V^2} \rightarrow \frac{aN^2}{V^2} - \frac{bNk_BT^{3/2}}{N_c} - \cancel{\left(\frac{N-Vb}{N_c}T^{3/2}\right)\frac{\partial S}{\partial V}} \to 0$

$S = \frac{5}{2}Nk_B + Nk_B \ln\left(\frac{N-Vb}{N_c}T^{3/2}\right)$

$dS = \frac{3}{2}Nk_B \left(\frac{N-Vb}{N_c}T^{3/2}\right)^{-1} dT - \frac{b}{N_c}Nk_BT^{3/2}\left(\frac{N-Vb}{N_c}T^{3/2}\right)^{-1} dV$

$\left(\frac{N-Vb}{N_c}T^{3/2}\right)\frac{\partial S}{\partial V} = \frac{3}{2}Nk_B\frac{\partial T}{\partial V} - \frac{b}{N_c}Nk_BT^{3/2}$

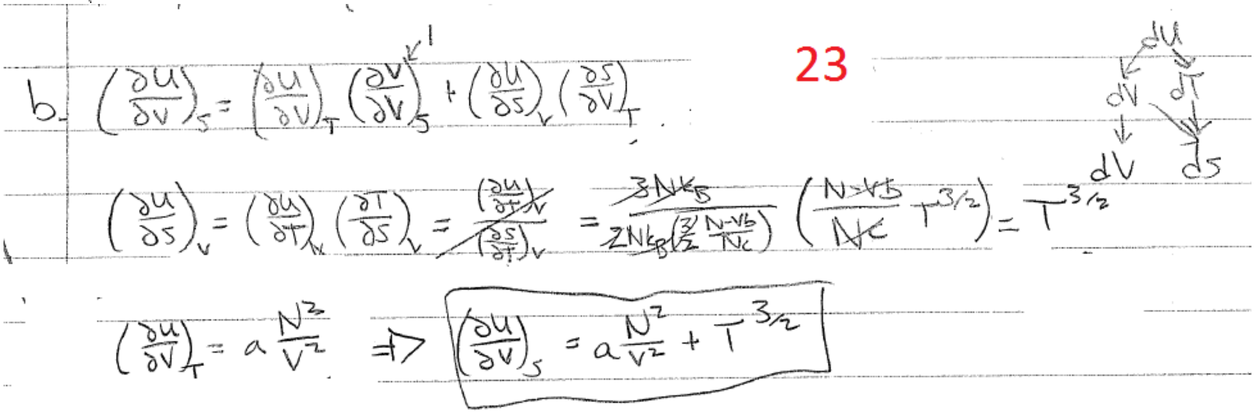
$\frac{3}{2}Nk_B\frac{\partial T}{\partial V} = -\frac{b}{N_c}Nk_BT^{3/2} - \left(\frac{N-Vb}{N_c}T^{3/2}\right)\frac{\partial S}{\partial V}$



b) find $\left(\frac{\partial U}{\partial V}\right)_S$ $\qquad U = \frac{3}{2}Nk_BT - a\frac{N^2}{V}$

$$\left(\frac{dU}{dV}\right)_S = a\frac{N^2}{V^2}$$

4



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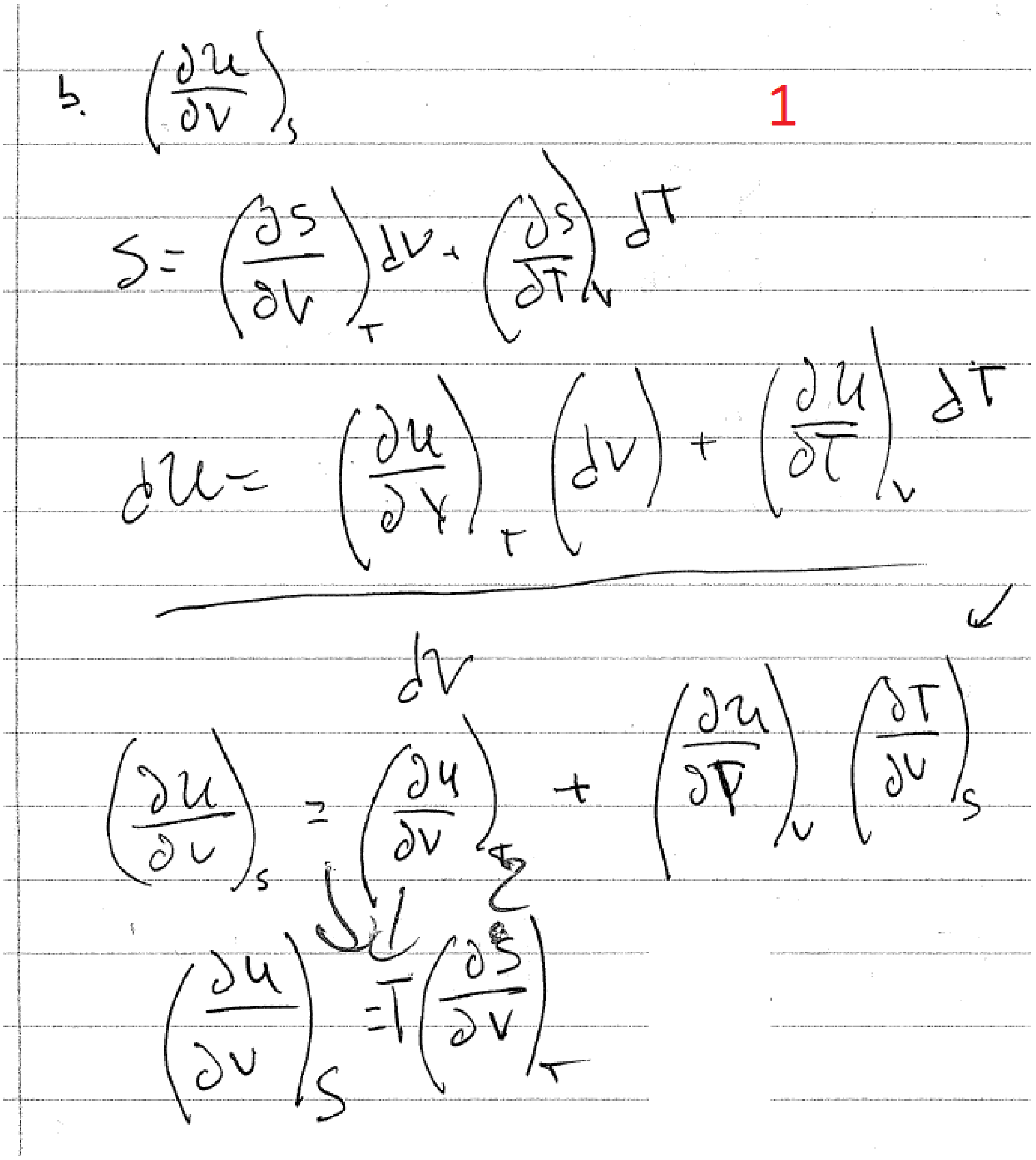
b. $\left(\frac{\partial U}{\partial V}\right)_S = \left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial V}{\partial V}\right)_S^{\swarrow 1} + \left(\frac{\partial U}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T$

$$\begin{tikzcd} & dU \arrow[dl] \arrow[d] \\ dV \arrow[d] & dT \arrow[d] \\ dV & dS \end{tikzcd}$$

$$\left(\frac{\partial U}{\partial S}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial S}\right)_V = \cancel{\frac{\left(\frac{\partial U}{\partial T}\right)_V}{\left(\frac{\partial S}{\partial T}\right)_V}} = \frac{\cancel{3Nk_B}}{\cancel{2Nk_B}\left(\frac{3}{2}\frac{N-Vb}{Nc}\right)}\left(\frac{N-Vb}{\cancel{Nc}}T^{3/2}\right) = T^{3/2}$$

$$\left(\frac{\partial U}{\partial V}\right)_T = a\frac{N^3}{V^2} \Rightarrow \boxed{\left(\frac{\partial U}{\partial V}\right)_S = a\frac{N^2}{V^2} + T^{3/2}}$$



b. $\left(\frac{\partial u}{\partial v}\right)_s$

1

$$S = \left(\frac{\partial s}{\partial v}\right)_T dv + \left(\frac{\partial s}{\partial T}\right)_v dT$$

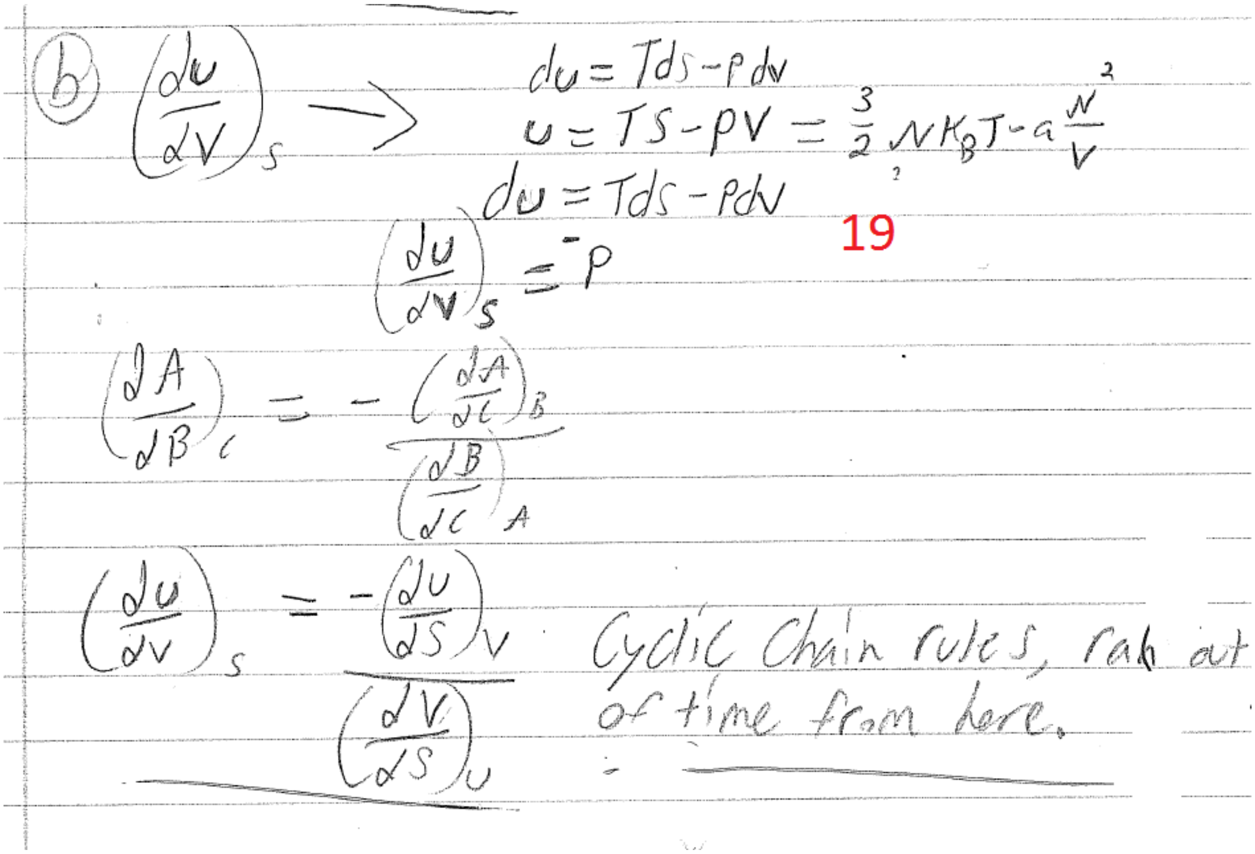
$$dU = \left(\frac{\partial u}{\partial v}\right)_T (dv) + \left(\frac{\partial u}{\partial T}\right)_v dT$$

$$\overline{}$$

$$dv$$

$$\left(\frac{\partial u}{\partial v}\right)_s = \left(\frac{\partial u}{\partial v}\right)_T + \left(\frac{\partial u}{\partial T}\right)_v \left(\frac{\partial T}{\partial v}\right)_s$$

$$\left(\frac{\partial u}{\partial v}\right)_S = T\left(\frac{\partial S}{\partial v}\right)_T$$



ⓑ $\left(\frac{du}{dV}\right)_S \longrightarrow$

$du = Tds - pdv$

$u = TS - pV = \frac{3}{2} NK_BT - a\frac{N^2}{V}$

$du = Tds - PdV$

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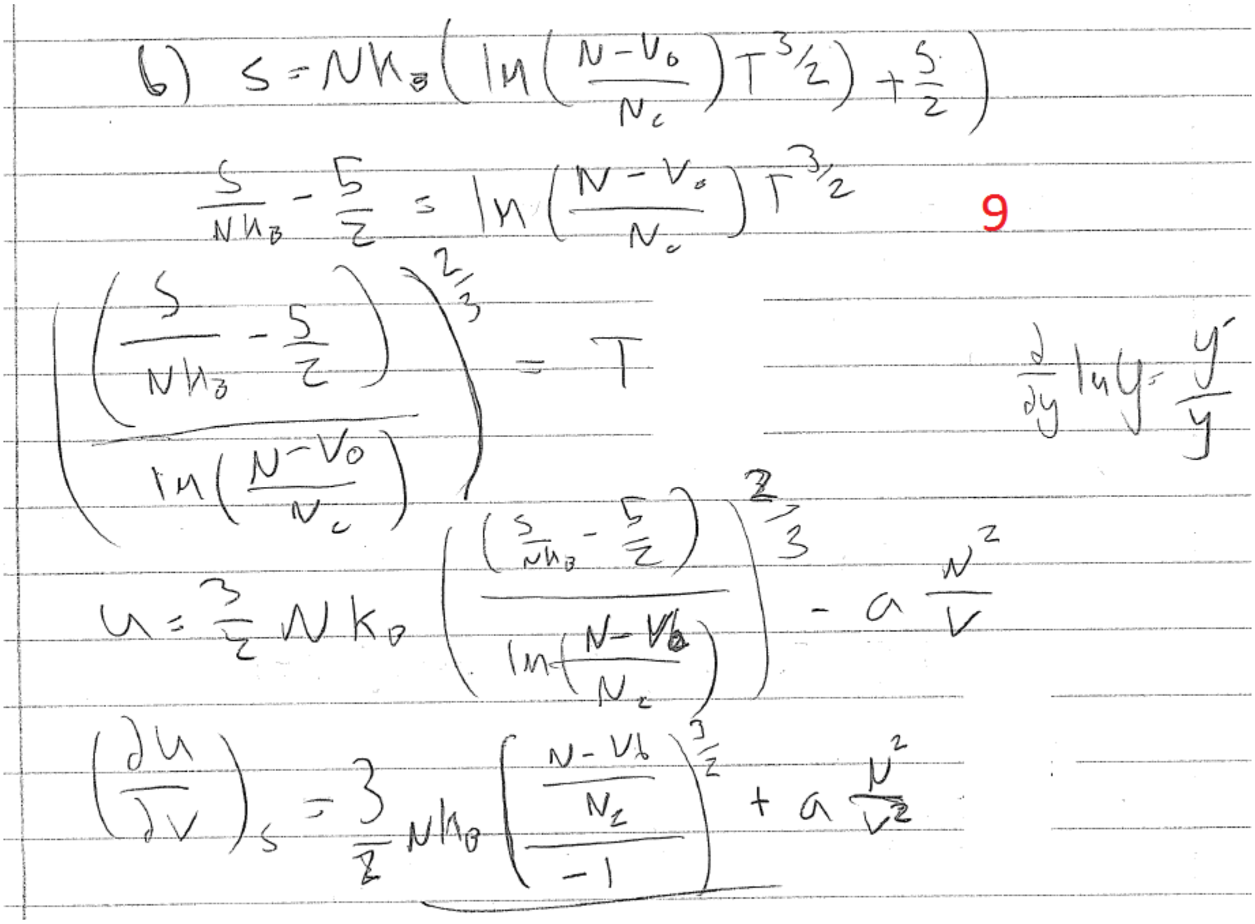
$$\left(\frac{du}{dV}\right)_S = -P$$

$$\left(\frac{\partial A}{\partial B}\right)_C = -\frac{\left(\frac{\partial A}{\partial C}\right)_B}{\left(\frac{\partial B}{\partial C}\right)_A}$$

$$\left(\frac{du}{dv}\right)_S = -\frac{\left(\frac{du}{dS}\right)_V}{\left(\frac{dV}{dS}\right)_U}$$

Cyclic chain rules, ran out of time from here.



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b) $S = Nk_B\left(\ln\left(\frac{N-V_b}{N_c}\right)T^{3/2}) + \frac{5}{2}\right)$

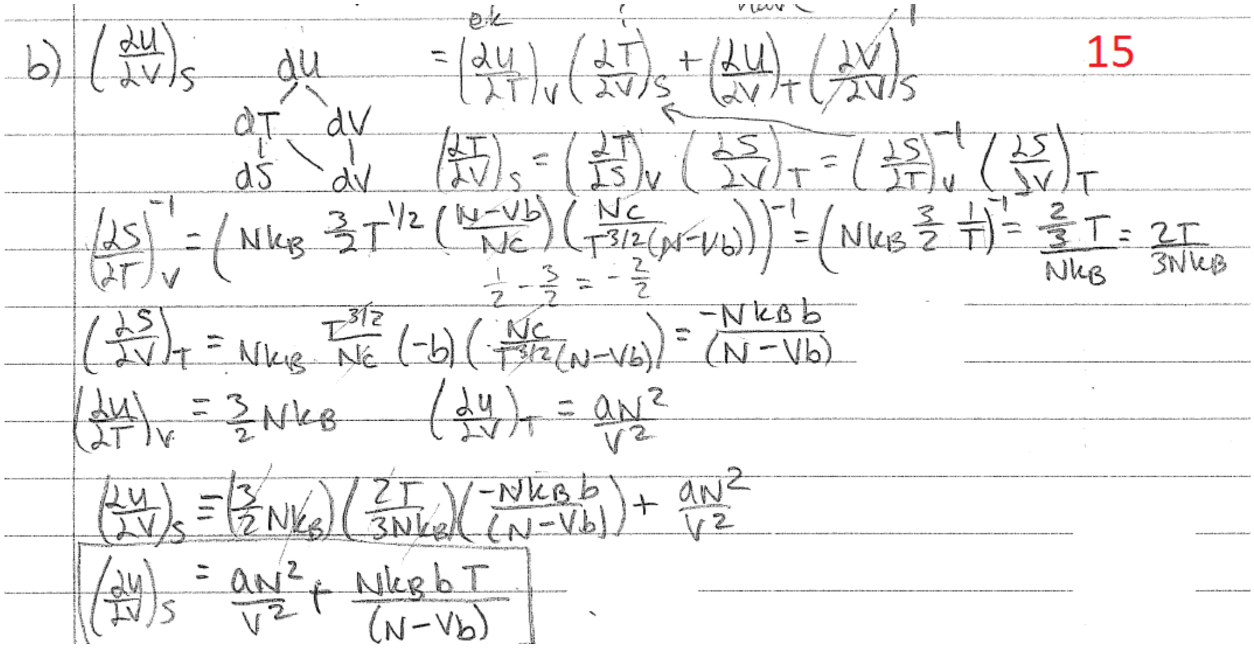
$$\frac{S}{Nk_B} - \frac{5}{2} = \ln\left(\frac{N-V_b}{N_c}\right)T^{3/2}$$

$$\left(\frac{\left(\frac{S}{Nk_B} - \frac{5}{2}\right)}{\ln\left(\frac{N-V_b}{N_c}\right)}\right)^{2/3} = T$$

$$\frac{d}{dy}\ln y = \frac{y'}{y}$$

$$U = \frac{3}{2}Nk_B\left(\frac{\left(\frac{S}{Nk_B} - \frac{5}{2}\right)}{\ln\left(\frac{N-V_b}{N_c}\right)}\right)^{2/3} - a\frac{N^2}{V}$$

$$\underline{\left(\frac{\partial U}{\partial V}\right)_S = \frac{3}{2}Nk_B\left(\frac{\frac{N-V_b}{N_c}}{-1}\right)^{3/2} + a\frac{N^2}{V^2}}$$



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b) $\left(\frac{\partial U}{\partial V}\right)_S$

$$\begin{tikzcd} & dU \arrow[dl, dash] \arrow[dr, dash] & \\ dT \arrow[d, dash] \arrow[drr, dash] & & dV \arrow[d, dash] \\ dS & & dV \end{tikzcd}$$

$$= \overset{\text{ok}}{\left(\frac{\partial U}{\partial T}\right)_V}\left(\frac{\partial T}{\partial V}\right)_S + \left(\frac{\partial U}{\partial V}\right)_T\overset{1}{\cancel{\left(\frac{\partial V}{\partial V}\right)_S}}$$

$$\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{\partial T}{\partial S}\right)_V\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial S}{\partial T}\right)_V^{-1}\left(\frac{\partial S}{\partial V}\right)_T$$

$$\left(\frac{\partial S}{\partial T}\right)_V^{-1} = \left(Nk_B\frac{3}{2}T^{1/2}\left(\frac{N-Vb}{Nc}\right)\left(\frac{Nc}{T^{3/2}(N-Vb)}\right)\right)^{-1} = \left(Nk_B\frac{3}{2}\frac{1}{T}\right)^{-1} = \frac{\frac{2}{3}T}{Nk_B} = \frac{2T}{3Nk_B}$$

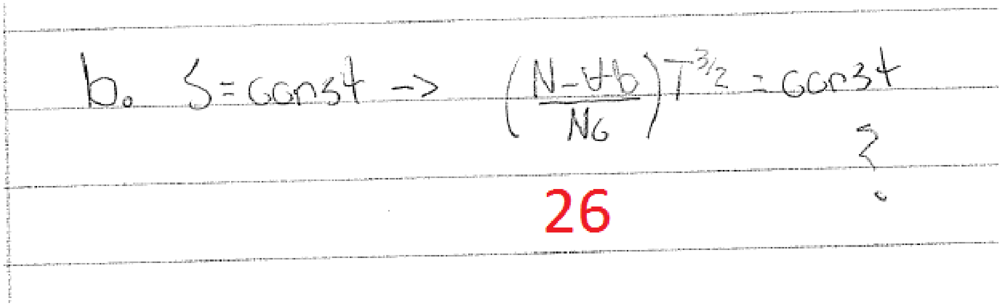
$$\frac{1}{2} - \frac{3}{2} = -\frac{2}{2}$$

$$\left(\frac{\partial S}{\partial V}\right)_T = Nk_B\frac{T^{3/2}}{Nc}(-b)\left(\frac{Nc}{T^{3/2}(N-Vb)}\right) = \frac{-Nk_Bb}{(N-Vb)}$$

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}Nk_B \qquad \left(\frac{\partial U}{\partial V}\right)_T = \frac{aN^2}{V^2}$$

$$\left(\frac{\partial U}{\partial V}\right)_S = \left(\frac{3}{2}Nk_B\right)\left(\frac{2T}{3Nk_B}\right)\left(\frac{-Nk_Bb}{(N-Vb)}\right) + \frac{aN^2}{V^2}$$

$$\boxed{\left(\frac{\partial U}{\partial V}\right)_S = \frac{aN^2}{V^2} + \frac{Nk_BbT}{(N-Vb)}}$$



b. $S = const \rightarrow \left(\frac{N-Vb}{NG}\right)T^{3/2} = const$

?

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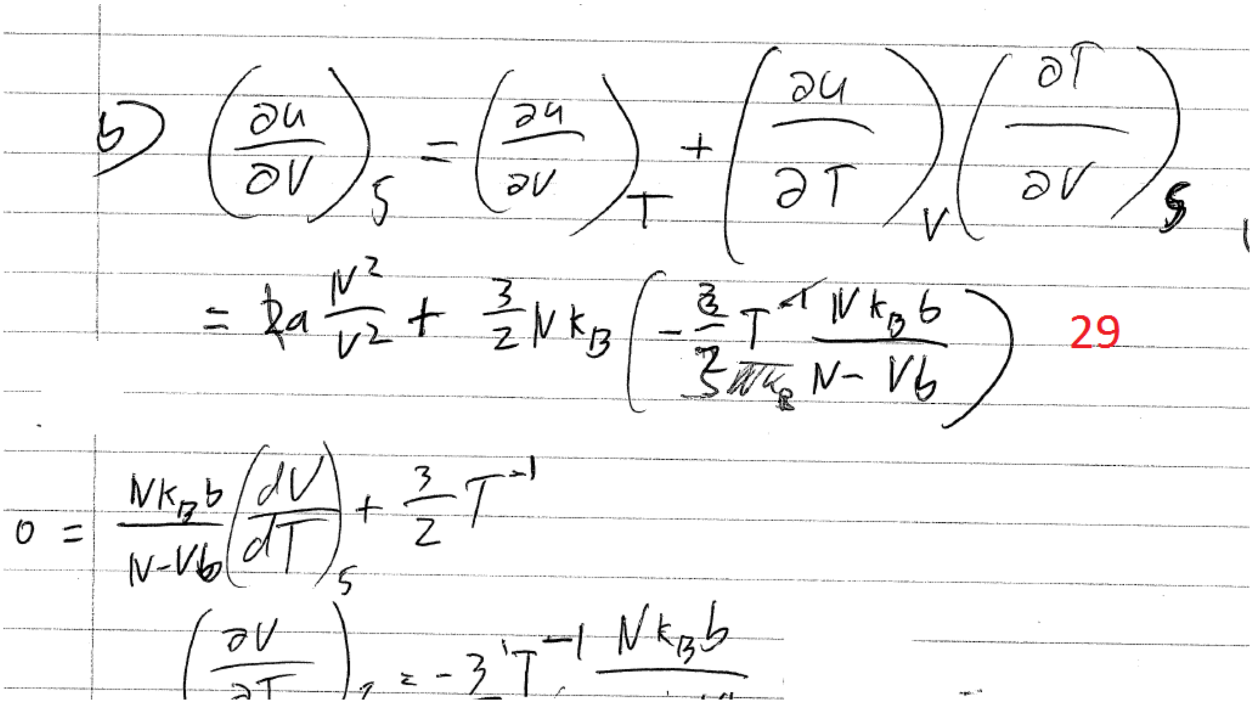

b) $\left(\frac{\partial u}{\partial V}\right)_S = \left(\frac{\partial u}{\partial V}\right)_T + \left(\frac{\partial u}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S$

$= \cancel{2}a\frac{N^2}{V^2} + \frac{3}{2}Nk_B\left(-\frac{2}{3\cancel{Nk_B}}T^{-1}\frac{Nk_Bb}{N-Vb}\right)$

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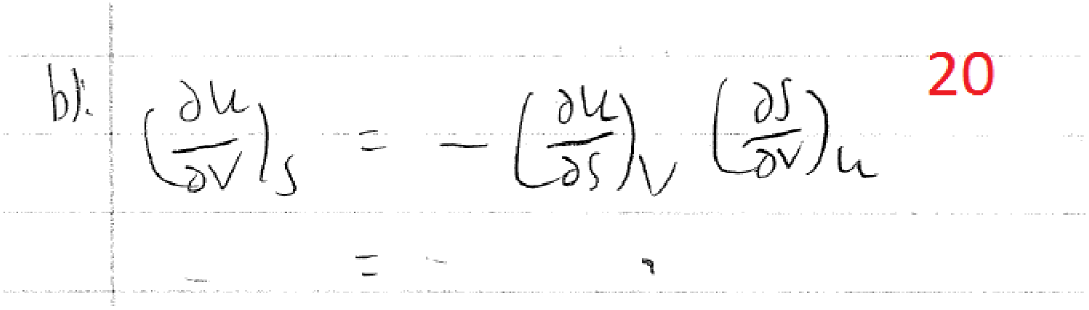
$0 = \frac{Nk_Bb}{N-Vb}\left(\frac{dV}{dT}\right)_S + \frac{3}{2}T^{-1}$

$\left(\frac{\partial V}{\partial T}\right)_S = -3T^{-1}\frac{Nk_Bb}{}$



b). $\left(\frac{\partial u}{\partial V}\right)_S = -\left(\frac{\partial u}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_u$

$=$

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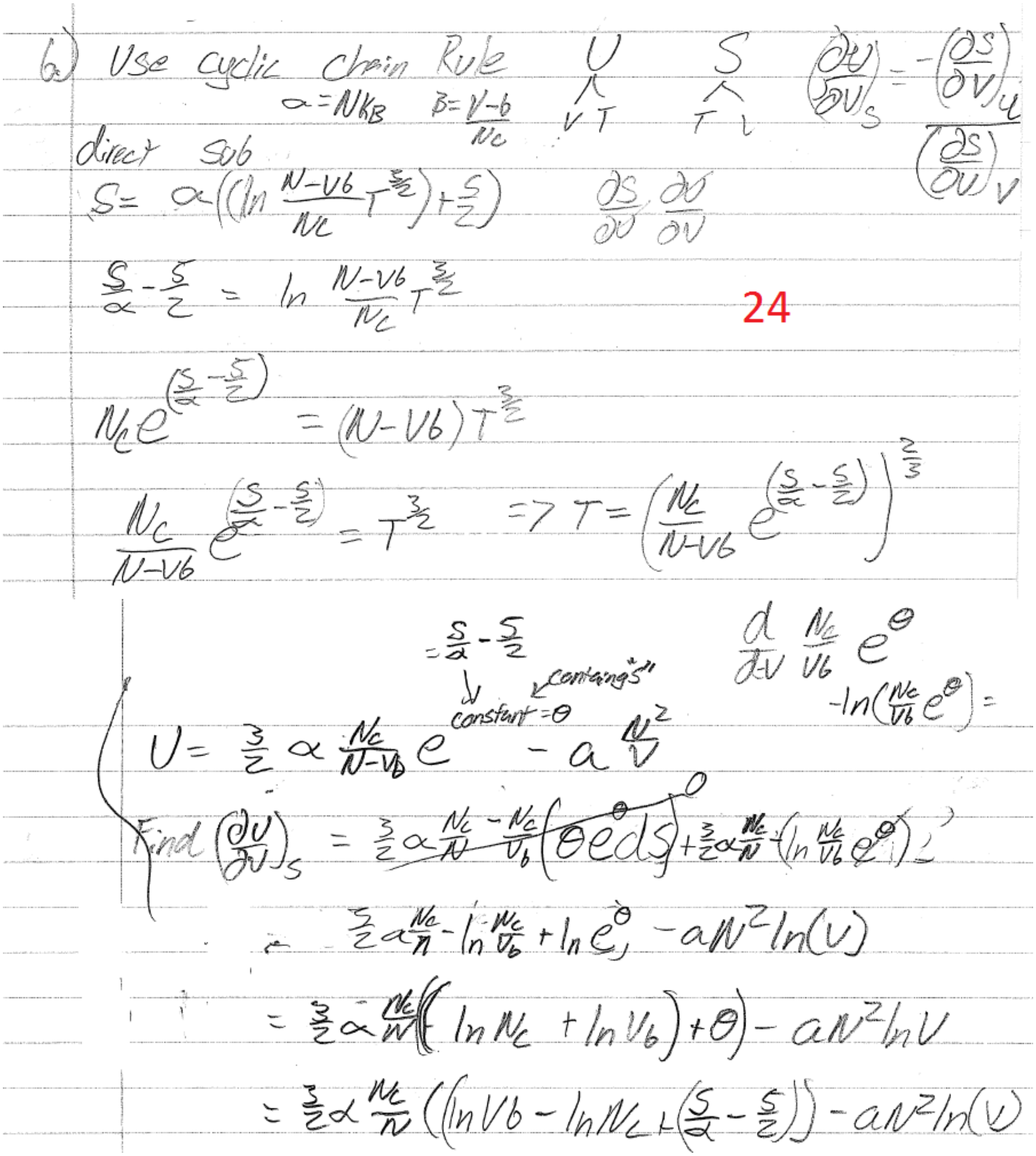

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b) Use cyclic chain Rule $\quad \alpha = Nk_B \quad B = \frac{V-b}{Nc}$

U over $V\ T$; S over $T\ V$

$$\left(\frac{\partial U}{\partial V}\right)_S = \frac{-\left(\frac{\partial S}{\partial V}\right)_U}{\left(\frac{\partial S}{\partial U}\right)_V}$$

direct sub

$$S = \alpha\left(\left(\ln \frac{N-Vb}{Nc} T^{\frac{3}{2}}\right) + \frac{5}{2}\right) \qquad \frac{\partial S}{\partial U}\,\frac{\partial U}{\partial V}$$

$$\frac{S}{\alpha} - \frac{5}{2} = \ln \frac{N-Vb}{N_c} T^{\frac{3}{2}}$$

$$N_c e^{\left(\frac{S}{\alpha} - \frac{5}{2}\right)} = (N - Vb)\,T^{\frac{3}{2}}$$

$$\frac{N_c}{N-Vb} e^{\left(\frac{S}{\alpha} - \frac{5}{2}\right)} = T^{\frac{3}{2}} \quad => T = \left(\frac{N_c}{N-Vb} e^{\left(\frac{S}{\alpha} - \frac{5}{2}\right)}\right)^{\frac{2}{3}}$$

$= \frac{S}{\alpha} - \frac{5}{2}$ ↓ constant $= \theta$ ← contains "S"

$$\frac{d}{dV}\,\frac{N_c}{Vb}\,e^{\theta}$$

$$-\ln\left(\frac{N_c}{Vb} e^{\theta}\right) =$$

$$U = \frac{3}{2}\alpha \frac{N_c}{N-Vb} e - a\frac{N^2}{V}$$

$$\text{Find} \left(\frac{\partial U}{\partial V}\right)_S = ~~\frac{3}{2}\alpha\frac{N_c}{N} - \frac{N_c}{V_b}\left(\theta e d S\right)~~ + \frac{3}{2}\alpha\frac{N_c}{N}\left(\ln\frac{N_c}{Vb} e^{\theta}\right)$$

$$= \frac{3}{2}\alpha\frac{N_c}{N} - \ln\frac{N_c}{V_b} + \ln e^{\theta} - aN^2\ln(V)$$

$$= \frac{3}{2}\alpha\frac{N_c}{N}\left(\left(\ln N_c + \ln V_b\right) + \theta\right) - aN^2 \ln V$$

$$= \frac{3}{2}\alpha\frac{N_c}{N}\left(\left(\ln Vb - \ln N_c + \left(\frac{S}{\alpha} - \frac{5}{2}\right)\right)\right) - aN^2\ln(V)$$

).) find $\left(\frac{\partial U}{\partial V}\right)_S$ as a function of V; T.

2

$$S = NK_B\left(\ln\left(\frac{N-Vb}{NC}T^{3/2}\right)+\frac{5}{2}\right)$$

$$= NK_B\ln\left(\frac{N-Vb}{NC}T^{3/2}\right)+NK_B\frac{5}{2} \to \frac{S-NK_B 5/2}{NK_B} = \ln\left(\frac{N-Vb}{NC}T^{3/2}\right)$$

$$e^{S/NK_B}e^{-5/2} = \frac{N-Vb}{NC}T^{3/2}$$

$$NCe^{S/NK_B}e^{-5/2} = N-VbT^{3/2}$$

$$VbT^{3/2} = N-NCe^{-5/2}e^{S/NK_B}$$

$$T^{3/2} = \frac{N}{Vb}\left(1-Ce^{-5/2}e^{\cancel{S}/NK_B}\right)$$

$$T = \left(\frac{N^2}{V^2b^2}\left(1-Ce^{-5/2}e^{S/NK_B}\right)^2\right)^{1/3}$$

$$T = \left(\frac{N}{Vb}\right)^{2/3}\left(1-Ce^{-5/2}e^{S/NK_B}\right)^{2/3}$$

$dU(V,T)$

$\left(\frac{\partial U}{\partial V}\right)_T$ $\left(\frac{\partial U}{\partial T}\right)_V$

dV dT

$$\left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT = dU$$

$dT(V,S)$

$\frac{\partial T}{\partial S}$

dS

$$dT = \left(\frac{\partial T}{\partial V}\right)_S dV + \left(\frac{\partial T}{\partial S}\right)_T dS$$

$$dU = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V\left[\left(\frac{\partial T}{\partial V}\right)_S dV + \left(\frac{\partial T}{\partial S}\right)_T dS\right]$$

$$= \left[\left(\frac{\partial U}{\partial V}\right)_T + \left(\frac{\partial U}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_S\right]dV + \left(\frac{\partial U}{\partial T}\right)_V\left(\frac{\partial T}{\partial S}\right)_T dS$$

1. Cont.

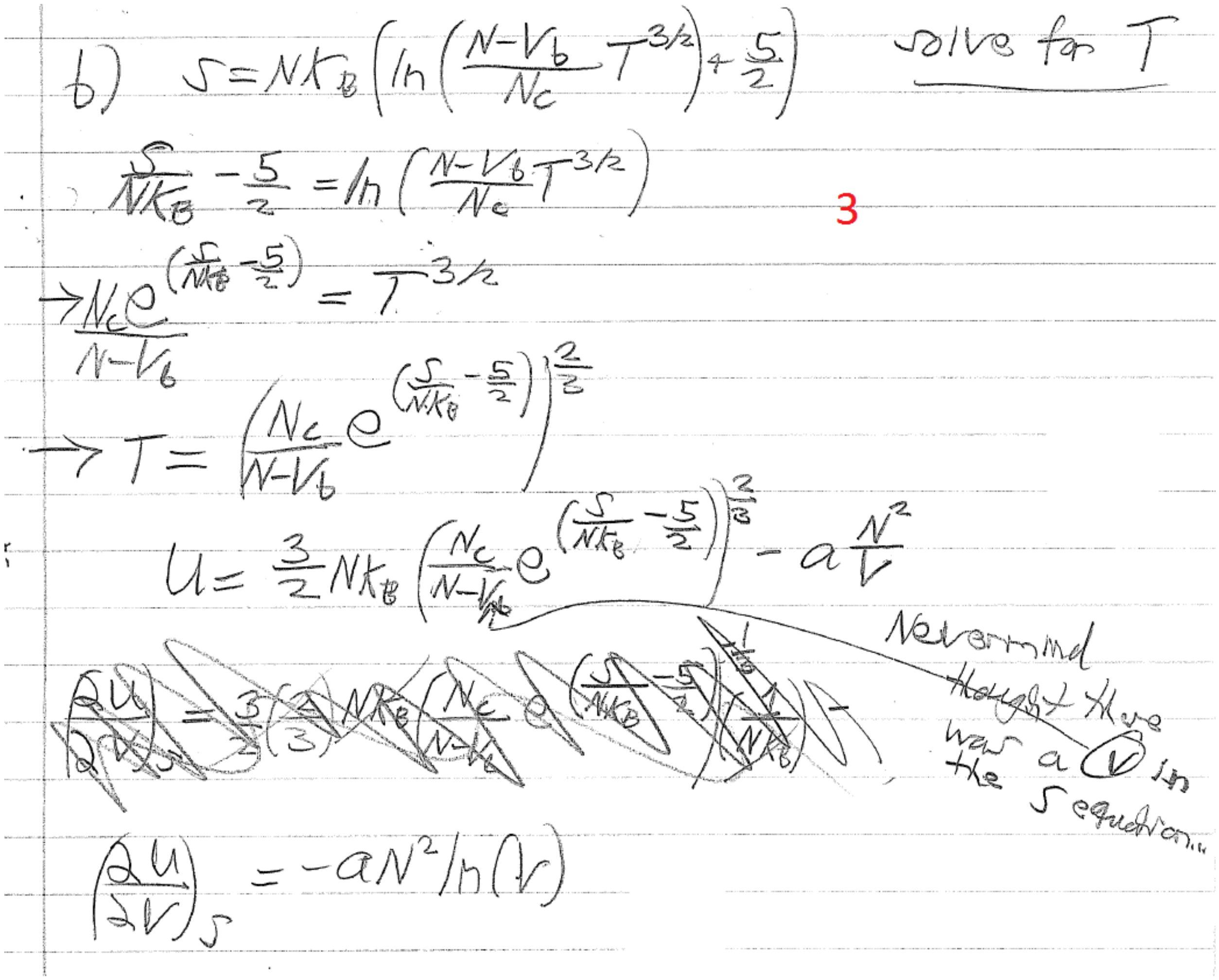
$$dU = \left[\left(\frac{\partial U}{\partial V}\right)_T + \left(\frac{\partial U}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_S\right]dV + \cancel{\frac{\partial U}{\partial t}}\left(\frac{\partial U}{\partial T}\right)_V\left(\frac{\partial T}{\partial S}\right)_T dS$$

a.) $\left(\frac{\partial U}{\partial V}\right)_T = -aN^2\ln V$

b.) $\left(\frac{\partial U}{\partial V}\right)_S = \left(\frac{\partial U}{\partial V}\right)_T + \left(\frac{\partial U}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_S$

$$= -aN^2\ln V + \left(\frac{3}{2}NK_B\right)\left[\left(1-Ce^{-5/2}e^{S/NK_B}\right)^{2/3}\left(\frac{N}{Vb}\right)^{-1/3}\left(\frac{2}{3}\right)\left(\frac{N}{b}\ln V\right)\right]$$

$$= -aN^2\ln V + K_B\left(1-Ce^{-5/2}e^{S/NK_B}\right)^{2/3}\left(\frac{N}{Vb}\right)^{-1/3}\left(\frac{N}{b}\ln V\right)$$



b) $S = Nk_B\left(\ln\left(\frac{N-V_b}{N_c}T^{3/2}\right)+\frac{5}{2}\right)$ solve for T

$\frac{S}{Nk_B} - \frac{5}{2} = \ln\left(\frac{N-V_b}{N_c}T^{3/2}\right)$

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$\rightarrow \frac{N_c e^{\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}}{N-V_b} = T^{3/2}$

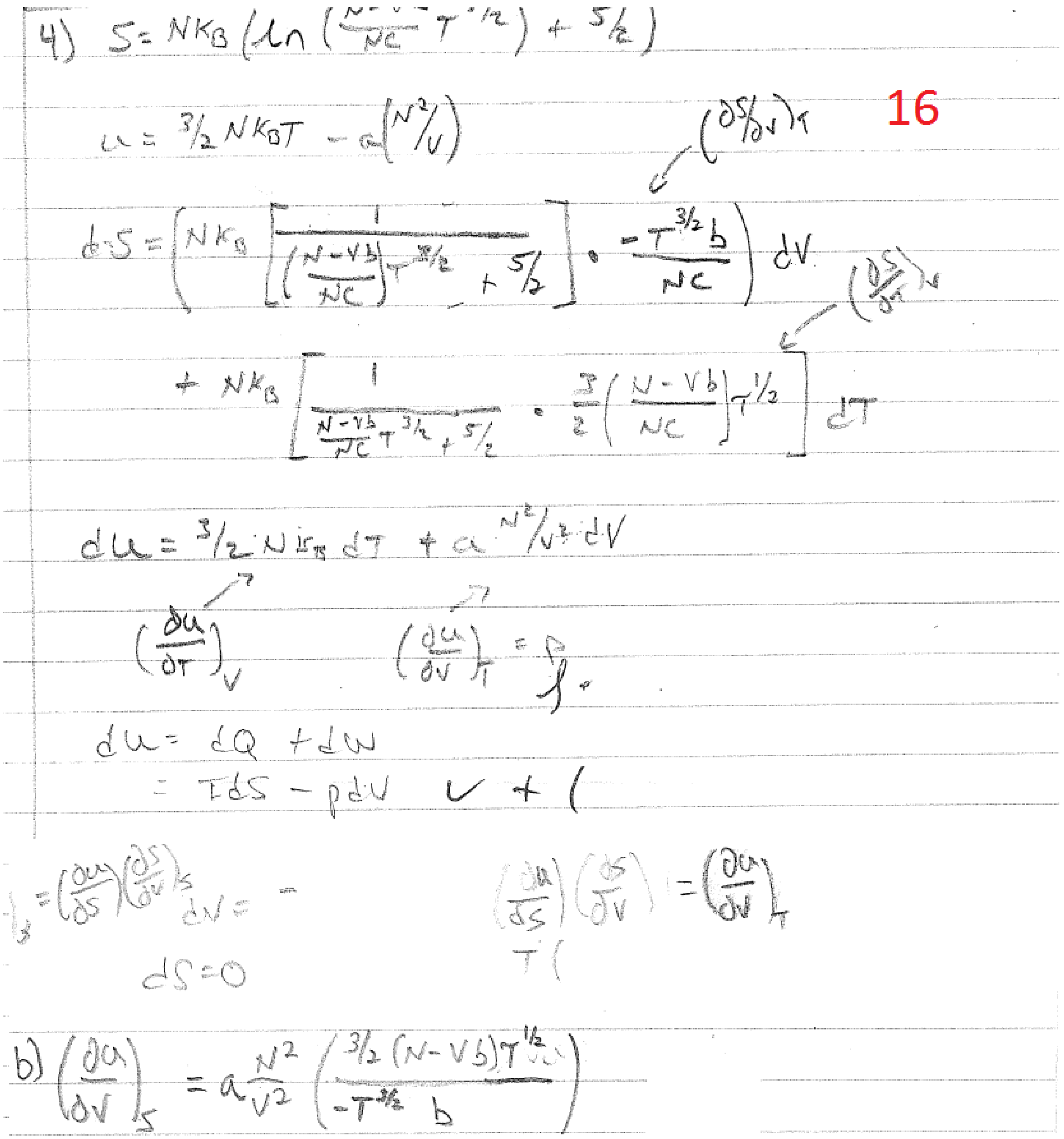
$\rightarrow T = \left(\frac{N_c}{N-V_b}e^{\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}\right)^{\frac{2}{3}}$

$U = \frac{3}{2}Nk_B\left(\frac{N_c}{N-V_b}e^{\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}\right)^{\frac{2}{3}} - a\frac{N^2}{V}$

~~$\left(\frac{\partial U}{\partial V}\right)_S = \frac{3}{2}\left(\frac{2}{3}\right)Nk_B\left(\frac{N_c}{N-V_b}e^{\left(\frac{S}{Nk_B}-\frac{5}{2}\right)}\right)^{\frac{1}{3}}\left(\frac{1}{Nk_B}\right) -$~~

Nevermind thought there was a V in the S equation…

$\left(\frac{\partial U}{\partial V}\right)_S = -aN^2\ln(V)$



4) $S = NK_B\left(\ln\left(\frac{N - V -}{NC} T^{\cdot/2}\right) + 5/2\right)$

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$u = 3/2\, NK_BT - a\left(N^2/V\right)$

$\left(\partial S/\partial V\right)_T$

$$dS = \left(NK_B\left[\frac{1}{\left(\frac{N - Vb}{NC}\right)T^{3/2} + 5/2}\right] \cdot \frac{-T^{3/2}b}{NC}\right)dV$$

$\left(\frac{\partial S}{\partial T}\right)_V$

$$+ NK_B\left[\frac{1}{\frac{N - Vb}{NC}T^{3/2} + 5/2} \cdot \frac{3}{2}\left(\frac{N - Vb}{NC}\right)T^{1/2}\right]dT$$

$du = 3/2\, N k_B\, dT + a\, N^2/V^2\, dV$

$\left(\frac{\partial u}{\partial T}\right)_V$ $\left(\frac{\partial u}{\partial V}\right)_T = p$

$du = dQ + dW$

$= TdS - pdV$ ✓ + (

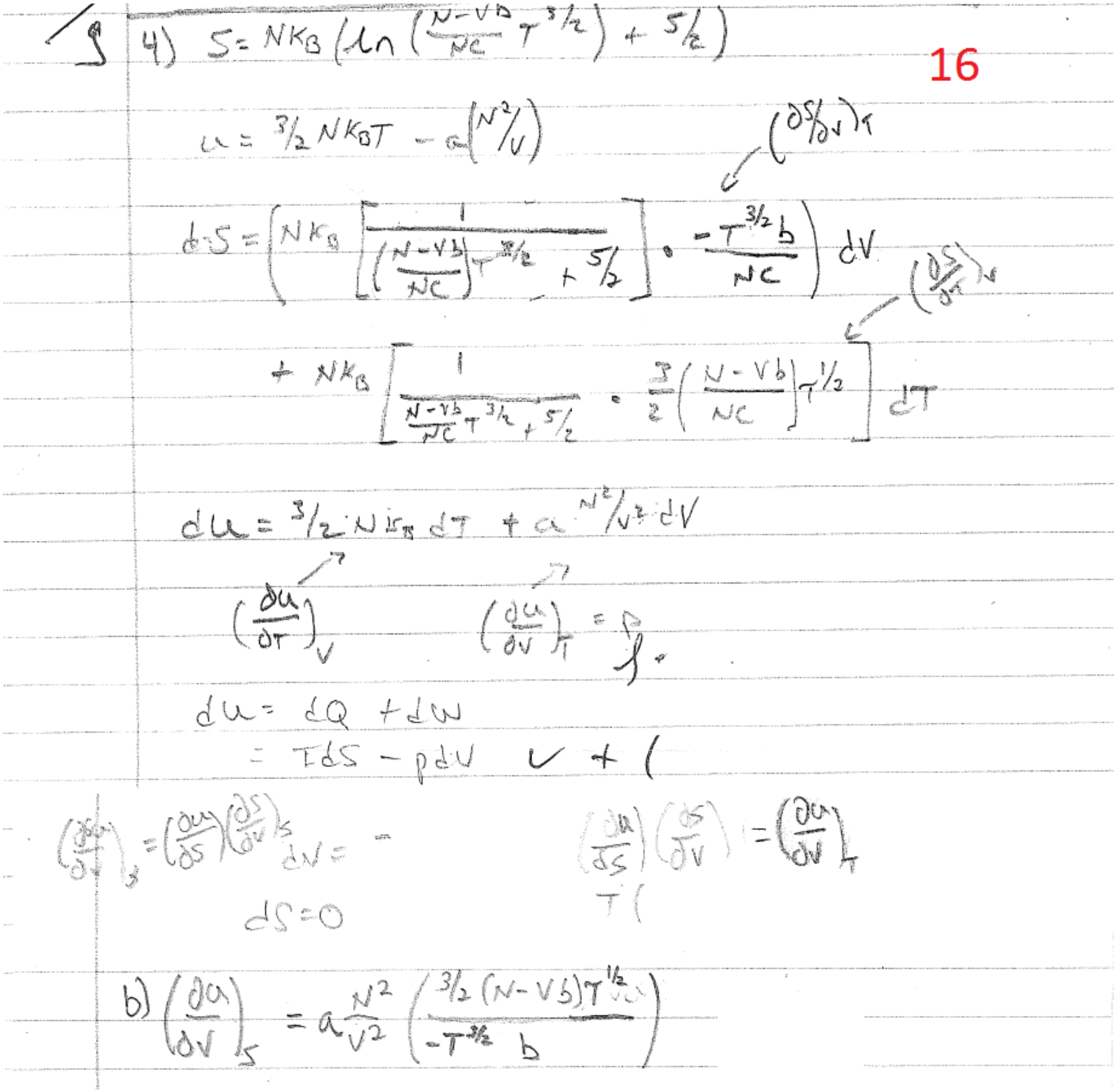
$p_S = \left(\frac{\partial u}{\partial S}\right)\left(\frac{\partial S}{\partial V}\right)_S dV = -$

$dS = 0$

$\left(\frac{\partial u}{\partial S}\right)\left(\frac{\partial S}{\partial V}\right) = \left(\frac{\partial u}{\partial V}\right)_T$

T (

b) $\left(\frac{\partial u}{\partial V}\right)_S = a\frac{N^2}{V^2}\left(\frac{3/2\,(N - Vb)T^{1/2}}{-T^{3/2}\ b}\right)$



16

4) $S = Nk_B\left(\ln\left(\frac{N-Vb}{Nc}T^{3/2}\right) + 5/2\right)$

$u = 3/2\, Nk_B T - a\left(N^2/V\right)$

$$dS = \left(Nk_B\left[\frac{1}{\left(\frac{N-Vb}{Nc}\right)T^{3/2} + 5/2}\right] \cdot \frac{-T^{3/2}b}{Nc}\right) dV \quad \left(\partial S/\partial V\right)_T$$

$$+ Nk_B\left[\frac{1}{\frac{N-Vb}{Nc}T^{3/2} + 5/2} \cdot \frac{3}{2}\left(\frac{N-Vb}{Nc}\right)T^{1/2}\right] dT \quad \left(\frac{\partial S}{\partial T}\right)_V$$

$du = 3/2\, Nk_B\, dT + a\, N^2/V^2\, dV$

$\left(\frac{\partial u}{\partial T}\right)_V \qquad \left(\frac{\partial u}{\partial V}\right)_T = p$

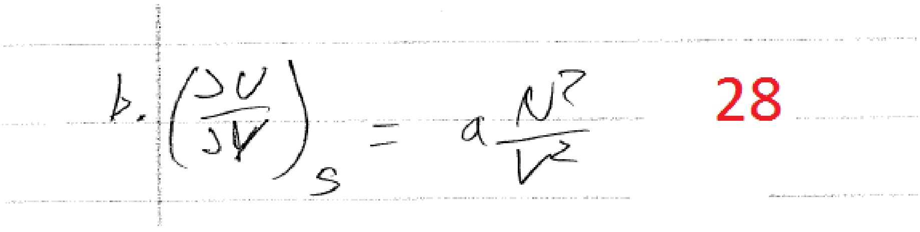
$du = dQ + dW$

$= TdS - pdV \quad \checkmark + ($

$\left(\frac{\partial u}{\partial V}\right)_S = \left(\frac{\partial u}{\partial S}\right)\left(\frac{\partial S}{\partial V}\right)_S \quad dV = \quad = \qquad \left(\frac{\partial u}{\partial S}\right)\left(\frac{\partial S}{\partial V}\right) = \left(\frac{\partial u}{\partial V}\right)_T$

$dS = 0 \qquad T($

b) $\left(\frac{\partial u}{\partial V}\right)_S = a\frac{N^2}{V^2}\left(\frac{3/2\,(N-Vb)T^{1/2}}{-T^{3/2}\, b}\right)$



b. $\left(\frac{\partial U}{\partial V}\right)_S = a\frac{N^2}{V^2}$

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