How Students Narrow to Different Paths when Finding Partial Derivatives

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The ability to determine and interpret partial derivatives is a valuable skill in physics. It is important for both experts and students to be able to use and understand partial derivatives within a variety of representations, including symbols, graphs, and words. To investigate how students determine and think about partial derivatives, we conducted open-ended problem-solving interviews with nine upper-division physics students, in which we asked students to determine derivatives from contour graphs with electrostatic and thermodynamic contexts. We analyzed the students' responses to determine how they chose which path to find the derivative along as well as what procedure(s) they used to find the derivative. We find that most students had strong procedural abilities for calculating derivatives from contour graphs, but that their deeper understanding of underlying mathematical concepts was somewhat weaker than their procedural ability. We also find that many students did not strongly link the process of finding a partial derivative from a contour graph with the idea of holding a variable constant, especially when dealing with a thermodynamics context. However, short targeted prompts from the interviewer were typically sufficient to help all but two of the nine students make connections that they did not come to spontaneously, and subsequently to express a richer understanding of both contour graphs and derivatives. Our results suggest that an instructional focus on exploring partial derivative concepts in multiple representations (including contour graphs) may be extremely beneficial for students studying upper-division physics.

I. INTRODUCTION

Derivatives are used throughout the sciences to express the relationships between pairs of variables. One way to describe a derivative is the instantaneous rate at which one variable (the dependent variable) changes when a corresponding change is made to another variable (the independent variable). Subjects like physics that frequently involve three or more variables require a generalization of the derivative to the partial derivative. Partial derivatives are fundamental to much of physics, particularly to advanced topics such as thermodynamics and electromagnetism, which inherently describe physical systems that possess more than two related variables.

Recent research in several different fields has shown that while students are typically able to perform calculations with partial derivatives, they often lack a deeper understanding of the concepts that underlie them [1–19]. For example, many students incorrectly believe that *all* variables other than the ones in the numerator and denominator are held constant when finding a partial derivative. In fact, it is only the other *independent* variables that are held constant, and in general there is a choice of which other variables might be considered independent.

Consider, for example, the partial derivative of V with respect to y, where V is a function of two other variables, x and y (Fig. 1 shows an example of such a function). In conventional Leibniz notation, such a derivative is written $\frac{\partial V}{\partial y}$. It would be most common (and natural) to choose x to be held constant, which in physics is written $\left(\frac{\partial V}{\partial y}\right)_x$. However, we might instead choose to hold some other variable constant, perhaps r, where $r^2 = x^2 + y^2$, which is written $\left(\frac{\partial V}{\partial y}\right)_r$. These two partial derivatives, $\left(\frac{\partial V}{\partial y}\right)_x$ and $\left(\frac{\partial V}{\partial y}\right)_r$, are *fundamentally different*, and in general they have different numerical values. In the first case, the limit in the definition of the derivative approaches the indicated point along a vertical, $x =$ constant line. In the second case the limit approaches the indicated point along the $r =$ constant curve. In this example, "holding a variable constant" signals that a function of two variables has been reduced to a function of only a single variable. This function of one variable is different if x is held constant than if r (or any other variable) is held constant.

"Holding a variable constant," however, is only one way of describing what must be done to find a derivative of a function of more than one variable. For example, it is not uncommon for students to describe such a derivative as "in the y-direction" [20], a phrase with two ambiguities. First, it is possible to go in the y-direction while also going in the x-direction; to be more precise, we might specify that "in the y-direction" actually means *not* in the x-direction. Second, it is possible to take a derivative "in the y-direction" with respect to a variable that is not y (perhaps $q = x + y$). This distinction is not simply a matter of language: any partial derivative is necessarily taken with respect to some variable *and* is taken "in a direction." In Leibniz notation (*e.g.*, $\frac{\partial V}{\partial y}$), the variable that the derivative is "with respect to" is indicated in the denominator, and is not necessarily related to the "direction." Leibniz notation must be augmented when the direction is not obvious from the context, as in the example above using parentheses and a subscript.

To further complicate matters, the word "direction" is a poor one when describing variables that are not spatial, such as in thermodynamics. In this paper, we will use the phrase "along a path" to describe more precisely how a derivative of a multivariable function might be found, regardless of any possible physical context.

Whether a derivative is described as "with a variable held constant," "along a path," or "in a direction," it necessarily involves the same idea: reducing the multivariable relationship to something one-dimensional that can be approximated linearly. We introduce the term *narrowing to a path* to refer to

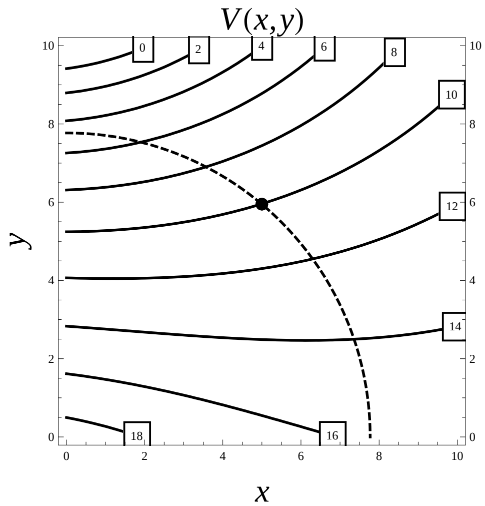


FIG. 1: An example contour graph of a function of two variables, $V(x, y)$, where each boxed number indicates the value of V along the corresponding line. It is typical to find the partial derivative of V with respect to y at the indicated point along a path of constant x. However, it is also possible to find the partial derivative of V with respect to y along any other path, including the dashed line, which represents a path along which r is constant, where $r^2 = x^2 + y^2$.

any such reducing of a multivariable relationship. The choice of how to narrow, and the fact that there is a choice, is therefore key to understanding partial derivatives. Narrowing can be explicit and intentional (*e.g.,* sketching a two-dimensional cross-section of a three-dimensional graph), but is often implicit and unstated.

Since partial derivatives are used across a variety of different contexts, we have been investigating how student understanding of derivatives progresses throughout courses in both multivariable calculus and upper-level physics [4, 20–29]. In this study, we explore physics students' knowledge about partial derivatives by focusing on instances of student-driven narrowing with information given in the form of contour graphs. Our specific research questions are:

1. What features of symbolic, graphical, and verbal representations do students attend to or use when finding a partial derivative?
2. How do students narrow to a path in order to find a partial derivative?
3. Do students narrow to a path intentionally and explicitly, and what representational features do they notice when they do narrow?

We investigate these questions by examining how students solve problems during interviews. Our prompts ask students to find a derivative from a contour graph but are ambiguous in the sense that they do not specify which (if any) variable should be held constant.

We begin in Section II by describing the relevant literature and giving background and context to the study. Section III provides the methods and methodology that we used to collect and analyze data. Our results for two different interview prompts are presented in Sections IV and V and finally discussed and interpreted in Section VI.

II. BACKGROUND

A. Prior literature

The body of research on the teaching and learning of partial derivatives is small but increasing. Much of this research has identified common student difficulties with determining, understanding, and/or interpreting partial derivatives in (or adjacent to) a thermodynamics context [1–6]. For example, Thompson, Bucy, and Mountcastle [1] found that upper-level students tended to have "a largely algorithmic, rather than conceptual, understanding" of partial derivatives in thermodynamics. That is, the students were able to perform calculations and manipulations with derivatives, but were not necessarily able to interpret them. In a follow-up study, Bucy, Thompson, and Mountcastle [2] found that students hold variables constant inappropriately when finding second derivatives and continued to observe that students were reluctant to reason physically. Becker and Towns [3] characterized students' responses to a broad, nine-question survey about partial derivatives in physical chemistry and found that many students struggled to transfer mathematics knowledge to physical contexts despite being able to perform the mathematical procedures. Research within the context of electromagnetism has tended to focus on student understanding not of partial derivatives themselves but of vector derivatives such as the gradient, divergence, and curl [7–16].

The mathematics education research community has also begun to characterize student ideas about derivatives of multivariable functions. In particular, Martínez-Planel, Gaisman, and McGee [17] investigated multivariable calculus students' ideas about the slopes of a function of two variables. They found that students had particular difficulty with directional derivatives and tangent planes. In a separate study, McGee and Moore-Russo [18] found that students did not easily generalize the idea of slope to multivariable functions.

There is substantial research on how students in physics or mathematics courses understand and interpret graphs of single-variable functions [30–37], including how students think about ordinary derivatives graphically [6, 38–45]. Much of this research has identified student difficulties with graphs and/or derivatives; for example, many studies have revealed that students often confuse value and slope when comparing graphs of functions [6, 33, 36, 45]. However, relatively little research has specifically focused on how students understand graphs of multivariable functions [46–48] or partial derivatives of multivariable functions that have been represented graphically [17–19].

B. Theoretical perspective

This study is part of a larger effort to develop a learning progression for partial derivatives that encompasses mathematics courses in multivariable calculus as well as upper-level physics courses [27]. Zandieh's concept image framework for examining student understanding of ordinary derivatives [49] has been foundational to this effort. A concept image is defined by Tall and Vinner as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" [50]. Zandieh's framework characterizes two complementary aspects of student understanding of derivatives: *representations* and *process-object layers*.

Representations are a major part of the framework because derivatives (particularly in physics) can be understood using a wide variety of representations. In fact, some research has focused on how students understand the use of different representations for multivariable functions and for partial derivatives [6, 51]. Zandieh identifies four prominent representations for ordinary derivatives: verbal, symbolic, graphical, and physical [49]. As an example, the two prevalent student interpretations of derivatives that we previously identified [52] are as a "slope" (graphical) or as "how a function changes as x changes" (verbal). Because physics often deals with data and measurements, our group has added a *numerical* representation to this framework [25]. An important facet of the numerical representation is that there are often experimental limits on the precision of measurements which prevent the use of truly infinitesimal differences.

The other aspect of Zandieh's framework is a set of three process-object layers: *ratio*, *limit*, and *function*. Each layer can be considered as a process in itself; for example, determining a ratio involves choosing two points, finding the change in two variables between those points, and dividing those differences. Alternatively, a layer might be reified into an object that can then be acted on by further processes—the ratio above is found repeatedly as the two points are chosen to lie closer and closer together.

In this article, we are primarily concerned with the first two layers, as *narrowing to a path* involves aspects of each layer that go beyond what has been considered for ordinary derivatives. Finding a derivative at a point involves choosing two points and finding differences (the ratio layer). For a multivariable function, many such pairs of points might be chosen. For example, in Fig. 1, the two points might be chosen along a vertical line, corresponding to $\left(\frac{\partial V}{\partial y}\right)_x$, or along the dashed line, corresponding to $\left(\frac{\partial V}{\partial y}\right)_r$. Each of these choices approximates a *different* derivative. Of course, the ratio calculated from two points is only a good approximation of a derivative if the points are chosen to lie close together along the path (the limit layer), where what constitutes close enough may vary with context. (It is even possible to imagine two different paths that pass through a single point in such a way that the derivatives are the same—but only at that single point, and not everywhere along the path.) The way in which the points are chosen specifies how the relationship is narrowed, including the fact that the points should be chosen such that they represent a roughly linear approximation to the narrowed relationship.

C. Instructional context

This research was conducted within the context of the *Paradigms in Physics Project* [53], the reformed upper-division physics program at Oregon State University (OSU). The core of this program is a year-long sequence of intensive courses at the junior level, which we refer to as Paradigms courses. Each Paradigms course is five weeks long and meets for a total of seven class hours per week. One of the five weeks in each course is dedicated to reviewing and extending the mathematical methods relevant to that course's physics content [54]. The class meetings are taught using a wide variety of interactive techniques, including the use of individual small white boards, kinesthetic activities and tangible metaphors, integrated laboratories, computer visualizations, and small-group activities [55]. The in-class focus is on helping students work through challenging, high-level problems that highlight the use of important mathematical procedures, physical laws, and underlying concepts. The Paradigms strategy is to provide opportunities for students to make common mistakes and face the most difficult elements of a problem in a setting with maximum available resources (*e.g.,* other students, TAs, and the instructor). Students also complete two hand-written homework assignments each week, in which they are encouraged to collaborate with each other, and where they practice the techniques learned in class and also tackle new and challenging applications of the conceptual ideas introduced in class.

Two of the Paradigms courses are particularly relevant to student understanding of partial derivatives: *Static Fields* (electro- and magneto-statics) and *Energy and Entropy* (thermodynamics and the beginnings of statistical mechanics). During the year when this study took place, the courses were taught back-to-back in the fall term by two different instructors. A third instructor (the second author) taught all class meetings dedicated to mathematical methods across the Paradigms. The first author served as an instructional assistant for both courses, and was present during all class meetings. Both classes also made use of the same undergraduate learning assistant. Although both courses involve partial derivatives, the term narrowing to a path was not used in class because we had not yet developed this language when the courses were taught.

Static Fields covers electro- and magneto-statics (roughly chapters 1, 2, and 5 of Griffiths [58]). The mathematical methods taught with this course therefore include a wide range of content related to partial derivatives. The course has a particular focus on coordinating multiple representations of multivariable functions when finding or interpreting

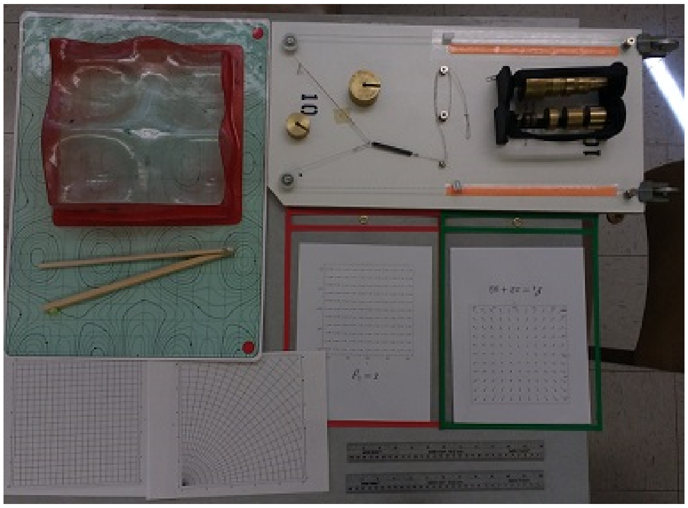


FIG. 2: Tools used in the Paradigms in Physics that were also provided for the interviewees: a plastic surface [56] and matching contour graph (top left), rectangular and polar grids (bottom left), a Partial Derivative Machine [57] (top right), two vector field maps in dry-erasable plastic sleeves (middle right), two clear plastic rulers (bottom right), and a protractor (not shown).

partial derivatives. Students work through several activities to develop a geometric understanding of partial derivatives along with gradient, divergence, and curl. One crucial aspect of this geometric understanding is the idea that a derivative can be approximated as a ratio of small changes. An example of the focus on multiple representations is that contour graphs and three-dimensional plastic surfaces (see the top left of Fig. 2) are used to help students develop a robust understanding of the gradient as the rate of change and the slope of a tangent line in the steepest direction [56]. Similarly, vector field graphs and other manipulatives, such as butterfly nets, are used to talk about divergence and curl [59].

The *Energy and Entropy* course teaches and makes use of partial derivatives in ways that are very different from *Static Fields*. Vectors are not an appropriate mathematical tool for thermodynamics because, unlike in electromagnetism, the variables are not spatial. It is common in thermodynamics to find partial derivatives of essentially any (state) variable with respect to any other (state) variable while holding any third such variable constant. These variables can be highly abstract (*e.g.,* entropy), and so they are sometimes harder for students to reason about and perform calculations with than variables in other physics courses. One of the ways in which *Energy and Entropy* helps students learn to reason and calculate using such variables is through the Partial Derivative Machine (see the top right of Fig. 2), a mechanical analogue to thermodynamic systems that was used to teach, among other concepts, the importance of holding variables constant [25, 57, 60–62].

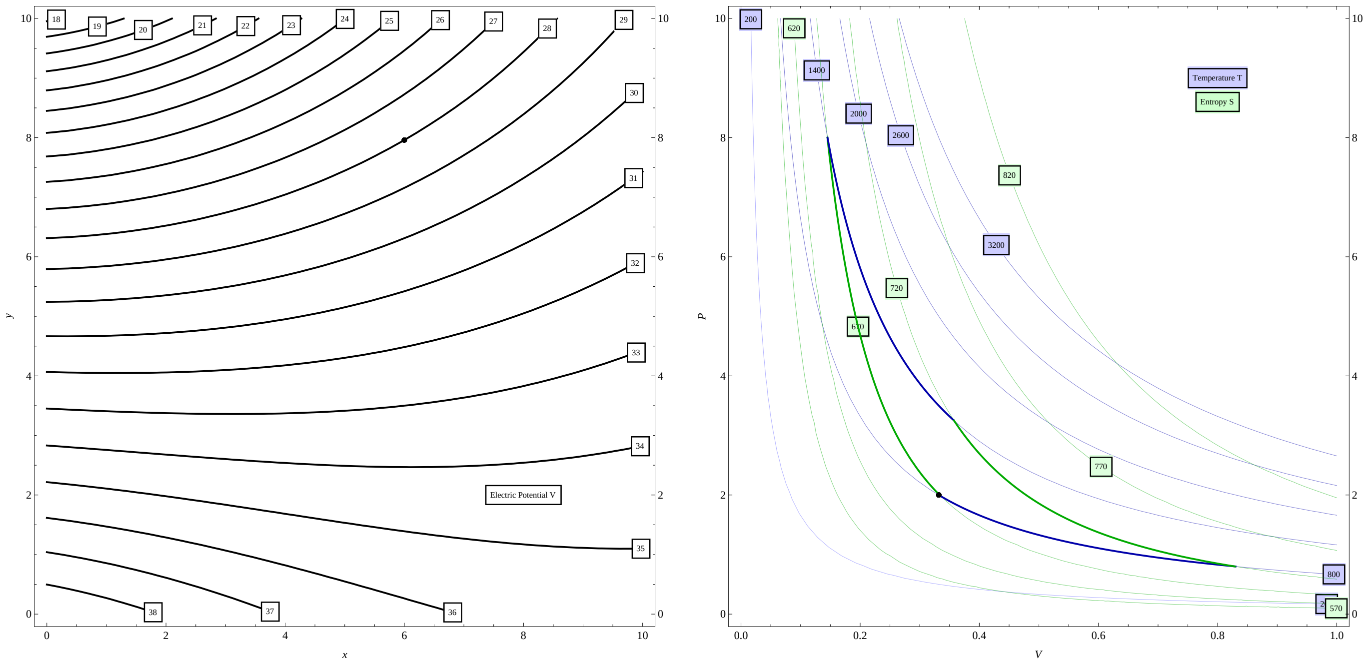
The *Static Fields* course made substantial use of contour graphs in the context of electromagnetism (*i.e.,* equipotential curves). Such graphs are common in other courses that cover electromagnetism, including introductory physics. In thermodynamics courses, including *Energy and Entropy*, pV-diagrams are frequently used for calculating work or in the context of cycles [63]. A pV-diagram shows one or more pressure *vs.* volume curves on the same set of axes for different physical processes. Each process typically corresponds to one part of a cycle for a heat engine and has a definitive beginning and ending point labeled on the graph. However, students did not typically use full contour graphs that extended across the full plot area for thermodynamic variables (pressure, volume, temperature, *etc.*).

III. METHODS

We conducted individual interviews with nine students enrolled in the final Paradigms course at the end of the junior year (spring term), identified throughout this paper by gender-neutral pseudonyms and pronouns. The interviews were carried out by the first author during the last two weeks of the academic year. Most of the interviewees were enrolled in the Paradigms throughout the year, and so experienced all of the instructional content detailed in the previous section. However, three students did not take the *Energy and Entropy* course that covers thermodynamics, and two of these students also did not take the *Static Fields* course that covers electromagnetism. All students who did not take one or both courses claimed to have instead completed physical chemistry courses that covered roughly equivalent content in thermodynamics and, except in the case of one student, electrostatics. Physical chemistry will have provided somewhat different learning experiences than is provided by the unique learning experiences of the Paradigms.

The interviews used a semi-structured think-aloud protocol and lasted approximately one hour. The protocol consisted of three phases: (1) an electrostatic potential prompt, (2) a thermodynamics prompt, and (3) several questions about connections between the two prompts. During each phase, the interviewer asked students to determine one or more derivatives, and also posed specific questions asking students to talk about the procedures they elected to use. A variety of tangible representations and tools were provided (see Fig. 2) and students were told that they could make use of any of them during the interview. This set of tools corresponds to those used throughout the Paradigms courses. Only the plastic surface (along with its matching contour graph), the rulers, and the protractor were referred to by any interviewee.

In Phase 1, the students were given the contour graph shown in Fig. 3a. They were told that the graph shows an electric potential V and were asked to determine the derivative of V with respect to y at the point indicated by the black dot. The marked point was intentionally chosen so that the tangent is neither vertical nor horizontal. After students arrived at a numerical answer, the interviewer prompted them to give an explicit description of how they determined the



(a) Contour graph for Phase 1 (Electric potential task)

(b) Contour graph for Phase 2 (Thermodynamics task)

FIG. 3: The graphs given to students during the first two phases of the interview. During Phase 1 (a), students are shown an electric potential V as a function of two spatial variables, x and y, and asked to find the derivative of V with respect to y at the indicated point. During Phase 2 (b), they are shown curves of pressure *vs.* volume for various constant values of temperature (blue) and entropy (green), and asked to find the derivative of p with respect to V at the indicated point. Students are told that the bold paths correspond to processes that are part of some cycle. Full-sized versions of the graphs may be found at the end of the article.

derivative.

In Phase 2, the students were given a new contour graph (shown in Fig. 3b) and told that it shows pressure *vs.* volume for a gas. They were also told that the bold paths indicate processes that are part of some cycle. Students were asked to determine the derivative of p with respect to V at the point indicated by the black dot. The marked point was intentionally chosen to lie on the intersection of a blue line of constant T and a green line of constant S. As in phase 1, the students were prompted to describe how they determined the derivative explicitly.

In both Phase 1 and Phase 2, the students were not given a written version of the prompt. We made this choice deliberately so that students would not be influenced by possible written notations that we might have chosen, such as $\frac{dV}{dy}$, $\frac{\partial V}{\partial y}$, or $\left(\frac{\partial V}{\partial y}\right)_x$. Furthermore, students were not explicitly told to determine a *partial* derivative in either phase, and they were not instructed to hold any variable constant or to find the derivative along any particular path. In Phase 2, students were asked about what variable is held constant only after they arrived at a numerical answer and described their method or when they brought up holding some variable constant on their own. (Four students mention holding x constant in Phase 1; they were not specifically asked to reflect on this statement until after completing Phase 2.)

Our use of prompts that did not specify what is held constant has allowed us particular insight into student thinking. Observing students' reactions to a question that does not have a definitive answer gave us a different window into their understanding than a more typical question with a well-defined answer might have. One particular affordance of our question is that students continued to talk about the task even after they had arrived at a numerical solution in order to check whether or not their solution did in fact answer the original task. We believe that this provided us with a robust view of the depth of the connections between different ideas in the concept images (as described in Section II B) of the students.

We note the subtle difference between the procedures necessary to compute the derivatives asked for during each phase. In Phase 1, students are asked for the derivative of V with respect to y: here y is one of the axis variables while V is given by the contours. In contrast, Phase 2 asks for the derivative of p with respect to V, the two variables that lie along the vertical and horizontal axes. In each phase, the derivative can be approximated by choosing two points near the indicated point, computing the change in each variable, and then dividing them. Because the prompts do not specify what should be held constant, the two points can be chosen to be consistent with narrowing to one of many different paths.

During the third and final phase of the interview, students were asked questions aimed at tying together the two prompts. First, if they had not done so already, students were asked to revisit the first prompt and to discuss what they held constant (if anything) and how they did so. Then, students were reminded that they had now found derivatives in two different physical contexts and asked whether their procedures were the same or different. If there was time available, students were asked to determine additional derivatives, such as $\left(\frac{\partial V}{\partial T}\right)_p$, $\left(\frac{\partial p}{\partial S}\right)_T$, or $\left(\frac{\partial V}{\partial y}\right)_r$, and to discuss similarities and differences between various derivatives, with an emphasis on the method they used to hold a variable constant. The results from Phase 3 are not discussed in this article but will be described in future work.

The interviews were video- and audio-recorded. We transcribed all segments relevant to our initial (broad) research questions: (1) What methods did students use when finding partial derivatives from contour graphs? and (2) How did students decide what to hold constant? We proceeded in the style of Thematic Analysis [64, 65] by examining these segments and identifying common themes within and across the interviews. These themes, along with our research questions themselves, were refined after extensive discussion and re-examination of the videos [66]. Eventually, we arrived at a research focus on how students narrow to different paths. The data were re-examined to identify all instances of narrowing to a path. We further analyzed the data in and around each instance of narrowing to identify all representational features associated with any graphical, symbolic, or verbal representation of either a function or a derivative.

The results of our analysis for Phases 1 and 2 are described in sections IV and V, respectively. We present each theme that emerged from the data with at least one example of student work. We also report the number of students who gave responses that we interpreted in the same way, though the total number of interviewees (nine) is small enough that we caution against interpreting these numbers as overly significant.

IV. ANALYSIS OF PHASE 1 (THE ELECTROSTATIC POTENTIAL TASK)

We find three stages common to students' responses in Phase 1. Each stage provides us with information about how students might be narrowing to a path. Students first tend to take a few minutes to orient themselves to the contour graph (*i.e.,* to figure out what information the graph shows and the manner in which it shows this information) and to the task itself. Then they attempt to find the derivative numerically. Finally, they reflect on their result in some way, either independently or in response to a prompt from the interviewer. Examples of student work for each of these three stages is presented in one of the following subsections, along with a brief discussion of the trends we observed across multiple students.

A. Orienting to the graph of *V*

The contour graph given in Phase 1 (see Fig. 3a) represents a relationship between three variables: V, x, and y. Eight of the nine students verbally identify the labels on the x- and y-axes within the first minute of receiving the graph. Similarly, eight students identify the contours as conveying some information about the potential: either the explicit values of V, the fact that V is constant along the lines, or the sense of increase given by the numbered labels on the graph. For example, after identifying x and y, Alex points at a contour and says:

Alex: "And then these lines are supposed to represent a two-dimensional V."

Three students question the meaning of the boxed numbers labeling each contour, as in the example below:

Lee: "Are these like the numbered values of V?"

Here, Lee asks if the numbers should be interpreted as values of the potential, rather than stating that they are values of the potential. When any student asks this question the interviewer confirms that this interpretation is correct. In contrast, no student asked for the meaning of the tick marks on the axes to be clarified.

Four of the students write the target derivative using Leibniz notation: three as $\frac{\partial V}{\partial y}$ and one as $\frac{dV}{dy}$. None of these students indicate symbolically that any variable should be held constant, such as by adding parentheses and a subscript as in $\left(\frac{\partial V}{\partial y}\right)_x$. When a students writes the derivative symbolically, they do so within the first minute of receiving the graph, often as the interviewer is stating or restating the prompt.

At some point during Phase 1, eight of the nine students give a verbal description for the derivative such as:

Pat: "How much the graph is changing in just the y-direction."

This language matches the *change* interpretation for the derivative that we identified in a previous study [20] as being common among students in multivariable calculus and especially common among upper-division physics students. Furthermore, the term "just" suggests that Pat means to consider a change in the y-direction involving no change in the x-direction. Six students use "change" language within the first minute of receiving the graph, while two do so after 2 to 4 minutes.

B. Approximating the derivative of *V* with respect to *y*

After students have oriented to the graph and the task, they begin calculating the derivative. We find three different paths to which students narrow from the full contour graph in order

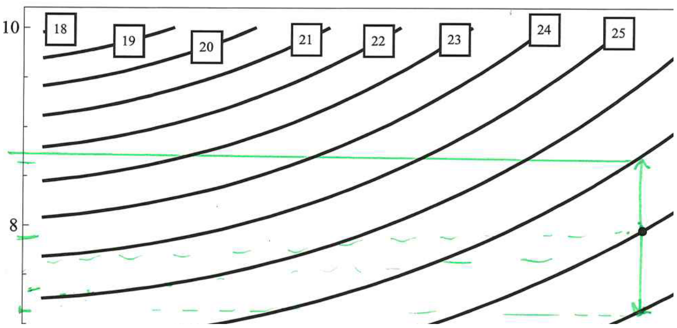


FIG. 4: Mel's work during Phase 1: a sketch of two arrows that constitute narrowing to a vertical path through the indicated point.

to make a calculation. For each of these ways, we provide a detailed example of one student's work followed by a summary of the similarities in the work of the other students.

1. *Narrowing to a vertical path*

Mel sketches the upward-pointing arrow on the right of Fig. 4 and explains as follows:

Mel: "So I looked at the y-direction of this graph [gestures at y-axis] and I saw that as you increase in the y [moves finger upward along y-axis], I drew an arrow in the y-direction to show, to help me determine how the potential changes in the y-direction."

Without measuring, Mel estimates the derivative to be about -1 because increasing in the y-direction causes V to go down by 1 unit, clarifying:

Mel: "I wanted to determine how much per unit y the potential changed."

Int.: "Okay."

Mel: "Yeah so that's why I drew this vector from the [indicated] point to another point on the line in the y-direction."

Here, Mel has chosen the two points on either end of the arrow and is using them to approximate the derivative, but has not yet written anything symbolic. Mel notes that the difference in y between the two points is a "unit change," not an "infinitesimal change," and pauses to think about whether or not this process accurately represents the derivative, eventually proceeding without resolving this tension. A similar hesitancy to view such an approximation as a true derivative has been observed among expert mathematicians [60].

At the interviewer's suggestion, Mel uses a ruler and finds the difference in y between the chosen points is 1.5. Mel then estimates the derivative mentally as $-2/3$ Volts (Mel introduces units for the potential but not for x or y). When the interviewer asks how they arrived at this answer, they write

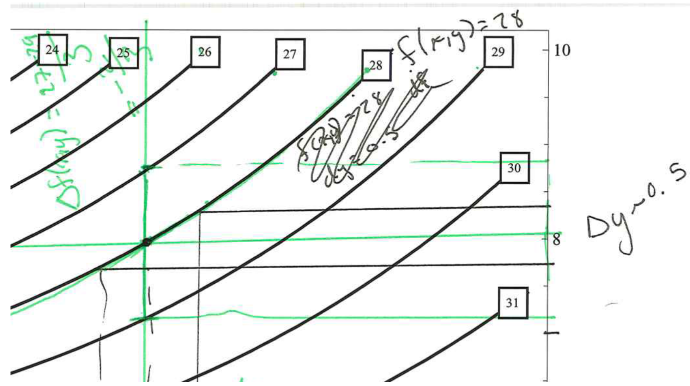


FIG. 5: Pat's work during Phase 1. Pat is initially narrowing to a path along the $V = 28$ contour line (work in black), then switches to narrowing to a vertical path (work in green).

$\frac{\Delta V}{\Delta y} = -\frac{1}{3/2}\text{V} = -\frac{2}{3}\text{V}$. Mel articulates the following change in perspective:

Mel: "Before I was looking for a one unit change in y but ... this time I'm focusing on a unit change in V and seeing how much y changed."

A few minutes later, the interviewer asks Mel to clarify their choice of points.

Int.: "You went from like the line labeled 28 to the line labeled 27 [gestures at labels]."

Mel: "Yeah."

Int.: "How did you decide to go from that point on the line to this point on the other line [gestures at the point Mel chose] and not to some other point on line number 27?"

Mel: "I decided that because every point that's perpendicular to the x-axis [gestures along the $x = 6$ line] has the same x-component so I didn't want to change x at all."

Int.: "Okay."

Mel: "Because I wanted to find the derivative with respect to y. So that was I guess how I held x constant while taking the derivative with respect to y."

Mel then wonders if the value would be the same using a point on the $V = 29$ contour instead of a point on the $V = 27$ contour. They draw a second vertical arrow (downward) and repeat the procedure described above.

Overall, Mel narrows from the given contour graph to a vertical path, denoting this graphically by drawing arrows. For Mel, the arrows appear to provide a focus for how the potential changes *in the y-direction* specifically. Mel's gesture perpendicular to the x-axis and claim that x is held constant together suggest that they are narrowing to this path explicitly and intentionally.

At some point during Phase 1, all nine of the students nar-

(a) Contour Graph (b) Gestures (c) Whiteboard

FIG. 6: Ira's work during Phase 1 (a) on the provided contour graph, (b) gesturing above the graph, and (c) on the large whiteboard. Ira first sketches the gradient but eventually narrows to a vertical path.

row to a vertical path. Seven students indicate this narrowing by drawing a line, arrow, or line segment on the provided graph. Five students, including Mel, use the phrase "in the y-direction" or "with respect to the y-direction" to refer to the derivative, language that we have previously identified [20] as common among both math and physics students in interpreting multivariable derivatives. Only four students, including Mel by the end of Phase 1, verbally specify that they are holding x constant at any point during Phase 1.

All but one of the students choose two points, separated vertically, for approximating the derivative. Some students choose the indicated point and a point on a neighboring contour while others choose points on the neighboring contours in each direction. (A few students, like Mel, even try more than one set of points to check that their value for the derivative is reasonable!) Students then go on to find differences or changes in the values of V and y between the chosen points and divide those changes to find a derivative. Six students explicitly write the derivative as a ratio of small changes, $\frac{\Delta V}{\Delta y}$. One instead writes Δf (having named the graphed function f instead of V) and Δy separately and divides the resulting numbers, while another appears to do the math mentally and later describes the same method.

Two students have trouble with some part of approximating the derivative. Blair finds Δy incorrectly by following the contour lines on which the chosen points lie to their intersections with the y-axis. Alex initially transposes x and y before noticing the error and deciding to change the derivative under consideration to be $\frac{\partial V}{\partial x}$. Eventually, eight students are able to find a numerical value for the derivative; the remaining student (Chris) admits being "not very familiar with these kinds of graphs." We note that Chris, who did not take the *Static Fields* Paradigm, claims only to have learned electromagnetism in introductory physics.

2. *Narrowing to a path along a contour*

Pat initially chooses points along the $V = 28$ contour line and measures y-coordinates and Δy (shown on the right side of Fig. 5). Having found Δy, Pat silently contemplates how to find ΔV, writing $f(x, y) = 28$, $dy = 0.5$, and df. Pat stares at this last expression for a few moments, smiles, and crosses out this work (see Fig. 5). Pat comments on maybe overcomplicating and maybe not remembering how contour plots work. After rewriting $f(x, y) = 28$ again, Pat appears to have an epiphany:

Pat: "I'm not looking at the individual curves, I'm supposed to be looking at how the whole field changes at that point."

Pat then draws a vertical line through the indicated point and chooses points on the neighboring contour lines that intersect this vertical line. Pat uses a difference quotient to evaluate the derivative numerically at the point (shown in green in Fig. 5).

Pat is the only student to narrow to the path specified by a single contour line. Labeling the contour line as $f(x, y) = 28$ strongly indicates intentional narrowing—that is, attention to the underlying path along which they chose points for approximating the derivative. However, Pat pauses after writing df, possibly discovering that the change in potential is zero, and reconsiders. At this point, Pat's use of the term "whole field" suggests a momentary return to viewing V as a function of both x and y before narrowing in a new way to a different (vertical) path.

3. *Sketching the gradient vector*

Ira begins by thinking about the gradient and its components.

Ira: "Okay so the first thing that I was thinking is maybe I can write the arrow that signifies which way the gradient would point, and then maybe break it down into components."

Ira sketches the gradient vector shown in Fig. 6a by noting that the gradient should point "perpendicular to [the equipotential lines] towards the direction of increasing potential." This statement supports an earlier sweeping gesture that Ira makes with one finger (see Fig. 6b left) to indicate the general direction in which the potential is increasing. Ira tries to determine the length of the gradient vector, but is not able to come up with an answer. (It is only when the interviewer asks for a reflection that Ira sketches the components of the gradient and the curved line also shown in Fig. 6a.)

Subsequently, Ira instead marks points above and below the indicated point (see Fig. 6a) and says:

Ira: "So in taking this partial derivative I'm imagining that I'm going to fix x to be constant and I'm gonna imagine how the value of the potential is changing with respect to the y-direction."

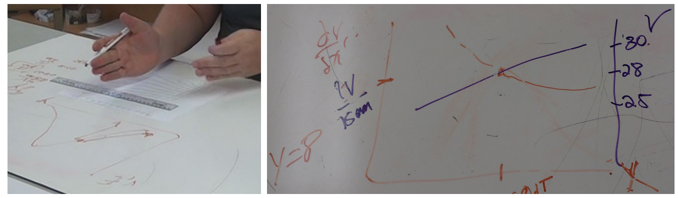
Ira uses one hand to indicate this new focus on the y-direction (see Fig. 6b right), similar to their gesture for the gradient. It is only at this point that Ira approximates the derivative as $\frac{\Delta V}{\Delta y}$ and calculates a numerical value (work shown in Fig. 6c).

Ira invokes the gradient vector, which can be thought of as narrowing to the path that points in the direction of greatest change (always perpendicular to the contour lines). However, Ira mentions the components of the gradient from the beginning and does not appear to have a method for determining the magnitude of the gradient, which together suggest that Ira is not narrowing explicitly to the path in the direction of the gradient. Only one other student (Drew) draws a gradient, and Drew also does not end up using it to find a derivative. Another student (Sam) briefly discusses the gradient (and its direction) and explicitly notes that the gradient is not the answer to the prompt.

C. Reflecting on the derivative of *V* with respect to *y*

After finding a numerical value for the derivative, it is common for students to reflect in some way on their answer. For example, four students assess whether or not the sign of their answer is reasonable, a common physics sensemaking strategy [67]. Some reflection is spontaneous, while some is in response to questions from the interviewer asking students to restate their procedure for finding the derivative, to explain interesting aspects of their overall responses, or to give a physical interpretation for the derivative. In some cases, the reflections provide us with new insight into how students narrow to different paths. In particular, several students reinterpret aspects of the given graph and the target derivative using an alternate graphical representation.

We begin by discussing Alex's reflection after narrowing to a vertical path. Aside from reversing x and y, Alex initially has no trouble approximating the derivative as $\frac{\Delta V}{\Delta y}$, finding the changes, and dividing them. Alex refers to the derivative as a "gradient" throughout the interview, but does not otherwise use any symbols or language appropriate to the vector quantity known as the gradient. After Alex's calculation, the interviewer asks Alex to give a physical interpretation of the derivative. Alex sketches a mock graph of V *vs.* y as a cross-section of a hill (see Fig. 7a), making a hand gesture to indicate the cross-section that is meant. Alex describes how this new sketch relates to the given graph:



(a) (b)

FIG. 7: Alex's whiteboard sketches during Phase 1: (a) an initial mock cross-section of V and (b) a more precise cross-section of V and its derivative.


(a) (b)

FIG. 8: During Phase 1, Drew (a) indicates how the contour graph can be viewed as a surface and (b) later interprets the derivative of V with respect to y as corresponding to the slope of a tangent line (represented by a finger) in the y-direction on a plastic surface.

Alex: "I mean right now I'm just doing linearly, cause that's what you, the gradient [points to $\frac{\partial V}{\partial y}$] is linear, we're keeping x constant right now."

Int.: "Can you say more about that?"

Alex: "Well you wanted the change in potential with respect to y [points at derivative again]. So in order for me to do that I made x constant and did a line straight across."

We clarify that Alex appears to use the word "linear" to mean "straight" rather than "varying linearly," which aligns with the language Alex uses at the end of the excerpt.

After this, Alex notices that they have mixed up x and y. After considering several options, Alex elects to change the derivative under consideration to be $\frac{\partial V}{\partial x}$ rather than recalculate the derivative with respect to y. They indicate that they would like to make a sketch at $y = 8$ and make a new graph (the orange curve in Fig. 7b) showing the derivative $\frac{\partial V}{\partial x}$ and how it changes from left to right. In response to a request from the interviewer, Alex adds a sketch of V to the same set of axes (the purple curve in Fig. 7b).

The fact that Alex coordinates the two different kinds of graphs reinforces our interpretation that Alex narrows, early and intentionally, to a path with constant y. Three other students draw a one-dimensional graph similar to the "cross-section" that Alex draws. We have previously observed students interpreting a multivariable derivative in this way [20], but it was much rarer than other interpretations.

Three students spontaneously use a plastic surface (provided by the interviewer) to interpret the derivative as the slope of a line tangent to the surface. For example, Drew calculates the derivative to be about $-4/3$ and states, unprompted, that the derivative "would be the slope of a tangent line on the surface there." (Earlier in the interview, Drew imagines a surface corresponding to the given contour graph, as shown in Fig. 8a.) Then, Drew demonstrates this idea using the plastic surface, representing the tangent line with a finger (see Fig. 8b) and clarifying as follows:

Drew: "So I think what I'm trying to say here is that if you, at some x value here [points to plastic surface], and you put something that sits tangent to the surface at this point here [points to contour graph], then the slope of that line [tilts finger] in the y-direction is $-4/3$."

For the students who sketch a one-dimensional cross-section graph or make use of a plastic surface, the alternative representation appears to help describe not only how the three variables (V, x, and y) are related, but also what the derivative is. Both Alex and Drew begin by establishing the new representation and then use it to give some meaning to the derivative that is not possible with the contour graph alone. The students are also able to demonstrate how they are narrowing the contour graph to find the derivative in new ways.

V. ANALYSIS OF PHASE 2 (THE THERMODYNAMICS TASK)

Students in Phase 2 tend to perform actions similar to those in Phase 1: first, the students orient to the new graph, then attempt to solve for the requested derivative, and finally reflect on their work. As in Section 3a, we provide examples of student work and discuss the patterns we observed across different students in the subsections below.

A. Orienting to the graph of pressure *vs.* volume

Many of the orienting steps in Phase 2 are similar to those in Phase 1: four students write the derivative as $\frac{\partial p}{\partial V}$ (with no subscript), three express the derivative as "how much p changes with respect to V," and five comment on the axis labels (p and V). Six students explicitly state that there are two different sets of contour lines, though several express uncertainty about what each set of contours represents. For example, Pat asks for clarification:

Pat: "So the blue lines are isotherms and the green ones are same entropy?"

The other three students do not comment on or appear to notice these features until much later in the interview.

Four students focus on the bold cycle (which the interviewer identifies in the prompt) and interpret it physically or experimentally at the beginning of Phase 2. For example, Blair describes the graph as representing a Carnot cycle and associates individual segments of the graph as "expanding" or "compressing," language that indicates a link between the given graph and common thermodynamic processes. Mel and Drew both refer to the compression of a gas for individual processes, whereas Sam refers to the whole cycle as a heat engine.

Five students reference a "cusp" or "kink" in the bold path at the indicated point and note that this implies that the derivative is different on the left (green) side than on the right (blue) side. For example, the following exchange occurs at the beginning of Phase 2, immediately after the interviewer asks Lee to find the derivative of p with respect to V at the indicated point:

Lee: "On like the bolded path? [Pause] Yeah so on the bolded path or [pause] which path? Cause there's two that go through that point."

Int.: "Does it matter which path you choose?"

Lee: "Yes, because at this point [the indicated point] there's a corner."

Int.: "Okay."

Lee: "So that either means that you would have to take one of these paths [gestures along each full path]."

Int.: "Can you tell me which paths you're referring to?"

Lee: "Oh um this one, this blue one, the bolded one, and then as it continues, and then this green one before it's bolded and then as it's bolded."

Int.: "Okay."

Lee: "Or I can't do a derivative at a corner unless you assumed it's like smooth."

Here, Lee describes three different paths: the bold path, the blue path, and the green path. Lee also identifies the "corner" at the indicated point (where the three paths intersect). Lee is unsure how to proceed until the interviewer asks Lee to pick one path for which to find the derivative. The other students who identify a cusp at the indicated point mention it prominently during their early attempts to find derivatives during Phase 2, either indicating that the derivative is undefined or that there are different derivatives from the two sides.

B. Approximating the derivative of *p* with respect to *V*

We begin by describing how students focus on the bold portions of the contours, which students are told correspond to processes that are part of some cycle. Then we describe how most of the students shift their perspective and narrow to paths along the contour lines, typically with some intervention from the interviewer. Lastly, we discuss three students who attempt to find the derivative using thermodynam-

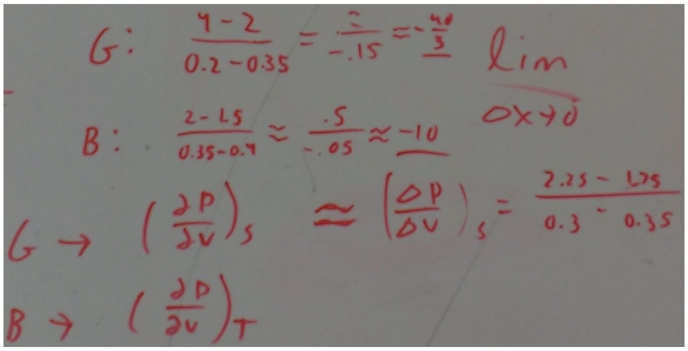


FIG. 9: Sam's whiteboard work during Phase 2. The top two rows (early in Phase 2) correspond to Sam narrowing to the bold path, while the bottom rows (later in Phase 2) represent narrowing to one path along a blue contour line and separately narrowing to one path along a green contour line.

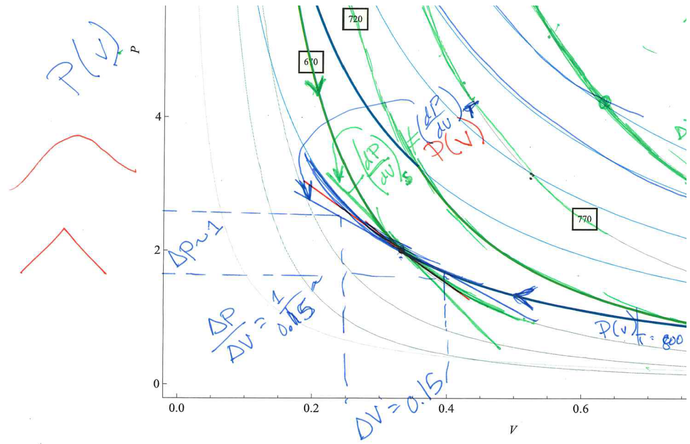


FIG. 10: Pat's work during Phase 2. Pat initially narrows to the bold path (work in red), including sketching a tangent line and labeling the path as a function $p(V)$. Later, Pat narrows to the blue and green paths and draws new, separate tangent lines to each path (work in blue and green).

ics equations such as the ideal gas law and the thermodynamic identity.

1. *Narrowing to the bold path*

Sam's initial method for approximating the derivative is very similar to the method that most students use in Phase 1:

Sam: "Okay so I'm going to use the Δ approximation and see if that gets me anywhere. So if we approximate $\Delta p/\Delta V$ at this point it does depend on the path we take since we have a bit of a cusp there. There could be, depending on which path we take ... we could get two different derivatives but I'll go ahead and see what I get along the green path."

Sam chooses a point on the bold green path and the indicated point, determines the values of p and V at each point, and uses them to find a ratio of small changes to approximate the derivative (see the top row of Fig. 9). Sam proceeds to follow the same procedure using a point on the bold blue path (the second row of Fig. 9). After finding two different numerical values, Sam observes:

Sam: "So then going along two paths I get two different derivatives depending on which limit I take since this is a cusp and technically a cusp is not technically differentiable at least classically."

Sam's attention is focused on the bold portions of the contours. In addition to using the terms "path" and "cusp" when attempting to find the derivative, Sam spends the first two minutes of Phase 2 describing how the thermodynamic variables (p, V, S, and T) change along each of the four bold processes. Sam's method in Phase 2 is different from Phase 1 in a key way. Though Sam still chooses two points and finds differences between them, the two points are now chosen to lie along the same contour line, while in Phase 1 the two points lie on different contour lines. We anticipated that some students might have difficulty with this transition, but no students did.

Pat uses a different method to try to find the derivative: sketching a single line and claiming it is "tangent" to the bold contour (the red line at the center of Fig. 10). This leads to the following exchange with the interviewer:

Pat: "So what I'm doing is I'm looking at this graph and just kind of imposing in my mind that this is a function $p(V)$ [shown in red at the center of Fig. 10]."

Int.: "Which thing is a function?"

Pat: "Uh the green and the blue lines [gestures at the bold paths]."

Int.: "The bolded paths?"

Pat: "Yeah the bolded paths. Um because I know that they, well they do represent the pressure of the system at a given point. I think I'm just overcomplicating what it means in my head cause I'm just going oh it's like an engine and yeah. So the derivative at that point is the slope of the line tangent to that and it doesn't really look like it's a smooth function there. So I'm not really sure if this will work."

Int.: "Can you say what you mean by smooth function?"

Pat: "Um it looks like ... instead of being ... continuous it looks like there's an edge [sketches example] which makes a discontinuity in the derivative. So I'm trying to figure out how to make that work. So what would be happening here? The cycle's going one direction and this would be changing from constant T to constant S. [...] So drawing the tangent thing isn't going to work because there's a discontinuity that doesn't quite look

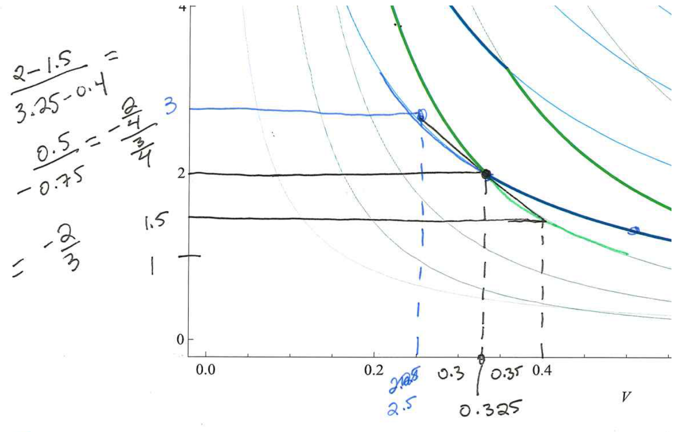


FIG. 11: Ira's work during the beginning of Phase 2. Ira sketches a line that is "tangent" to the bold path and then calculates the derivative by finding the slope of the line.

like a discontinuity."

In this excerpt, Pat describes the bold portions of the blue and green contours as a single function, labeling this function symbolically as $p(V)$ in red at the center of Fig. 10. Pat then discusses using the slope of the red tangent line to find the derivative, but immediately notes that the function does not look smooth. Pat draws only one tangent line on the graph itself at this point, then sketches a pair of one-dimensional functions (in red to the far left of Fig. 10), the top function smooth and the bottom function not smooth. After thinking for some time, Pat concludes that a tangent line will not work because of what Pat calls a "discontinuity in the derivative." Pat then switches to attempting to find a Maxwell relation (see Section V B 3).

Although Pat is only asked to find the derivative at a single point, the term "discontinuity" suggests Pat may be thinking of the derivative as a function along the bold path, which has different values as the indicated point is approached from the right and from the left. Pat states that the tangent line will not work, but does not formally note that a line tangent to the bold path does not actually exist.

Ira starts in a way that is similar to Pat:

Ira: "So the derivative of p with respect to V on this graph should tell me the slope of the tangent line to this point."

Ira's tangent line and slope calculation are shown in Fig. 11. While calculating, Ira compares the thermodynamics graph with the electrostatic graph from Phase 1.

Ira: "This is sort of I guess using a similar strategy to what I did over here [points to electrostatic graph] except for now on this graph there isn't really a direction I need to worry about, if that makes sense."

Int.: "Why not?"

Ira: "Because I'm concerned about the value of p and V and in this [electrostatic] graph I had x values, y values, and the potential value. So in order to determine this derivative I really just need to consider the values on these axes as opposed to ... entropy and temperature."

Unlike Pat, Ira does not focus on the cusp at the indicated point and finds a numerical answer for the derivative. Ira observes the values of temperature and entropy during the calculation, but explicitly states that these variables do not matter—only the "axis" variables do.

Eight students narrow to the bold path at the outset of Phase 2. Five of these students draw a single "tangent" line to the bold curve at the indicated point, as both Pat and Ira do. Six students recognize that the derivative is not the same on either side of the indicated point. Three students acknowledge there are two derivatives at the point without recognizing that there are two tangent lines, which may indicate some misunderstanding of what a tangent line is and/or a weak conceptual link between slopes of tangent lines and derivatives, a surprising result that is worthy of further investigation. All of these actions, along with those described in section V A, suggest that most of the students are consciously attending to the bold path when they narrow to it. Once a given student finds numerical values using this method, that student typically stops until the interviewer prompts them to think about holding something constant (see Section V B 2), or the student attempts to find the derivative symbolically (see Section V B 3).

2. *Narrowing to paths along contour lines*

While discussing the two derivatives Sam found from narrowing to the bold path (the underlined numbers in Fig. 9), Sam's thinking changes spontaneously:

Sam: "But if I take two limits I get two different derivatives ... they're close to each other but not quite from these two different approximations. So I'm not quite sure what to say beyond that, other than I have approximately two derivatives [underlines numerical values] at that point if I were to just look at some curve along this green line and this blue line then uh it's not a smooth curve if I were to go along these two. [Jerks head.] Oh that's another way to look at it! So another way of saying this is there's different derivatives holding different pieces constant. So moving along the green curve that's the same as saying a derivative of pressure with respect to volume holding entropy constant [points to box label] since we're moving along a curve of constant entropy [gestures at the whole green contour line]."

The focus of Sam's attention on the graph changes dramatically over the course of this excerpt. Sam is first clearly talking about the bold path as a single, continuous curve that has different derivatives from the left and right at the indicated point. Sam does not initially know what else to say, but seems to have an epiphany while looking at the graph (accompanied by a change in body language and in tone of voice). After this moment, Sam explicitly indicates two of the graph's features: the green box specifying the entropy contours (which Sam had observed previously) and the complete green contour passing through the indicated point. This is strong evidence that how Sam's narrowing has changed.

Sam goes on to write the two partial derivatives with subscripts, as shown at the bottom of Fig. 9, and verbally describes each derivative as "holding constant" either entropy or temperature. These are symbolic and verbal representations that are not present in any of Sam's earlier work during Phase 2. Sam even recalculates $\left(\frac{\partial p}{\partial V}\right)_S$ by choosing two new points on the green path, one on each side of the indicated point, and computing the ratio $\frac{\Delta p}{\Delta V}$. Sam concludes:

Sam: "So depending on which derivative we're looking at, since there's two unknown variables that could be held constant or allowed to vary with this, there's two different derivatives."

While Sam changes perspective spontaneously, most students do not narrow to the individual contour lines on their own. Pat, for example, spends the first part of Phase 2 considering the cusp in the bold curve (Section V B 1), followed by manipulating some thermodynamic equations (Section V B 3). After those efforts prove unproductive, the interviewer prompts Pat to take the derivative from only one side or the other, leading to the following exchange:

Pat: "Oh that discontinuity wouldn't be there, and you could figure out what dp/dV is with respect to that direction."

Int.: "Could you show me how you would do that?"

Pat: "So ... if you were to extend, for the blue line, if you just look at the graph [traces the blue contour line] of um constant temperature and take a tangent line there [sketches tangent line]."

Pat labels this curve $p(V)_{T=800}$ (in blue at the bottom right of Fig. 10). This echoes Pat's labeling of the bold curve as simply $p(V)$ when initially narrowing to the bold path. Pat chooses points on the new tangent line, writes Δp and ΔV, and finds the ratio $\Delta p/\Delta V$ to approximate the derivative (work in blue in Fig. 10). Pat then switches to a green pen to trace the green contour and sketches another tangent line, noting:

Pat: "And obviously like the lines are different here."

Int.: "Different tangent lines?"

Pat: "Yeah. So dp/dV with a constant S [writes $\left(\frac{dp}{dV}\right)_S$] is not going to be the same as [switches to blue pen] dp/dV with a constant T [writes $\neq$ and $\left(\frac{dp}{dV}\right)_T$]. But you could find both [derivatives]."

Int.: "So you just labeled those derivatives as being with something held constant. What made you decide to do that?"

Pat: "The graph says [gestures at boxed label] that the blue curves are constant temperature and the green ones are constant entropy so the curve represents p as a function of V [writes $p(V)$]."

As shown in blue and green at the center of Fig. 10, Pat writes each derivative with a subscript (these derivatives are written with d's, while Pat writes other derivatives with ∂'s), and specifies that they are not equal. Pat later draws arrows from each derivative to the corresponding tangent line to clarify their meaning. Pat refers to the temperature and entropy labels on the graph for the first time when justifying the new notation.

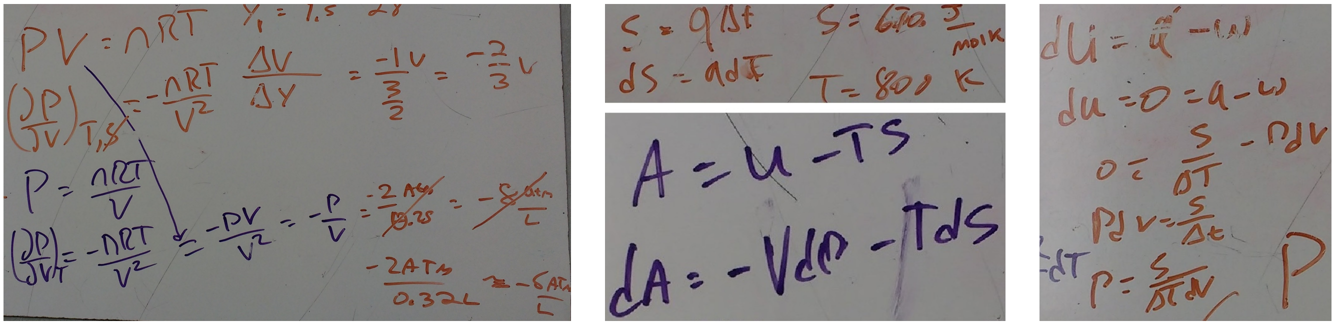
Four other students each follow a trajectory very similar to Pat's at this point in Phase 2: the interviewer asks if they thought about holding something constant (as specified in the interview protocol outlined in Section III), the student switches from narrowing to the bold path to narrowing to the contour lines, and then the student calculates two separate derivatives and interprets them as derivatives with T and S held constant. As part of this process, some students identify one or more new features of the relevant representations. For example, Blair observes the temperature and entropy labels on the provided graph and also adds subscripts to previously written derivatives.

Two students have substantial difficulty determining the partial derivative(s) at the indicated point, despite the fact that each of them appears to narrow to the individual contours at some point during Phase 2. Alex requires substantial intervention from the interviewer to proceed, and eventually finds two separate derivatives after comparing the thermodynamics graph to the graph of electric potential given in Phase 1. The other student, Chris, is discussed in more detail at the end of Section V C.

3. *Manipulating thermodynamic equations symbolically*

During Phase 2, three students attempt to use knowledge from thermodynamics to determine the requested derivative. Each of these students decides to use thermodynamics knowledge *after* narrowing to the bold path (Section V B 1) but before narrowing to the contour lines (Section V B 2). For example, Pat mentions "partial derivative relations" early in Phase 2, but does not attempt to derive one until after trying to find the derivative from a line tangent to the bold curve. We begin by discussing the case of Mel in detail.

After processing the thermodynamics graph for about five minutes, including discussing the "kink" at the indicated point, Mel decides to assume that the system can be treated as an ideal gas. (Although the contour graph provided in the interview matches the equations of state for a monatomic ideal gas, students are not told this at any point during the interview.) Mel writes the ideal gas law and attempts to



(a) (b) (c)

FIG. 12: Mel's symbolic manipulations during Phase 2. Mel begins (a) by finding the derivative (with respect to both T and S) from the ideal gas law. When reminded that the system under consideration may not be an ideal gas, Mel attempts to use the definition of entropy (top b), the Helmholtz free energy (bottom b), and the thermodynamic identity (c) to find the derivative.

calculate the derivative symbolically (in orange at the upper left of Fig. 12a). At this point, Mel writes the derivative as $\left(\frac{\partial p}{\partial V}\right)_{T,S}$. Mel clarifies that this notation (introduced for the first time) refers to the derivative of p with respect to V at constant T and constant S. Both Pat and Ira also use subscript notation while doing similar symbolic manipulations, and had not done so at any previous point during their interview.

Mel solves the ideal gas law for p and takes the derivative of it with respect to V. At this time, Mel expresses uncertainty about whether or not both T and S are actually constant. (In Fig. 12a Mel has crossed out the S and erased an S subscript on the purple derivative as a result of later reflection—see Section V C.) Mel manipulates the resulting derivative symbolically (in purple at the bottom left of Fig. 12a) and then plugs in values of p and V from the graph to determine a numerical value for the derivative (in orange at the bottom right of Fig. 12a).

Mel's procedure is not incorrect for an ideal gas, though it gives only the derivative at constant T. After this calculation, the interviewer reminds Mel that the system may not be an ideal gas, and asks if it is possible to find the derivative from the graph alone. Mel looks at the graph but then returns to working symbolically on the whiteboard, proceeding through a succession of equations and concepts from thermodynamics (see Figs. 12b and 12c), none of which prove helpful.

Pat and Ira each also invoke thermodynamic equations and attempt to manipulate them symbolically. All three of these students misremember some parts of the thermodynamic equations they attempt to use. In addition to working with the thermodynamic identity, both Pat and Ira try to derive a Maxwell relation that could be used to relate partial derivatives. Pat appears to understand the purpose of a Maxwell relation despite not remembering some details and gives the following explanation:

Pat: "Yeah and I was going to do that [relate two partial derivatives] because I can't easily get $\partial p/\partial V$ so I was going to look for something like this [a Maxwell relation] and then have it related to a different partial [derivative] and then I would figure out whatever that partial [derivative] is at that point and since they're equal then it works out."

Though it turns out not to be possible to find a Maxwell relation involving any derivative of p with respect to V, it is in general a reasonable and useful method for finding partial derivatives. It is interesting that Pat does not recognize that even if able to express $\partial p/\partial V$ in terms of another derivative, it would still be necessary to read some derivative from the provided graph.

Ultimately, the thermodynamic equations invoked by each student do not prove immediately useful in determining derivatives. However, it is interesting that all three students begin to use subscript notation for partial derivatives while engaged with the thermodynamic equations. As we discuss in the next section, this notation is common in the interviews only after a student has identified that it is possible to find two partial derivatives at the indicated point.

C. Reflecting on the derivative of p with respect to V

As discussed in Section V A, some students identify either the contour graph they are given or thermodynamics in general as complicated. As Phase 2 progresses, many students identify specific features of the contour graph (that are not explicitly present in the graph of electric potential) that may provide insight into why the students view thermodynamics as complicated.

We begin by considering when the interviewer asked if Blair thought about holding anything constant. Blair immediately claims that "temperature has to be held constant," but then stops to think more. Blair points to the boxed legend and traces some of the lines, appearing to really notice the full contour lines (not just the bold portions) for the first time. Blair spends several minutes interpreting what each contour

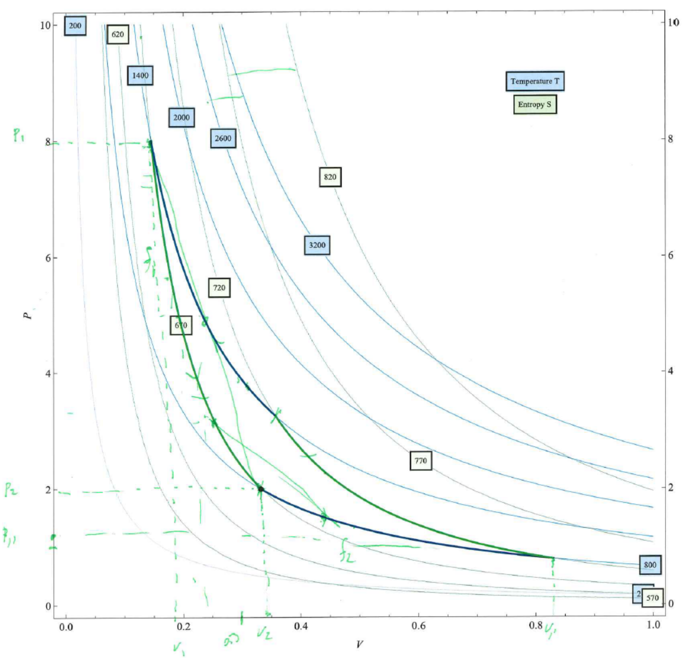


FIG. 13: Blair's work during Phase 2.

line represents (*i.e.,* what is actually constant along each line), as shown below, with sketches at the top of Fig. 13.

Blair: "So I guess I'm trying to figure out, does that mean [entropy] is constant or is it changing? But since it's graphed and that is entropy being graphed I'd imagine it has to be changing, so it has to mean that temperature is constant, right? Yeah cause these [points to successive green lines] have to be them changing. So because entropy see here to here [draws a line from green curve to green curve] is a change, and then here to here [draws a line from blue curve to blue curve] has to be a change in temperature. Oh, so if it's on the line then it has to be held constant. Okay."

Int.: "Can you say that again?"

Blair: "So if it's on the line it has to be held constant. Because if these are two entropy lines [points to green curves] that means here to here [along a blue line] is a change in entropy, and if these are two temperature lines [points to blue curves] then here to here [along a green line] has to be a change in temperature. So if it's on the entropy line that means it's on the same entropy and if it's on the temperature line so it has to be the same temperature. So that would mean that when I take these derivatives, so if I'm taking this derivative here [changes one $\frac{\partial p}{\partial V}$ into $\left(\frac{\partial p}{\partial V}\right)_S$] that means entropy is held constant and here [changes the other $\frac{\partial p}{\partial V}$ into $\left(\frac{\partial p}{\partial V}\right)_T$] it was temperature."

Int.: "Okay."

Blair: "But then I'm concerned because now I'm thinking ... my [constant T] derivative isn't accounting for entropy. But then if I'm taking the derivative of pressure to volume assuming entropy is constant, and then the derivative ... I guess accounting for the change in temperature wouldn't have to matter because we're only concerned with how pressure changes with volume."

This episode shows Blair explicitly recognizing two important features of the given contour graph: (1) that the temperature is constant along any given blue curve, and (2) that the entropy is *not constant* but is *changing* along any given blue curve. In general, this is a feature of thermodynamics related to the number of independent variables that a system has [61, 68], and is taught as part of the *Energy and Entropy* course taken by most of the interviewees (including Blair). However, the dialogue strongly suggests that Blair is discovering this feature of a thermodynamics graph for the first time, implying either that they did not learn it or that it did not make a strong enough impression to be remembered about six months after the end of the course. It is instead possible that Blair understands this idea in a *non-graphical* context and is simply transferring that knowledge to the graphical representation.

Four additional students comment on the number of variables that can change and/or be held constant, typically near the end of Phase 2 when the interviewer is asking them to reflect on specific portions of their work. For example, Mel initially attempts to find a derivative (symbolically) with both T and S constant (see Fig. 12a). Later in Phase 2, Mel reflects on this effort in response to a prompt from the interviewer.

Mel: "I don't think that derivative has entropy constant."

Int.: "The one that you found earlier?"

Mel: "Yeah."

Int.: "Why not?"

Mel: "Well yeah because at constant entropy T is changing, or T can change."

Int.: "How do you know?"

Mel: "I guess based off of this relation [$S = q\Delta t$ in Fig. 12b] because if entropy is a constant then, if the temperature increased by some amount the heat lost or gained would just decrease by some amount to compensate for that."

At first, Mel relies on a symbolic equation to argue that temperature and entropy cannot both be held constant. The interviewer encourages Mel to use the graph to justify why there is no derivative with both S and T constant, eventually leading to the following conclusion about the derivative:

Mel: "But then I guess what that's telling me is I have to follow a line tangent to both the constant temperature and constant entropy lines at that point and I guess seeing that these two [contour lines] have different slopes at that point tells me that they aren't, there isn't a line tangent to both of them."

Pat makes an observation about the interdependency of variables in thermodynamics when the interviewer asks about Pat having labeled the partial derivatives with subscripts.

Int.: "Was it important to label those derivatives as having something constant?"

Pat: "Yeah, because they have different physical meanings. Entropy and temperature aren't the same thing. So in this system temperature's probably changing specifically to keep the entropy constant because you can't just change the volume and only get a change in pressure in thermodynamics because everything's dependent on everything else in the system."

Int.: "Okay. Can you say more about that?"

Pat: "Um [pause] thermodynamics is complicated. [Pat and the interviewer laugh.]"

Int.: "Is there something on the graph that tells you that that's happening?"

Pat: "Um well [pause] the curves are different for constant T and constant S. I guess the biggest indicator of that would be that there's different curves, like different isotherms for different temperatures. So if you have the temperature held at 3200 degrees then that pressure *vs.* volume graph is going to be different than if you held it at 1400. So there's other dependencies that aren't listed when you just plot pressure and volume [points at axes labels]. You have to have specific other conditions in order to say definitively this is what the curve looks like."

Pat describes the same feature of the system as Blair and Mel—that moving along a contour line with one variable (entropy) held constant necessitates a change in the other free variable (temperature). Pat's reasoning for this claim is first general (all thermodynamic variables depend on each other) and then specific to the given context (the contour lines are different).

All five of the students who discuss how many variables can change appear to be discovering or rediscovering this feature of thermodynamics, rather than repeating something they already know. Each also requires some prompting from the interviewer to think about what variables can be held constant and/or how each variable is held constant on the contour graph. This suggests that interpreting and understanding graphs of relationships between thermodynamic variables (like the provided contour graph) is an involved process, especially given that the relationships between those variables are often nontrivial.

One additional student, Chris, initially uses a line "tangent" to the bold curve to approximate a single value for the derivative. (Chris is one of the students who completed a physical chemistry course instead of *Energy and Entropy*.) Chris eventually notices the S and T labels and the different curves, but ends up concluding that the derivatives are the same because the curves intersect. This response seems analogous to a previously identified student difficulty conflating the value of a function and the value of its derivative at a point [45].

Chris eventually claims that their derivative is with both S and T held constant. The interviewer asks if it is possible to find a derivative with only one of those variables held constant, leading to the following statement:

Chris: "I don't think so. Because I know in thermodynamics you have to set things equal to a constant to solve for you know V or p or yeah. So I think you have to set both of them constant."

Unlike the five students discussed above, Chris does not appear to recognize the ways in which thermodynamic variables are interrelated, even by the end of the interview. This suggests that this feature of thermodynamics is not necessarily an easy one to identify, especially when other features of thermodynamics graphs, notation, or terminology are distracting or confusing.

VI. DISCUSSION AND IMPLICATIONS

We observe several patterns in the results described in sections IV and V. Below, we focus on those patterns that are directly related to our research questions: what representational features do students attend to, how do students narrow to a path, and do students narrow intentionally? We also discuss possible implications for upper-division physics instruction indicated by each pattern.

A. Patterns in the representational features students observed when orienting to the contour graphs

There are some similarities in how students approached solving the tasks given during each phase of the interviews, especially at the beginning of each phase, and in the ways that students used and interacted with the various verbal, symbolic, and graphical representations available to them. Essentially all students began by processing the task and the representation given to them. It was especially common for students to attend to the labels on the horizontal and vertical axes.

However, when students referred to the contour lines, to the label V, or to the boxed numbers indicating values of V, they often phrased their identification as a question or in a manner that seemed to be seeking confirmation of their interpretation, as in the example below.

Sam: "I'm guessing the boxed numbers refer to the potential along these lines."

Students' lack of comfort with the numerical labeling of contour graphs aligns with previous research we have done regarding student sensemaking about equipotential graphs [69].

In Phase 2, students who commented on the two sets of contour lines, indicated by different colors, were often uncertain about what the lines represent. Two students did not explicitly identify the box labeling the curves as temperature and entropy until ten or more minutes into Phase 2. Some students misremembered the proper thermodynamic names for the curves—for example, Ira mislabeled an isotherm (*i.e.,* a temperature contour) as "adiabatic." Similarly, four students did not reference the numbered values labeling individual contours during Phase 2, although these numbers are not necessary to solve the given task.

This pattern suggests that while the students may have developed some expert-like behavior for identifying the features of traditional graphs (such as horizontal and vertical labels), certain features of the contour graphs (especially the labels and colors on the thermodynamics graph) were not as salient to the students. The fact that many students did not appear to attend to the non-axis variables (*e.g.,* temperature and entropy) may partially explain why students initially attempted to find a derivative without any variable held constant.

Since our population of students completed a junior-level thermodynamics course, we speculate that students who have not yet taken thermodynamics are even more likely to have trouble interpreting and using contour-like graphs. It therefore strikes us as worthwhile for instructors to devote time at the beginning of thermodynamics courses to allowing students to investigate the features of such graphs, and the features of thermodynamics that such graphs reveal. The realization that, for example, the entropy changes when moving along a temperature contour, was a particularly powerful moment that would be valuable for students to experience in the classroom.

B. Patterns in how students narrowed to find a derivative

We find that most students struggled less with the procedure of calculating derivatives as ratios of small changes and more with understanding what derivatives it is possible to find from a contour graph. This is somewhat consistent with prior research that has found students to be less proficient at understanding partial derivatives that at calculating them symbolically [1–19]. Our results indicate that physics students remain capable of approximating derivatives numerically but that *understanding* those derivatives is challenging.

In Phase 1 all but one of the students approximated the derivative as a ratio of small changes, $\frac{\Delta V}{\Delta y}$, using values of V and y for two points at or near the indicated point. A few students initially pursued an alternate strategy, either narrowing to one of the contour lines or thinking about the gradient, but they all then spontaneously shifted to the strategy outlined above after failing to make progress using an alternate method. About half the students used a phrase like "holding x constant" to describe their procedure for finding the derivative.

Students' initial procedures for finding the thermodynamics derivative involved narrowing to the bold path. This may not be surprising, as the pV-curve for a cycle is a particularly common representation in thermodynamics courses, including *Energy and Entropy*, and the cycle was bolded in the given graph and highlighted in the verbal prompt. Several students attempted to find a single derivative at the indicated point, with most concluding that no single derivative exists because there is a "cusp" in the bold function at that point.

Even when students found two distinct derivatives (usually at the interviewer's suggestion), they often stopped at that point without further interpreting or interrogating their answers independently. Only Sam spontaneously realized that the two derivatives (one for the blue curve and one for the green curve) can be reified into partial derivatives with the corresponding variables held *constant*. The other students were uncertain about how to proceed. Only after the interviewer asked a question such as "Did you think about holding anything constant?" did some students proceed to have the same realization as Sam. For one student, this realization took many more questions from the interviewer, and it did not occur at all for two students even by the end of the interview.

Even after interpreting the two numbers as partial derivatives and labeling them with subscripts, three students attempted to combine their answers in some way in an attempt to find a single derivative at the indicated point. Their attempts to continue may have been due to the fact that the prompt directed students to "determine *the* derivative" combined with a desire to please the interviewer. Nevertheless, that we were able to follow their subsequent behavior is one advantage of our decision to use a prompt asking for an ambiguous derivative. For example, Lee wrote a sum of the two partial derivatives while thinking about a total derivative. When asked what it represents, Lee made the following claim:

Lee: "I don't think that represents the derivative here. I'm not sure. I'd put that on a test as my answer, cause I don't know what else it would be."

Lee was uncertain, but did not really appear to think the total derivative can be written as a sum. However, Lee did still seem to think that there might be an answer (that could be put on a test) but did not know what that answer might be.

Like Lee, many students were troubled by the thermodynamics derivatives: for example, some failed to find two separate derivatives, some did not recognize that there are two separate tangent lines at the point, and most did not spontaneously associate the idea of a "constant" variable with these derivatives. To help students develop stronger connections between derivatives concepts, we suggest that instructors of thermodynamics might leverage students' understandings of derivatives in other physical contexts like electromagnetism. Similarly, instructors of electromagnetism might foreshadow ideas that become important in thermodynamics. For example, being more explicit about what is held constant in circumstances where it seems obvious (as with $\partial V/\partial y$), and possibly using subscript notation, may help students transfer knowledge from one context to another.

It is important, however, to note that some thermodynamics courses are taken by students who have not completed electromagnetism. For such courses, we suggest that connections to derivative concepts from multivariable calculus or introductory physics may still be valuable, especially if students are asked to re-explore fundamental derivatives concepts prior to using them in rich thermodynamic contexts. These students might then be able to use what they learn in thermodynamics to help understand derivatives when they get to electromagnetism!

C. Patterns in representational features associated with partial derivatives

There is a particularly interesting pattern regarding the subscript notation for partial derivatives, which is commonly introduced in thermodynamics (and does not tend to be present in mathematics courses or even in physics courses other than thermodynamics). No students used this notation in Phase 1 of the interviews, or in the beginning of Phase 2. Interestingly, students who attempted to use thermodynamic equations to solve the problem did use this notation—but only after they switched to manipulating equations symbolically, and not for the original derivative they were tasked with finding. It was only after each student recognized that they might think about partial derivatives with temperature or entropy held constant that the student introduced the subscript notation to the symbolic representations they chose to use.

Students' use of subscript notation is of profound interest to thermodynamics instructors, since it is the standard expert way of denoting what is held constant, and yet the students essentially did not invoke it until after solving the problem. This indicates that the students had not internally adopted this notation in the same way that experts have. Students knew the notation, but it did not appear to be connected to their other understandings of derivatives. It is interesting to note that several students made use of the ∂ symbol at some point during the interview, suggesting that this symbol has been more internalized than subscripts. However, using ∂'s to write a derivative did not spur the students to think about holding a variable constant, indicating that this symbol may have a richer meaning to experts than it does to students.

The interviewer intentionally did not ask students to find a derivative with something held constant in the initial prompts. In Phase 1, four students referred to a derivative as "with x held constant." In addition, Pat and Lee used "constant" to refer to the contour lines, but not to the derivative or to x or y. In Phase 2, only Blair and Mel used the word "constant" in reference to the contour lines within five minutes of receiving the thermodynamics graph, though Pat did use the term "isotherm." Only Sam referred to a derivative as holding S or T constant in phase 2 without intervention from the interviewer.

Again, this finding has possible implications for instructors in upper-level physics courses. While these students were clearly familiar with the use of "constant" to describe both partial derivatives and certain features of representations such as contour graphs, it is not necessarily a term that they used spontaneously, or that was strongly linked to other understandings of derivatives. This pattern appears to hold across the verbal, symbolic, and graphical representations—within each representation, the prominent features used to indicate that a variable is constant (for a derivative) were not cued by a prompt that asks (only) for a derivative.

We suggest that students should be asked to think about partial derivatives using a wide variety of representations, and that instruction should explicitly aim at helping students make connections—and then deepen those connections—between the language, notation, and ideas related to derivatives in each of those representations. This echoes calls from the mathematics education research community supporting the use of multiple representations in the classroom [70] and aligns with our findings in Section IV C that some students introduced new representations to support and justify their answers. It also agrees with prior research at OSU on students' ability to transfer knowledge between contexts and representations in thermodynamics [71]. In addition to ordinary and contour graphs, we have found that representations such as the Partial Derivative Machine [25] and three-dimensional plastic surfaces [56] are pedagogically useful for making and deepening connections because they are tangible, manipulatable, and memorable.

D. Patterns in whether or not students narrowed intentionally

Students identified the paths to which they narrowed in different ways in response to each of the two prompts. In Phase 1, students typically narrowed to a vertical path between V and y. However, most students did not specifically talk about this path as a relationship between variables. Instead they tended to draw something vertical to indicate the range of points they were considering, and about half the students mentioned that they were holding x constant. There is a sense in which this is intentional (they mean to narrow) but not explicit (they are not focusing on the path as a relationship). The strongest instances of explicit and intentional narrowing were when students talked about a cross-section of the graph, in some cases even making a sketch of a generic function or the given graph, and in other cases making use of the plastic surface to demonstrate their meaning (Section IV C).

In Phase 2, on the other hand, almost all students initially narrowed to the bold path. (It is worthwhile to note that the paths in the thermodynamics graph are contour lines, which are represented by actual lines on the page, while the paths in the electrostatics graph are the implicit grid lines.) In this case, students did tend to view the bold path as a new relationship, either labeling it symbolically or describing how a system would proceed physically along the path. Students paid particular attention to the "cusp" and its implications for the derivative. Students' attention to the bold path as its own relationship was clearly explicit, though it may not have been intentionally chosen, as the interview prompt highlighted this combined path. It is also the case that the bold path is not a straightforward relationship between variables; it cannot be nicely summarized as holding a single variable constant in the way that the blue and green paths can be. Students' attention to those separate paths was typically minimal, at first. After the interviewer asks each student to think about holding something constant, however, students tended to identify the individual contour lines as distinct relationships.

The results indicate that students are capable of narrowing in different ways but that they need more practice recognizing there is more than one way to narrow, and that narrowing differently will produce a different derivative. We suggest introducing language (like "narrowing") that describes this process in detail for a wide variety of representations. Such language would give students a way to unify the ways in which different representations express related ideas, namely: the words "holding a variable constant," the subscript on a partial derivative, and the meaning of a contour line.

E. Additional implications for instruction

It was encouraging to see that prompts from the interviewer were often sufficient to shift the paths that students narrowed to dramatically. The prompts consisted of some variant of "Did you think about holding anything constant?" After such an intervention, most students were able both to solve the problem and to make sense of it—and they proceeded to use different language (*e.g.,* "partial" derivatives with some variable "held constant") and different notation (*e.g.,* subscripts).

Furthermore, some students appeared to build new understanding or observed new features of the given thermodynamics graph (that they did not appear to have considered prior to the interviews). For example, Blair was initially troubled (see Section V C) by the fact that temperature changes when calculating $\left(\frac{\partial p}{\partial V}\right)_S$. After considering which changes are relevant to actually calculating that derivative, Blair concluded that only "how pressure changes with volume" matters—the fact that T is changing is already accounted for by the fact that S is constant. This is particularly encouraging because several students made explicit comments during the interviews agreeing with the sentiment of Pat's repeated claim that "thermodynamics is complicated." Thermodynamics *is* complicated—but students were able to make sense of why it is complicated when asked to consider the different thermodynamic variables together, along with how those variables change. Classroom activities that ask students to work with the broad set of relevant thermodynamic variables, to reason about those variables graphically, and to consider explicitly how variables change with respect to one another may prove particularly beneficial to building foundational understanding in thermodynamics. We have found that contour graphs are a particularly useful pedagogical representation because they encompass each of these three features.

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FIG. 14: Full-sized electrostatics graph

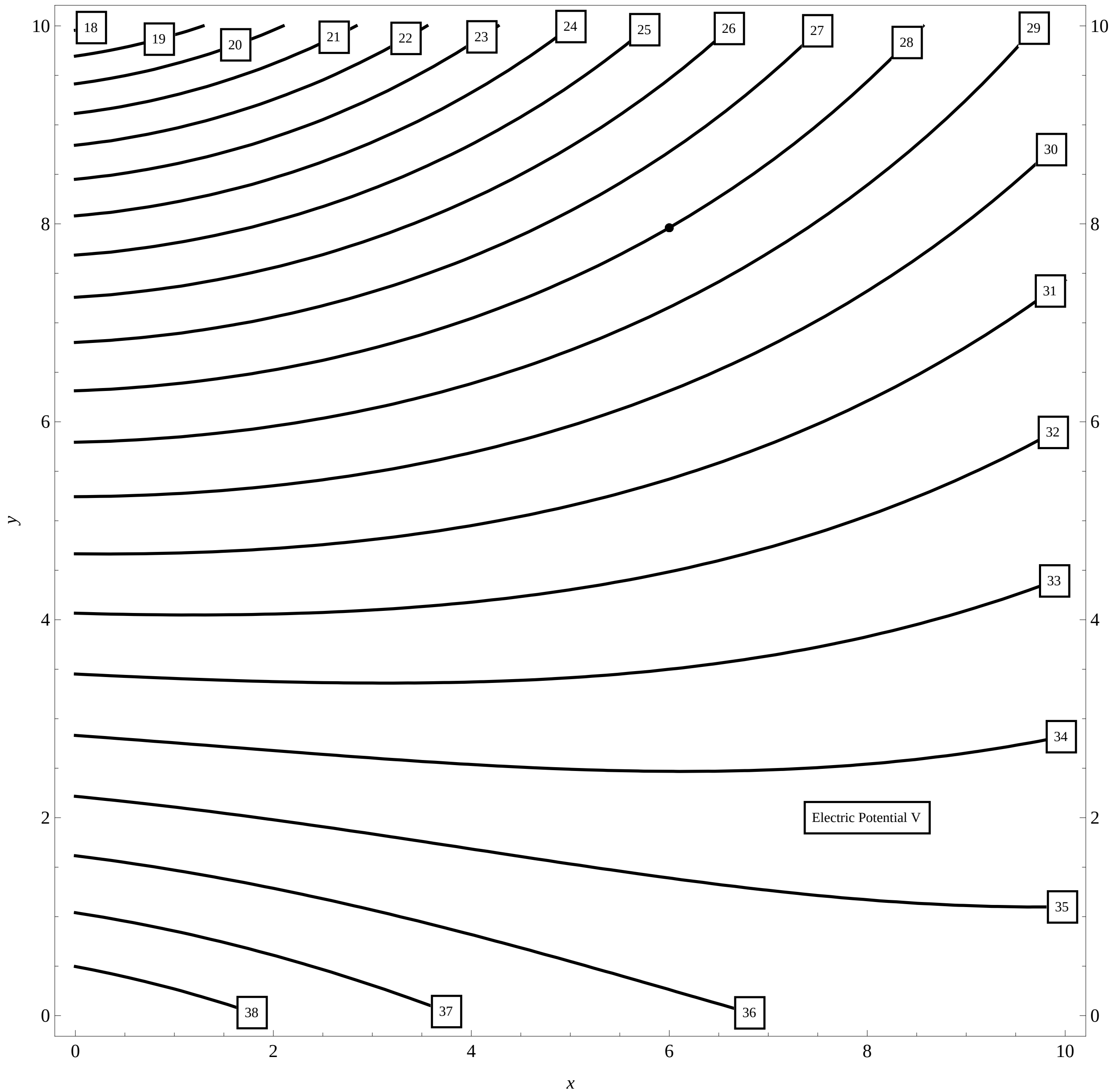

FIG. 15: Full-sized thermodynamics graph