Electrostatic Potential Due to a Ring of Charge (Code:1D)

The problem I was asked to solve was to find the electrostatic potential due to a ring of charge. I was told that the ring had a radius R and a total charge Q. In order to solve this problem I started with the general equation:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r_i}|} \tag{1}$$

Where q_i is the individual charge, $|\vec{r}-\vec{r_i}|$ is the distance between the point we are measuring the potential at ($\vec{r}$) and the charge ($\vec{r_i}$); ϵ_0 is the permittivity of free space.

Dr. Manogue gave us the next equation which was V for a linear charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\vec{r'})|d\vec{r'}|}{|\vec{r}-\vec{r'}|} \tag{2}$$

Where $\vec{r'}$ is the position of the piece of charge, and $|d\vec{r'}|$ is the little distance used to integrate around the ring.I also knew the charge distribution was constant, so I had:

$$\lambda = \frac{Q}{2\pi R} \tag{3}$$

After plugging this into Eqn (2) I had:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi R}\int\frac{|d\vec{r'}|}{|\vec{r}-\vec{r'}|} \tag{4}$$

I used cylindrical coordinates because of the geometry of the ring. In this system $|d\vec{r'}|$ becomes $Rd\phi'$ and the limits of integration then become $[0, 2\pi]$. Applying this to Eqn (4) yeilds:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{|\vec{r}-\vec{r'}|} \tag{5}$$

Because $\vec{r}$ and $\vec{r'}$ won't always point in the same direction, I needed to write them out explicitly. Using the solution from our homework assignment to write out $|\vec{r}-\vec{r'}|$ in cartesian coordinates converted to polar components I had:

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{(r^2+R^2+z^2-2rR\cos(\phi-\phi')}} \tag{6}$$

This is an elliptic integral that can be evaluated numerically with computer software. I was then asked to find an expression for V along the z-axis. This makes r equal to 0 and Eqn (7) becomes:

$$V(r{=}0,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{R^2+z^2}} \tag{7}$$

This is easily integrable to give:

$$V(r{=}0,\phi,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2+z^2}} \tag{8}$$

I was then prompted to expand this in a power series to approximate V at points very close to zero. After recognizing that I needed to use the power series

$$(1+c)^p = 1 + pc + \frac{p(p-1)}{2!}c^2 + ... \tag{9}$$

I factored out an R from the denominator so that $c << 1$. I then had:

$$V = \frac{Q}{4\pi\epsilon_0 R}\left(1+\frac{z^2}{R^2}\right)^{-\frac{1}{2}} \tag{10}$$

Using Eqn (10) and recognizing that $p = -\frac{1}{2}$ and $c = \frac{z^2}{R^2}$, I obtained the following:

$$V(z) = \frac{Q}{4\pi\epsilon_0 R}\left(1 - \frac{z^2}{2R^2} + \frac{3z^4}{8R^4} + ...\right) \tag{11}$$

I learned that applying information to get the equation you want is really hard, and that you have to know a lot of tricks or else you will get stuck along the way. I discovered that working in a group can also make things a lot easier, because up until this assignment I didn't have much difficulty with our group activities.