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Bridging the Gap between Mathematics and the Physical Sciences

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Abstract

What makes mathematics different from other sciences? Mathematics may be the universal language of science, but other scientists speak a different dialect. We discuss here some of what we have learned from the NSF-funded *Vector Calculus Bridge Project*, an effort to bridge this language gap at the level of second-year calculus, and then suggest implications for the teaching of mathematics at all levels.

1 Introduction

We are a husband-and-wife team of mathematical physicists; we've done research together ever since we met. While our working styles are quite different, we have successfully collaborated on numerous joint projects, including seeking external funding for our research, and writing up the results for publication.

Far and away the hardest thing we have ever done is to teach together. We recently taught multivariable calculus jointly, and spent nearly every evening reacting in disbelief to what the other planned to say in the classroom the next day.

We now realize that the way mathematicians view and teach mathematics, and the way mathematics is used by physicists and other scientists, are completely different; we speak different languages, or at least different dialects.

While our own primary interest is the gap between lower-division mathematics and upper-division physics courses at the college level, we believe much of what we have learned from each other is applicable in other contexts. We begin with an example, then briefly describe our own work, and finally offer some suggestions for improving communication. Along the way, we will raise three main issues: that context is everything, that units matter, and that geometric reasoning is important.

2 An Example

Here's our favorite example:

Suppose $T(x, y) = k(x^2 + y^2)$. *What is* $T(r, \theta)$*?*

We often ask this question of mathematicians and other scientists. Some mathematicians say "$k(r^2 + \theta^2)$". Many mathematicians refuse to answer, claiming that the question is ambiguous. Everyone else, including some mathematicians, says "kr^2". One colleague, who holds a split appointment in mathematics and physics, simply laughed, then asked which hat he should wear when answering the question. What's going on here?

Just as x and y have standard meanings as rectangular coordinates, r and θ are the standard labels for polar coordinates, with r denoting the distance from the origin and θ the angle counterclockwise from the positive x-axis. So $r^2 = x^2 + y^2$, and if you express $k(x^2 + y^2)$ in polar coordinates you get kr^2.

But wait a minute; that wasn't the question! $T(x, y)$ is a function of two variables, x and y. It doesn't matter what you call them; r and θ are as good as any other names. So replace x by r and y by θ; the answer is clearly $k(r^2 + \theta^2)$.

This is of course exactly what mathematicians teach their students about functions, so it is especially noteworthy that many mathematicians nonetheless give the polar coordinate answer.

What is the point? That the mathematics we teach tends to be about formal manipulation of symbols according to well-defined rules, whereas the mathematics we use always has a context. In this example, many mathematicians recognize the context and use this additional information when answering the question. Nonmathematicians have to do this with every problem, but this skill is rarely taught. Students often express their inability to exploit the context with the words, "I just don't know how to get started."

And yes, a physicist really will write $T(x, y) = k(x^2 + y^2)$ for, say, the temperature on a rectangular metal slab, and $T(r, \theta) = kr^2$ for the same temperature in polar coordinates, even though the mathematician would argue that the symbol T is being used for two different functions. This is not sloppy mathematics on the part of the physicist; it's a different language. T is the temperature, a physical quantity which is a function of position; the letters which follow merely indicate which coordinate system one is using to label the position. This can be rigorously translated into the differential geometer's notion of a scalar field, or phrased more informally as:

Science is about physical quantities, not about functions.

So not only do other scientists speak a different language, they use the same vocabulary! [1]

[1] Ask a mathematician and a physicist to compare notes on the conventions they use for spherical coordinates [1]. Stand back!

3 Units

What is k in the preceding example, and why is it there?

To take an extreme example, scientists will rightfully balk at an equation like $y = x^2$. Do you see why? Try to provide a physical context for this equation. A typical answer would be that if x is the length of one side of a square, the y is the area of the square. But x and y are normally used to denote lengths, and it makes no sense to compare meters with square meters!

In practice this particular example doesn't actually cause much trouble, although we believe this is because most scientists are bilingual, having first seen such expressions in their math classes. But there is an important issue of principle here:

Physical quantities have dimensions.

This gives the scientist a tool unfamiliar to many mathematicians: equations can be checked for reasonableness by seeing whether both sides have the same dimensions. For instance, students often think that angles measure distance, even though the units aren't right. We like to ask our students, "What sort of a beast is it?"

Units are especially important in power series expansions. It doesn't make sense to expand in x, or t, since those variables are not usually dimensionless. In physics, one expands in kx, or ωt, where the constants k and ω have appropriate units. This also means that $\sin x$ doesn't make sense, nor e^x. Why do the mathematicians take out all the constants? They really do matter, and hiding them now makes life difficult for many students later.

Here is a classic example of this confusion. [2] Physically measure the area of a rectangle by placing unit squares inside it, then counting the number of squares used. Looking for a followup activity? The same squares might be used as rulers to measure the perimeter of the rectangle, by placing them around the outside. But many students will get the wrong answer, since they add four extra squares to fill in the corners! They are trying to use area to measure length. Instead, the students should use unit rods, perhaps toothpicks, to measure perimeter. In general, measurement involves counting the number of standard units that "fit" inside the thing being measured; the standard unit always has the same dimensions as the thing being measured.

Yes, there is value in extracting and analyzing the mathematical content of physical examples. But this can be carried too far. We don't insist that every problem have a physical context, but it should be possible to add one. Our rule of thumb is that if we can't easily (in principle) assign units to the symbols in a problem, we don't use it.

[2]This actually happened in our son's elementary-school classroom.

4 The Bridge Project

We started the *Vector Calculus Bridge Project* [2] in order to bridge the language gap at the level of second-year calculus [3]. The creation of this project was strongly influenced by our work with the *Paradigms in Physics Project* [4], a major NSF-supported reform of the upper-division physics major, which in turn was motivated in part by trying to help students make the transition from lower-division mathematics to upper-division physics.

We initially compared syllabi between a vector calculus class and the junior-level physics class which “reviews” similar material. After getting nowhere — the list of math topics was nearly identical to the list of physics topics — we finally compared the actual content. We eventually realized that we could sum up the differences in a single sentence:

Mathematicians teach algebra; physicists do geometry.

Physicists (and other scientists) tend to reason geometrically, rather than (or, more precisely, in addition to) “mere” symbol pushing. The temperature is a physical quantity, whose representation as a function is secondary to the fact that the temperature here is, say, 70°, while over there it might be 75°.

Returning to the example, the physicist “sees” the temperature as living on the (2-dimensional) metal slab itself, perhaps by associating a color with different temperature ranges, or equivalently in terms of isothermal contour lines — something like a topographic map. This is very different from the graph of the corresponding function, which in this case is a (3-dimensional) paraboloid. This difference in viewpoint is especially important since physical quantities usually depend on three spatial variables; the corresponding graphs would therefore require four dimensions.

By emphasizing geometry, we have been able to unify our traditional vector calculus class around a single idea (the infinitesimal displacement vector; see [5]). Our other main ingredient is the use of small group activities with open-ended problems, similar but not identical to the MathExcel [6] and Peer Led Team Learning [7] programs. Not only do we now cover more material in more depth than before, but students seem to be coming away with a deeper (and yes, more geometric) understanding.

Both the Bridge and Paradigms Projects offer faculty workshops on the use of their materials, and the reasons for developing them. Further details are available on their respective websites [2, 4].

5 Suggestions

What can be done to better prepare students for a career in science? Emphasize geometric reasoning! This is not a plea for a better version of a traditional course in Euclidean geometry, but rather a plea for a greater emphasis on geometry in *all* mathematics courses. Students need to know how to calculate, but they also need to know just what they are calculating. Encouraging geometric understanding is an excellent way to develop this ability.

A significant step in the right direction would be the use of appropriate units, which helps establish a context for the mathematics. Context is everything! For further steps along these lines, we heartily recommend the notion of Context Rich Problems [8], developed at the University of Minnesota for physics courses, but easily adaptable to other settings.

One final remark: Don't feel as though you suddenly need to teach the underlying science in order to use interesting applications. Simple examples, using temperature, for instance, are fine. Scientists dont want mathematicians teaching science anymore than mathematicians want scientists teaching mathematics. The goal is not to change fields, but to improve communication and understanding.

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[7] Peer Led Team Learning, `http://www.pltl.org`.

[8] Context Rich Problems, `http://groups.physics.umn.edu/physed/Research/CRP/crintro.html`