

BRIDGING THE GAP BETWEEN MATHEMATICS AND PHYSICS

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Ask physicists to write down the magnetic field around a current-carrying wire, and the response may well be

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta}.$$

Ask them where they learned about $\hat{\theta}$, and they'll most likely say, "In a math class." Yet most mathematicians have never heard of $\hat{\theta}$! So ask your mathematician colleagues, "Don't you teach students about spherical coordinates?" "Yes," they'll respond. But watch out; to a mathematician, working in spherical coordinates means just that—using the coordinates r , θ , and ϕ (which are called ρ , ϕ , and θ), but still using rectangular basis vectors \hat{i} , \hat{j} , and \hat{k} .

Here's another example of the disconnect between the language of a mathematician and that of a physicist. Suppose that

$$T(x, y) = k(x^2 + y^2).$$

What is $T(r, \theta)$? Most physicists say $T(r, \theta) = kr^2$; many mathematicians would argue instead that $T(r, \theta) = k(r^2 + \theta^2)$. Physicists are thinking of some physical quantity T , perhaps the temperature on a tabletop. Writing $T = T(x, y)$ refers to this temperature in rectangular coordinates, whereas $T = T(r, \theta)$ is the temperature in polar coordinates. Mathematicians have great difficulty taking this seriously, since T is being used as the name for two different functions. Instead, they would write $T = f(x, y)$, but $T = g(r, \theta)$, which physicists have great difficulty taking seriously, since it's the same temperature.

These two examples illustrate some of the pitfalls involved when mathematicians and physicists try to communicate. We speak different languages—but the basic vocabulary is the same! The first step in learning to communicate is for both sides to acknowledge that the languages are indeed different. The mathematician's claim that physicists are sloppy is no more true than the physicist's claim that mathematicians split hairs. Mathematical precision is important, especially in the absence of a physical context, but physics always has such a context.

At Oregon State University, the mathematics and physics departments are working to bridge this gap. Our first goal has been to revise the vector calculus course taught by the mathematics department. The material in this course is important for physicists, yet the math language is so different from that used in the standard physics applications that students are often unable to make the connection.

With support from the National Science Foundation, we have developed a series of guided group activities emphasizing the geometry of vector calculus, and an instructors' guide to accompany them. The materials discussed here have been used primarily in second-year calculus courses, at Oregon State University and elsewhere. We are actively developing similar materials for use in appropriate physics courses, especially mathematical methods courses or upper-division electricity and magnetism courses—look for an update!

In the process of developing these materials, we have concluded that there are two essential differences between mathematicians and physicists:

- Physics is about things;
- Physicists can't change the problem.

We discuss each of these in turn.

Physics is About Things

We like to ask students, "What sort of a beast is it?" Vector or scalar? Large or small? What are the units? This is not only a good way to quickly check the reasonableness of an equation, but also emphasizes the context. Mathematicians tend to ignore things like the "obvious" units; you can't equate a length to a squared length. Similarly, the argument of a trigonometric or exponential function, and the parameter in a power series expansion, must all be dimensionless. Putting in some extra constants is a small price to pay to get this idea across to students.

The question of units shows up again when dealing with graphs. Graphs are about the relationships between physical quantities. This means that hills are not the best examples of functions of two variables, since in this case the units for the domain and range are, atypically, the same. More importantly, physics is three-dimensional. It's hard to apply the intuition developed by graphing functions of two variables to a problem involving, say, the temperature in a room. Hence, we believe that more time should be spent on alternative ways of conveying the same information, such as the use of color, or contour diagrams, both of which do generalize to three dimensions.

Furthermore, few physics problems come neatly packaged with a coordinate system, since the physical world is independent of coordinates. It is therefore crucial to treat vectors as arrows in space, not just triples of numbers, and equally important to emphasize the geometric interpretation of the dot and cross products, not just how to compute them. In addition, physics tends to be highly symmetric. Paraboloids are the favorite surface in a vector calculus class, but how many paraboloids are there in physics? Interesting physics problems often involve elementary math. It is at least as important to understand the simple examples as it is to know how to generalize them. It is the desire to exploit symmetry that leads physicists to use adapted basis vectors such as $\hat{\theta}$, a skill mathematicians neglect in favor of more generality—those paraboloids again. In fact, we have found that the paraboloid is better handled in cylindrical coordinates!

Physicists Can't Change the Problem

Mathematics tries to chop learning up into neat packages, identifying each skill and refining it as far as possible. Physics involves the creative synthesis of multiple ideas. The problem drives the methods, not vice versa. In short, physics problems don't fit templates, so skill at solving template problems is not enough.

This can make it hard to get started. Physics problems are not usually as well-defined as math problems. There may be no preferred coordinates or independent variables, and certainly no parameterization of curves or surfaces. Unknowns don't have names. Getting to a well-defined math problem is often the hardest part of a physics problem.

Rather than a plethora of formulas for different cases, physicists need a few key ideas that will be remembered later on. The traditional vector calculus course is crammed full of formulas, most usually forgotten after the exam. We have instead built the entire course around a single idea, that of the infinitesimal vector displacement between points. By emphasizing the unity of the subject, we provide students something they may actually remember years later. Our favorite student complaint is that there doesn't seem to be enough material for an exam—we've made things too easy.

Here is a problem that illustrates some of these ideas. A helix with 17 turns has height H and radius R . Charge is distributed on the helix so that the linear charge density increases like the square of the distance up the helix. At the bottom of the helix the linear charge density is 0 Coulombs/meter. At the top of the helix, the linear charge density is 13 Coulombs/meter. What is the total charge on the helix?

We give this problem to students who have just learned about line integrals. They hate it. First of all, they don't know what "increases like" means. Second, they don't see the linear relationship between the polar angle and the height. And third, they're not comfortable finding arclength in cylindrical coordinates. None of these issues would arise in a traditional math class—the first two, because they're associated with setting up the problem, not solving it, and the last one because cylindrical coordinates are not emphasized in mathematics courses so the students are unlikely to see the problem at all.

The bottom line is that physicists tend to think geometrically, but lower-division mathematics classes have become increasingly algebraic. Perhaps the single most important goal of our work is to improve students' geometric visualization skills, thus helping to bridge the gap.

We offer workshops using materials inspired by these ideas. As of May 5, 2004, space was still available for summer workshops in Corvallis, Oregon for both the Bridge and Paradigms projects. Further information about the projects and workshops, including copies of papers and talks, can be found at <http://www.math.oregonstate.edu/bridge> and <http://www.physics.oregonstate.edu/paradigms>, respectively. These projects are supported by NSF grants DUE-0231032 and DUE-0231194.

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