

1 Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis

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Conceptual and procedural knowledge of mathematics represents a distinction that has received a great deal of discussion and debate through the years. Questions of how students learn mathematics, and especially how they should be taught, turn on speculations about which type of knowledge is more important or what might be an appropriate balance between them. Additionally, discussions of conceptual and procedural knowledge extend beyond the boundaries of mathematics education. The distinction between concepts and procedures plays an important role in more general questions of knowledge acquisition. In some theories of learning and development, the distinction occupies center stage. Although the types of knowledge that are identified from theory to theory are not identical, there is much overlap. The differences are primarily in emphasis rather than kind. For example, Piaget (1978) distinguishes between conceptual understanding and successful action; Tulving (1983) distinguishes between semantic memory and episodic memory; Anderson (1983) distinguishes between declarative and procedural knowledge. Parallel distinctions are made in philosophical theories of knowledge. For example, Scheffler (1965) distinguishes between the propositional use of "knowing that" and the procedural use of "knowing how to." The distinction between conceptual and procedural knowledge that we elaborate in this chapter is not synonymous with any of these distinctions, but it draws upon all of them.

Mathematics, with its tightly structured and clearly defined content, has provided an arena for much discussion of conceptual and procedural knowledge. Over the past century, considerations of these two kinds of mathematical knowledge have taken different forms using different labels. Probably the most widely recognized distinction has been that between skill and understanding. Often the discussions of skill and understanding have taken the form of a debate about which should receive greater emphasis during instruction. McLellan and Dewey

(1895) argued for understanding and presented a mathematics curriculum they felt would raise the level of understanding beyond that existing in classrooms at the time. Thorndike (1922) presented the case for skill learning and described in detail how skills should be taught to maximize retention. Brownell (1935) opposed the emphasis on learning isolated skills and argued forcefully for an increased emphasis on understanding. Essentially the same debate was carried on periodically through Gagné's (1977) emphasis on skill learning versus Bruner's (1960) case for understanding. Along the way there have been many additional voices addressing the issue from a variety of perspectives (see Jones, 1970; Shulman, 1970).

Currently, cognitive psychologists and mathematics educators are looking again at conceptual and procedural knowledge in mathematics learning. Sometimes, the discussions are couched in terms different from those used in the past. For example, Resnick (1982) talks about semantics and syntax, and Gelman and Gallistel (1978) distinguish between principles and skills. Even within this volume, a variety of terms are used to differentiate between types of knowledge. Baroody and Ginsburg describe differences between meaningful and mechanical knowledge, and VanLehn distinguishes between schematic and teleologic knowledge. But, regardless of the labels, the division between types of knowledge lies in approximately the same place today as it has in the past.

There are, however, three important differences between current discussions of conceptual and procedural knowledge and the historic discussions of understanding and skill. First, the essays of the past have treated understandings and skills as instructional outcomes and have dealt with them in the context of advocating instructional programs. The issue has been whether skills, or understandings, or both should be emphasized during classroom instruction. The context for addressing the question of the relative importance of skills and understandings often has been the *prescription* of instructional programs. Today, many of the writings *describe* the acquisition of knowledge and the relationships between different kinds of knowledge. The implicit assumption is that more complete descriptions are a first step on the road to better prescriptions. Detailed descriptions are believed to provide a sound basis from which to develop effective instructional programs.

The second difference between past and current discussions of conceptual and procedural knowledge is found in the current attention to *relationships* between concepts and procedures. Historically, the two kinds of knowledge have been viewed as separate entities, sometimes competing for the teacher's attention, at best coexisting as disjoint neighbors. Little interest has been shown in studying the relationships between the two kinds of knowledge. In contrast, there is a growing interest today in how concepts and procedures are related. Current discussions treat the two forms of knowledge as distinct, but linked in critical, mutually beneficial ways. The new language of cognitive science has facilitated this approach because a single language can now be used to deal with both forms

of knowledge (Anderson, 1983; Davis, 1984; Norman & Rumelhart, 1975). It is no longer the case that different theories are needed to express the principles guiding the acquisition and application of each kind of knowledge; a single, theoretical orientation can handle both conceptual and procedural knowledge.

The third difference between past and present discussions is that past distinctions between conceptual and procedural knowledge focused on mathematics learning in school, whereas recent discussions of the issue have broadened the scope to include preschool mathematics learning in informal settings. Although it has long been recognized that children enter school with significant mathematical competencies (Brownell, 1941; McLaughlin, 1935), it is only recently that these competencies have been analyzed in great detail. Some of the analyses have revealed that the distinction between conceptual and procedural knowledge is as appropriate and useful for understanding the acquisition of informal mathematics as for formal mathematics. Preschool children acquire certain procedures or skills along with concepts, understandings, or intuitions about mathematics. The relationships between these kinds of knowledge, even at this level, appears to be complex.

Although the recent orientation to the issue of conceptual and procedural knowledge promises to provide significant insights into mathematics learning and performance, the relationship between these forms of knowledge are not yet well understood. A primary reason for the intractable nature of the problem is that the types of knowledge themselves are difficult to define. The core of each is easy to describe, but the outside edges are hard to pin down.

Our position is that the distinction between conceptual and procedural knowledge is useful for thinking about mathematics learning, and the clearer we can be about the distinction, the better. We do not believe, however that the distinction provides a classification scheme into which all knowledge can or should be sorted. Not all knowledge can be usefully described as either conceptual or procedural. Some knowledge seems to be a little of both, and some knowledge seems to be neither. Nevertheless, we believe that it is possible to distinguish between the two types of knowledge and that such a distinction provides a way of interpreting the learning process that helps us better understand students' failures and successes.

DEFINITIONS OF CONCEPTUAL AND PROCEDURAL KNOWLEDGE

Conceptual Knowledge

Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of

information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information.

The development of conceptual knowledge is achieved by the construction of relationships between pieces of information. This linking process can occur between two pieces of information that already have been stored in memory or between an existing piece of knowledge and one that is newly learned. It may be helpful to consider each of these phenomena in turn. The literature of psychology and education is filled with accounts of insights gained when previously unrelated items are suddenly seen as related in some way. Such insights are the bases of discovery learning (Bruner, 1961). We characterize this as an increase in conceptual knowledge. Two illuminating accounts of this kind of conceptual knowledge growth in elementary mathematics are found in Ginsburg (1977) and Lawler (1981). Ginsburg describes many points in the learning of number and arithmetic where understanding involves building relationships between existing bits of knowledge. For example, Jane (age nine) understood multidigit subtraction for the first time when she recognized the connection between the algorithm she had memorized and her knowledge of the positional value of each digit (p. 155). Relationships can tie together small pieces of information or larger pieces that are themselves networks of sorts. When previously independent networks are related, there is a dramatic and significant cognitive reorganization (Lawler, 1981).

A second way in which conceptual knowledge grows is through the creation of relationships between existing knowledge and new information that is just entering the system. The example of Jane cited above would fit here if Jane had recognized the connection between algorithm and place value immediately upon being taught the algorithm. Again, this phenomenon has been described with a variety of labels. Perhaps "understanding" is the term used most often to describe the state of knowledge when new mathematical information is connected appropriately to existing knowledge (Davis, 1984; Skemp, 1971; Van Engen, 1953). Other terms, like "meaningful learning," convey similar sentiments (Ausubel, 1967; Brownell, 1935; Greeno, 1983b). Regardless of the term used, the heart of the process involves assimilating (Piaget, 1960) the new material into appropriate knowledge networks or structures. The result is that the new material becomes part of an existing network.

It is useful to distinguish between two levels at which relationships between pieces of mathematical knowledge can be established. One level we will call primary. At this level the relationship connecting the information is constructed at the same level of abstractness (or at a less abstract level) than that at which the information itself is represented. That is, the relationship is no more abstract than the information it is connecting. The term abstract is used here to refer to the

degree to which a unit of knowledge (or a relationship) is tied to specific contexts. Abstractness increases as knowledge becomes freed from specific contexts.

An example may help to clarify the idea of primary level relationships. When students learn about decimal numbers, they learn a variety of things about decimals, including the following two facts. First, the position values to the right of the decimal point are tenths, hundredths, and so on; second, when you add or subtract decimal numbers you line up the decimal points. Usually, it is expected that students will relate these two pieces of information and recognize that when you line up decimal points in addition you end up adding tenths with tenths, hundredths with hundredths, and so forth. If students do make the connection, they certainly have advanced their understanding of addition. But a noteworthy characteristic of this primary relationship is that it connects two pieces of information about decimal numbers and nothing more. It is tied to the decimal context.

Some relationships are constructed at a higher, more abstract level than the pieces of information they connect. We call this the reflective level. Relationships at this level are less tied to specific contexts. They often are created by recognizing similar core features in pieces of information that are superficially different. The relationships transcend the level at which the knowledge currently is represented, pull out the common features of different-looking pieces of knowledge, and tie them together. In the example cited earlier, the learner might step back mentally and recall that you line up numerals on the right to add whole numbers and end up adding units with units, tens with tens, hundreds with hundreds. When adding common fractions, you look for common denominators and end up adding the same size pieces together. Now the connection between the position value and lining up decimal points to add decimal numbers is recognized as a special case of the general idea that you always add things that are alike in some crucial way, things that have been measured with a common unit. Lining up decimal points results in adding together the parts of the decimal fractions that are the "same size." This kind of a connection is at a reflective level because building it requires a process of stepping back and reflecting on the information being connected. It is at a higher level than the primary level, because from its vantage point the learner can see much more of the mathematical terrain.

There are other ways to describe the different kinds of relationships that are part of one's conceptual knowledge in mathematics, but the primary and reflective levels provide a useful distinction. The analysis is similar in some important ways to the different types of understanding described by Greeno (1980, 1983b) and the different types of intelligence proposed by Skemp (1971). Although this distinction is not always made explicit in the remaining discussion, it is important to remember that not all relationships are of a single kind.

Procedural Knowledge

Procedural knowledge, as we define it here, is made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks. The first part is sometimes called the "form" of mathematics (Byers & Ertwanger, 1984). It includes a familiarity with the symbols used to represent mathematical ideas and an awareness of the syntactic rules for writing symbols in an acceptable form. For example, those who possess this aspect of procedural knowledge would recognize that the expression $3.5 \div \square = 2.71$ is syntactically acceptable (although they may not know the "answer") and that $6 + \square = 2$ is not acceptable. At more advanced levels of mathematics, knowledge of form includes knowledge of the syntactic configurations of formal proofs. This does not include the content or logic of proofs, only the style in which proof statements are written. Notice that, in general, knowledge of the symbols and syntax of mathematics implies only an awareness of surface features, not a knowledge of meaning.

The second part of procedural knowledge consists of rules, algorithms, or procedures used to solve mathematical tasks. They are step-by-step instructions that prescribe how to complete tasks. A key feature of procedures is that they are executed in a predetermined linear sequence. It is the clearly sequential nature of procedures that probably sets them most apart from other forms of knowledge. The only relational requirement for a procedure to run is that prescription n must know that it comes after prescription $n - 1$. Actually, at the barest minimum it needs to recognize as input only the outcome of prescription $n - 1$.

The procedures we are describing can be characterized as production systems (Anderson, 1983; Newell & Simon, 1972) in that they require some sort of recognizable input for firing. For the completion of a task, the initial procedure operates on the input and produces an outcome that is recognized by the next procedure in sequence. In this way, the sequence of procedures moves the given state (the statement of problem) to a goal state (the answer).

It is useful to distinguish between two kinds of procedures by noticing the objects upon which they operate. A basic distinction can be drawn between objects that are standard written symbols (e.g., 3, +, $\sqrt{\quad}$) and objects that are nonsymbolic (e.g., concrete objects or mental images). After students have been in school for a few years, the objects often are symbols. Students are presented with problems in the form of symbol expressions, such as adding whole numbers, translating from common to decimal fraction notation, or solving algebraic equations. The task is to transform the symbol expression from the given form to an answer form by executing a sequence of symbol manipulation rules. In production system terms, the given state is a pattern of symbols (e.g., $\frac{1}{3} + \frac{2}{3} = \square$). Each step in the procedure recognizes the pattern it receives and changes it to another pattern, which in turn is recognized by the next step, and so on until a

number is produced that is recognized as the answer. Procedures whose input and output are visual symbol patterns have been labeled "visually-moderated sequences" (Davis, 1984, p. 35). Such procedures make up the lion's share of school mathematics.

That school tasks most often involve symbol manipulation procedures is a fact whose importance should not be underestimated. It means that examinations of students' procedural knowledge often deal with a rather narrow but critical kind of procedure. Because of their importance in school learning, we believe it is useful to distinguish procedures that essentially are syntactic maneuvers on symbols. Chapters in this volume that focus on school-age children reflect this emphasis, especially those by VanLehn, Hiebert and Wearne, and Silver.

A second kind of procedure is a problem-solving strategy or action that operates on concrete objects, visual diagrams, mental images, or other objects that are not standard symbols of our mathematical systems. Such procedures are used extensively by preschool children, by older children on "nonschool" tasks, and occasionally by students in school. Young children, for example, use a variety of counting strategies to solve verbally presented addition and subtraction problems (Carpenter, this volume). More elemental counting and number procedures are described by Gelman and Meek, Sinclair and Sinclair, and Baroody and Ginsburg (this volume). Examples of school tasks that require nonsymbol procedures are straightedge and compass constructions in geometry (Schoenfeld, this volume). The important point here is that procedures, like concepts, are not all of one kind. Some procedures manipulate written mathematical symbols, whereas others operate on concrete objects, visual diagrams, or other entities.

An important feature of the procedural system is that it is structured. Procedures are hierarchically arranged so that some procedures are embedded in others as subprocedures. An entire sequence of step-by-step prescriptions or subprocedures can be characterized as a superprocedure. The advantage of creating superprocedures is that all subprocedures in a sequence can be accessed by retrieving a single superprocedure. For example, to apply the superprocedure "multiply two decimal numbers" (e.g., $3.82 \times .43$) one usually applies three subprocedures: one to write the problem in appropriate vertical form, a second to calculate the numerical part of the answer, and a third to place the decimal point in the answer. The second of these is itself made up of lower level subprocedures for (whole number) multiplication. Often it is possible to identify several levels of subprocedures that comprise a single superprocedure. The subprocedures are accessed as a sequential string once the superprocedure is identified.

In summary, procedural knowledge of mathematics encompasses two kinds of information. One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating sym-

hols. However, procedural knowledge also includes strategies for solving problems that do not operate directly on symbols. Perhaps the biggest difference between procedural knowledge and conceptual knowledge is that the primary relationship in procedural knowledge is "after," which is used to sequence subprocedures and superprocedures linearly. In contrast, conceptual knowledge is saturated with relationships of many kinds.

Meaningful and Rote Learning

Lurking just below the surface is the question of how the notions of conceptual and procedural knowledge are related to the issue of meaningful versus rote learning. There are some clear similarities, to be sure, but there are also some important differences. In addition to setting (momentarily) the question of how these ideas are related, a discussion of their similarities and differences may help to clarify our view of conceptual and procedural knowledge.

Meaningful learning, as we indicated earlier, often has been used to convey many of the same ideas that we presented as part of conceptual knowledge (see Brownell, 1935; Davis, 1984; Greeno, 1983b). Meaning is generated as relationships between units of knowledge are recognized or created. Conceptual knowledge, by our definition, must be learned meaningfully. Procedures, on the other hand, may or may not be learned with meaning. We propose that procedures that are learned with meaning are procedures that are linked to conceptual knowledge. This is such an important idea in mathematics learning that its discussion takes up most of the remaining part of this chapter.

Rote learning, on the other hand, produces knowledge that is notably absent in relationships and is tied closely to the context in which it is learned. The knowledge that results from rote learning is not linked with other knowledge and therefore does not generalize to other situations; it can be accessed and applied only in those contexts that look very much like the original. Conceptual knowledge, as we have described it, cannot be generated directly by rote learning. Facts and propositions learned by rote are stored in memory as isolated bits of information, not linked with any conceptual network. Of course at some later time the learner may recognize or construct relationships between isolated pieces of information. In this case, conceptual knowledge is created from information that was learned initially by rote.

In contrast, procedures can be learned by rote. Procedures can be acquired and executed even if they are linked tightly to surface characteristics of the original context. In fact, many procedures, especially those that operate on symbol patterns, are triggered by surface features similar to those of the original context. The sequential nature of procedures also is not violated by rote learning. In fact, learning a predetermined linear sequence of actions seems to lend itself to rote memorization as an instructional method. Thus procedures can be learned by rote and probably often are learned by rote since they seem especially susceptible to this form of instruction.

No sooner than we propose definitions for conceptual and procedural knowledge and attempt to clarify them, we must back up and acknowledge that the definitions we have given and the impressions they convey will be flawed in some way. As we said earlier, not all knowledge fits nicely into one class or the other. Some knowledge lies at the intersection. Heuristic strategies for solving problems, which are themselves objects of thought, are examples. Rather than operating on symbols, heuristic strategies seem to have concepts as their arguments. Consider young children who invent strategies to solve addition problems (e.g., $8 + 9 = \square$) by combining their knowledge of doubles (e.g., $8 + 8 = 16$) with their knowledge of number relationships (e.g., 9 is one more than 8) (Carpenter, this volume). Here it is not always clear where conceptual knowledge ends and procedural knowledge begins.

Furthermore, it is difficult to imagine someone possessing conceptual and procedural knowledge as *entirely* independent systems. Some connections are inevitable (Nantais, Herscovics, & Bergeron, 1984). In fact, although it is possible to consider procedures without concepts, it is not so easy to imagine conceptual knowledge that is not linked with some procedures. This is due, in part, to the fact that procedures translate conceptual knowledge into something observable. Without procedures to access and act on the knowledge, we would not know it was there.

A final caveat about our definitions is that not everyone working in this area—even the contributors to this volume—will agree with them. There are explicit differences between our definitions and those contained in other chapters, and some contributors suggest other distinctions that are not part of our definition. For example, Gelman and Meek distinguish between conceptual, procedural, and utilization competence, and Schoenfeld discusses differences between empiricism and deduction, a distinction that has some similarities but certainly is not identical with our discussion of concepts and procedures. Rather than viewing these differences as a statement of discord, we believe they are a sign of a healthy, vital discussion about very complex issues.

POTENTIAL RELATIONSHIPS BETWEEN CONCEPTUAL AND PROCEDURAL KNOWLEDGE

Mathematical knowledge, in its fullest sense, includes significant, fundamental relationships between conceptual and procedural knowledge. Students are not fully competent in mathematics if either kind of knowledge is deficient or if they both have been acquired but remain separate entities. When concepts and procedures are not connected, students may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers but not understand what they are doing. Critical links between conceptual and procedural knowledge not only would prevent these deficit cases from developing but also would contribute in many other ways to the development of a sound knowledge base.

Benefits for Procedural Knowledge

Linking conceptual and procedural knowledge has many advantages for acquiring and using procedural knowledge. The advantages apply to both kinds of procedural knowledge identified earlier. Building relationships between conceptual knowledge and the formal symbol system of mathematics is the process that gives meaning to symbols. Building relationships between conceptual knowledge and the procedures of mathematics contributes to memory (storage and retrieval) of procedures and to their effective use.

Developing Meaning for Symbols. Few would deny that meaningful symbols provide an essential foundation for genuine mathematical competence. But students do not always establish meanings for the symbols they use. Given our description of procedural knowledge, it is possible to acquire knowledge of symbols purely as visual patterns that conform to certain syntactic constraints. Symbols acquired in this way make no demands on conceptual knowledge. For symbols to develop meaning they must be connected to the conceptual knowledge they represent.

The process of relating symbols to conceptual knowledge is described by Van Engen (1949). Symbols like 5, 2, +, =, $\frac{3}{4}$ all represent ideas that can be encountered in concrete or real-world experiences. Once experienced, the ideas can be represented as conceptual knowledge. It is these conceptual, concretely based ideas that provide the referents for symbols. For example, the symbol "+" represents the joining idea in the story, "Sue has three marbles. Her mother gave her five more marbles. How many marbles does Sue have altogether?" If the joining idea is connected with "+," the symbol takes on meaning. Similar links must be made between each symbol and appropriate referents in order for the formal language of mathematics to become meaningful.

The notion of connecting symbols with their referents is presented in a more formal way by Schoenfeld (this volume). Schoenfeld identifies a reference world, a symbol world, and mappings between them. The mappings connect the referents and operations on them with symbol representations and analogous operations on the symbols. A critical part of mathematical competence stipulates that entities in the symbol world must represent (for the learner) entities in the reference world.

Recalling Procedures. Doing mathematics requires the execution of hundreds of different procedures, probably too many to memorize as individual pieces of information. In order to make the task manageable, the learner must call on some additional intellectual equipment. A good choice would be the available store of conceptual knowledge. If procedures are related to the underlying rationale on which they are based, the procedures begin to look reasonable. It is possible to understand how and why the procedures work. Because it is easier

to remember things that make sense (Chase & Simon, 1973; Chi, 1978), procedures that are meaningful, that are understood by their users, are more likely to be recalled.

There are several reasons to believe that connecting procedures with their conceptual underpinnings is the key in producing procedures that are stored and retrieved more successfully. First, if procedures are linked with conceptual knowledge, they become stored as part of a network of information, glued together with semantic relationships. Such a network is less likely to deteriorate than an isolated piece of information, because memory is especially good for relationships that are meaningful (Anderson, 1983; Rohwer, 1973; Skemp, 1976) and highly organized (Baddeley, 1976; Bruner, 1960; Hilgard, 1957). Second, retrieval is enhanced because the knowledge structure, or network, of which the procedure is a part comes equipped with numerous links that enable access to the procedure. The "conceptual" links increase the chances that the procedure will be retrieved when needed, because they serve as alternate access routes for recall (Anderson, 1983). For example, suppose students are learning to add decimal numbers and the teacher says, "When you add decimals you must first line up the decimal points." If this is all the information students acquire about setting up decimal addition problems, the line-up-the-decimal-points rule likely will be stored as an isolated piece of information with retrieval dependent on retracing a single link between the procedure and the perception of an addition problem as one involving decimal numbers. However, if students also learn that the concept underlying the procedure is the adding together of things that are alike, and they are able to recognize the similarity between this rationale and that used in adding whole numbers or common fractions, they are in a much better position to remember the rule. The likelihood of recalling the appropriate procedure is increased, partly because now the retrieval process can be triggered by several external *and* internal cues, and the procedure can be accessed by crossing a number of different conceptual bridges (e.g., ideas about place value or about common denominators, or intuitive notions about relative sizes of quantities). In fact, with this sort of conceptual base, the rule could be reconstructed extemporaneously.

Using Procedures. There also is reason to believe that in addition to enhancing memory for procedures, linking conceptual and procedural knowledge facilitates the effective use of procedures. This may occur in at least three different ways. If conceptual knowledge is linked to procedures it can: (a) enhance problem representations and simplify procedural demands; (b) monitor procedure selection and execution; and (c) promote transfer and reduce the number of procedures required.

The logic behind the first claim, that related conceptual knowledge enhances problem representations and thus makes the problems easier to solve, is as follows. Problems are solved by building mental representations of the problems

and then dealing with the representations to select appropriate procedures. Relevant conceptual knowledge can be brought to bear on the task by elaborating the problem context (Silver, 1982). Related conceptual knowledge is accessed, and the problem representation is enriched (Larkin, 1983). The advantage of representing the problem conceptually is that it allows one to reason directly about the quantities involved rather than reasoning about the symbols of a mediating language (Greeno, 1983a). In this way, conceptual knowledge can turn a difficult problem into a simpler one, which can be solved by available procedures (Davis, 1984).

Data supporting the importance of problem representations that are heavily conceptual come from two different sources. First, comparisons between experts and novices suggest that expert problem solvers in a particular domain represent problems by using underlying structure and conceptual features of the problem context, whereas novices focus more on superficial features and specific symbol manipulation rules that might apply (Chi, Feltovich, & Glaser, 1981; Larkin, McDermott, Simon, & Simon, 1980; Silver, this volume). The implication is that conceptually enriched problem representations facilitate successful performance.

A different data source that delivers the same message comes from observing adults solve what they perceive to be "real-life" problems (Lave, Murtaugh, & de La Rocha, 1984; Lesh, Landau, & Hamilton, 1983). It appears that problem representations drawing on conceptual knowledge to place the problem in a meaningful context influence the selection of procedures and raise the accuracy and efficiency of the applied procedures. More specifically, problems that lack a conceptual representation, that are solved only by selecting and applying memorized procedures, are more susceptible to error than problems for which a rich conceptual knowledge representation can be built (Carragher & Schliemann, 1985; Lave et al., 1984).

A second way in which links with conceptual knowledge can enhance the use of procedures is by executive control. Conceptual knowledge, if linked with a procedure, can monitor its selection and use and can evaluate the reasonableness of the procedural outcome. With regard to selection, conceptual knowledge serves (a) as an aid in the choice of appropriate procedures (Gelman & Meck, this volume; Piaget, 1978) and (b) as a constraint that discourages the selection of unacceptable procedures (Gelman, 1982; Gelman & Meck, this volume; Greeno, 1980). Piaget (1978) argues that when conceptual knowledge is on par with procedural actions, the conceptualization of a task enables one to anticipate the consequences of possible actions. This information can be used to select and coordinate appropriate procedures.

Conceptual knowledge can also inform a user that a procedure is inappropriate. This happens when the procedure itself violates conceptual principles. Gelman and Meck (this volume) argue that children learn to count relatively early in life *because* conceptual principles provide constraints against which they

can evaluate their counting behaviors and reject them or alter them to conform to the conceptual principles. For an example later in life, consider again the student who is adding decimals (Hiebert & Wearne, this volume). Suppose the student is presented with $3.5 + 1.76$. One procedure that specifically would be rejected by conceptual considerations is adding the 5 to the 6 and the 3 to the 7. It would be rejected because these operations would combine quantities of different denomination or size. So, in addition to assisting with the selection of an appropriate procedure, conceptual knowledge acts as a screening agent to reject inappropriate procedures.

A second and related executive control function for conceptual knowledge is monitoring procedural *outcomes*. Conceptual knowledge fulfills this function by playing the role of a validating critic (Brownell, 1947; Davis & McKnight, 1980). The critic judges the reasonableness of the answer; it checks whether the answer "makes sense." Consider, for example, a word problem described by Silver (this volume) that involves long division. The problem asked students to find how many buses would be needed to transport a given number of people. Most eighth-grade students computed correctly but consistently provided answers (involving remainders) that were unreasonable solutions to the problem. Conceptual knowledge of the problem situation would warn students that their answers were inappropriate. Another example comes from an earlier National Assessment of Educational Progress (Post, 1981). Students were asked to estimate the answer to $\frac{11}{3} + \frac{1}{3}$ and were given choices of 1, 2, 19, 21, and "I don't know." A conceptual knowledge critic would warn the student that 19 and 21, the two most frequent responses, were unreasonable. Ideally, the warnings in both of these examples would encourage the student to reevaluate the solutions and perhaps the choice of procedures.

On complex mathematical problems, executive decisions must be made at a macroscopic strategic level as well as at the tactical level (Schoenfeld, 1983). Both functions described above, selecting a procedure and checking its outcome, are tactical decisions. Strategic decisions involve planning the direction in which a solution will be pursued and managing intellectual resources to keep the pursuit running. Conceptual knowledge undoubtedly plays a strong role in strategic decision making as well. Schoenfeld (1983) has illustrated that effective use of procedures requires conceptually informed decisions at both the tactical and the strategic levels.

Up to this point, the ways in which conceptual knowledge can improve the use of procedures have dealt with using procedures on a single problem. Another benefit has to do with using procedures across two or more problems. It has long been recognized that if procedures are understood, or learned in a meaningful way, they transfer more easily to structurally similar problems (Brownell, 1947; Dewey, 1910). It now is possible to describe the phenomenon in more detail. Problems that are structurally similar have problem representations with some conceptual elements in common. The links between the procedure and related

concepts connect the procedure, by way of the common elements, to many problem representations. Therefore, the appropriateness of the procedure for many superficially different problems is recognized, and the procedure "transfers" from one problem to another (Greeno, 1983a). In other words, conceptual knowledge releases the procedure from the surface context in which it was learned and encourages its use on other structurally similar problems.

An example of how conceptual knowledge facilitates transfer of procedures is provided by Carpenter (this volume). As young children's concept of subtraction is enriched to include different interpretations (such as take away, difference, and adding on), they are able to use a particular strategy learned for solving one type of problem to solve problems of a different type. The key is building a rich store of conceptual knowledge that covers a variety of task situations and, through its interconnections, becomes linked to a single, efficient procedure.

The real significance of increasing transfer of procedures is that it reduces the number of procedures that must be learned. Procedures that can be used flexibly, that are not tied to specific tasks, are procedures that have at least some generality. Generalized procedures eliminate the need to learn different procedures for each task, thereby reducing the number of procedures that must be learned and remembered. For instance, consider the process of multidigit subtraction with regrouping. If the process is learned at a purely syntactic level, different procedures must be learned for different kinds of problems (e.g., borrowing across zeros is syntactically different from borrowing across nonzeros). However, at the semantic level a single process of regrouping governs all cases. The slight syntactic variations that are executed for different tasks need not be treated as different procedures to be memorized independently; instead they simply appear as syntactically different manifestations of the same process applied to structurally similar tasks. Mathematics is filled with examples of a single, conceptually linked procedure that replaces numerous syntactically distinct ones.

To reiterate, a linking relationship between conceptual and procedural knowledge appears to increase the usefulness of procedural knowledge. The benefits accrue for both kinds of procedural knowledge, symbols and procedures. Linking conceptual knowledge with symbols creates a meaningful representation system, an essential prerequisite for intelligent mathematical learning and performance. Linking conceptual knowledge with rules, algorithms, or procedures reduces the number of procedures that must be learned and increases the likelihood that an appropriate procedure will be recalled and used effectively.

Benefits for Conceptual Knowledge

Linking conceptual and procedural knowledge benefits conceptual knowledge as well as procedural knowledge. The benefits for conceptual knowledge are cited less often but are equally significant. Under an organizational scheme similar to

the previous section, some benefits for conceptual knowledge arise from the formal language system and syntax conventions, whereas others emerge from the use of rules and procedures.

Symbols Enhance Concepts. The formal language system of mathematics provides a powerful tool for dealing with complex ideas. Symbols that have been connected with meaningful referents can be used to think about the concepts they represent. "It is largely by the use of symbols that we achieve voluntary control over our thoughts" (Skemp, 1971, p. 83). Thought is aided by the fact that symbols can represent complex or densely packed concepts; in these cases, cognitive effort in dealing with the concepts is reduced by focusing on the symbols. In fact, one of the powerful features of mathematics is the effortlessness with which complex ideas can be manipulated by moving symbols.

Viewed as cognitive aids, symbols help to organize and operate on conceptual knowledge. But that is not all. The symbol system can also *produce* conceptual knowledge. Byers and Ertwanger (1984) draw attention to the fact that the notation system, or syntax of mathematical symbols, is responsible for the development of some key mathematical concepts. Place-value notation and Leibniz's integral notation are two especially apt examples. Advances in form often bring with them advances in related concepts (Strunk, 1967).

Procedures Apply Concepts to Solve Problems. Conceptual knowledge is useful for solving mathematical tasks only when it is accessed and converted into appropriate form. Anderson (1983) describes one way in which this might happen. Problems for which no routine procedures are available are solved initially by applying facts and concepts in an effortful, laborious way. As similar problems are solved repeatedly, conceptual (declarative) knowledge is gradually transformed into set routines (condition-action pairs) for solving the problems. The condition-action pairs constitute the basic elements of the procedural system. Thus knowledge that is initially conceptual can be converted to knowledge that is procedural.

Gelman and Gallistel (1978) describe another way in which procedures can bring conceptual knowledge into the problem-solving arena. They believe that young children possess a significant store of conceptual knowledge about counting and number. However, they can only use their conceptual knowledge to reason about specific numerosities, about quantities to which a numerical value has been assigned. Counting *procedures* bring number within the purview of conceptual knowledge by generating specific numerosities with which conceptual knowledge can reason.

Third, procedures can facilitate the application of conceptual knowledge because highly routinized procedures can reduce the mental effort required in solving a problem and thereby make possible the solution of complex tasks. Case

(1985) explains this phenomenon by pointing out that efficient procedures require less of one's limited cognitive processing capacity. This frees additional space for more effortful processes, such as planning (Kotovsky, Hayes, & Simon, 1985), or looking for relationships between novel aspects of a problem and relevant conceptual knowledge. In other words, automated and efficient strategies make room for applying conceptual knowledge.

Procedures Promote Concepts. Just as new symbol notation occasionally generates or advances concepts in the discipline of mathematics, new procedures can trigger for individuals the development of concepts. It is clear that during the early years, relationships between conceptual and procedural knowledge are intricate and dynamic (Baroody & Ginsburg, this volume; Gelman & Meek, this volume; Sinclair & Sinclair, this volume). It appears that on occasion procedural knowledge takes the lead and spurs the development of new concepts. For example, Gelman and Meek (this volume) present a scenario in which children use already acquired counting skills to promote the development of an ordinal concept of number. Baroody and Ginsburg (this volume) describe other instances in which young children's conceptual development is motivated by the application of procedures.

Summary

Linking conceptual knowledge and procedural knowledge has many advantages. Usually the advantages are claimed for procedural knowledge. Procedural knowledge that is informed by conceptual knowledge results in symbols that have meaning and procedures that can be remembered better and used more effectively. A closer look reveals theoretical advantages for conceptual knowledge. Procedural knowledge provides a formal language and action sequences that raise the level and applicability of conceptual knowledge. These are the theoretical claims. In reality, the advantages are not always realized.

ACTUAL RELATIONSHIPS BETWEEN CONCEPTUAL AND PROCEDURAL KNOWLEDGE

If students of mathematics naturally and routinely connected their conceptual and procedural knowledge, the issue (and this book) would be of little interest. The skill versus understanding debate of the past would certainly not have persisted and probably would not have arisen at all. Examining the relationships between conceptual and procedural knowledge is a worthwhile pursuit only because students often fail to recognize or construct the relationships. Being competent in mathematics involves knowing concepts, knowing symbols and procedures, and knowing how they are related. Why is it that the relationships frequently are not

constructed? What are the factors that inhibit the creation and recognition of relationships between conceptual and procedural knowledge?

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Factors That Inhibit the Construction of Relationships

Building relationships between pieces of information does not always occur spontaneously. Even if the relationships are made explicit, they are not always recognized or internalized. Many factors may contribute to the failure to establish relationships between units of knowledge; we address three of them that may have special importance for mathematics learning.

Deficits in the Knowledge Base. Relationships between items of knowledge cannot be constructed if the knowledge does not exist. This observation is obvious, but it is worth elaborating because of its importance in mathematics learning. Three examples illustrate the point. Silver (this volume) argues that the common error in adding fractions of adding numerators and adding denominators (e.g., $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$) may not be due so much to a separation of concepts and procedures as to an erroneous and incomplete conceptual base. Many visual diagrams and other conceptual models unwittingly contribute to a conceptual structure that *reinforces* (rather than discourages) this error. Hiebert and Wearne (this volume) suggest that a critical relationship in decimal fractions is the equivalence relation between a decimal fraction and its associated common fraction. For many students, such relationships are impossible to establish because the students enter instruction on decimals with deficient knowledge of common fractions. A final example, from a slightly different perspective, comes from Carpenter (this volume). Carpenter points out that children's increasing proficiency in solving simple addition and subtraction problems can be traced to increases in conceptual knowledge. Furthermore, meaning for procedures cannot be developed unless a rich conceptual knowledge base is in place.

All three examples suggest that a sound knowledge base is necessary for useful relationships to be established. Deficiencies in concepts or procedures, although sometimes hidden, can be the source of weak or missing connections.

Difficulties of Encoding Relationships. A second factor that inhibits the construction of relationships is young children's tendency to overlook or fail to encode relationships that may be obvious to adults. Research on young children's encoding of information, specifically studies on elaboration in paired-associate learning, suggests that sometimes children have trouble constructing relationships between items of information (Ackerman, 1985; Pressley, 1982; Rohwer, 1973). Even when conceptual relationships between items are obvious to adults, children do not readily encode them. It may be that the failure to encode and construct relationships between units of information is not limited to these

rather special paired-associate tasks but extends to more complex school learning tasks as well. Relationships between units of mathematical knowledge, although taught by adults using seemingly appropriate methods, may not be picked up and internalized by children.

Tendency to Compartmentalize Knowledge. A third general factor that seems to impede the construction of relationships between units of knowledge is that knowledge just acquired often is context bound (Bruner, 1973; Tulving, 1983). Things learned in a particular context are initially tied to surface characteristics of that context. This prevents one from noticing similarities between the newly acquired knowledge and previously acquired knowledge held in memory. Context-bound knowledge is not looking for relationships outside the immediate context. The phenomenon is illustrated well by Lawler's (1981) description of a young child learning to compute in several different contexts. "Different" methods were learned in each context, and the methods remained faithful to the context in which they were acquired. Money methods were used only with money, LOGO methods with LOGO, paper and pencil methods with paper and pencil. For some time the methods remained encapsulated by the context. Later, similarities between the methods were recognized, but then only by detecting regularities in the outcome of the different methods rather than by detecting similarities in the methods themselves.

The experience of acquiring information as isolated, context-specific bits of knowledge is known to both older and younger learners and pervades all content domains. In whole numbers (Carpenter, this volume), fractions (Silver, this volume), decimals (Hiebert & Wearne, this volume), algebra (Kaput, 1982), and geometry (Schoenfeld, this volume), students acquire knowledge in one context and hold it separate from knowledge acquired in other contexts. Knowledge compartments can remain isolated even when they are well developed individually. Furthermore, the content of these knowledge compartments can be mutually incompatible, a condition that does not necessarily fluster the learner. Evidence for the isolated nature of students' knowledge often is collected by observing students solve the same problem in two different contexts, produce conflicting solutions, and not reconcile the differences. Indeed, most investigators report that many students do not recognize that the differences must be reconciled.

A further explanation for students' tendency to compartmentalize knowledge is offered by Posner, Strike, Hewson, & Gertzog (1982). They suggest that because learners often resist conceptual change, the new knowledge is compartmentalized so that it does not interfere with existing concepts. This is especially true when new knowledge is perceived to be unrelated to or conflicts with existing knowledge. Specific compartments are accessed only when surface characteristics of the problem context are recognized as similar to those in which the knowledge was acquired.

The effects on mathematics learning of acquiring units of knowledge but not establishing relationships between them are serious. Important insights into the acquisition and application of mathematical competencies can be gained by examining the nature of these breakdowns and the points at which they occur. On the basis of recent work in this area, it now is possible to trace the changing nature of the relationships between conceptual and procedural knowledge as children proceed through the preschool and school years. What follows is a simplified sketch. The remaining chapters in this volume describe in more detail the nature of the relationships at succeeding points in a person's mathematical career.

Ontogeny of Relationships During the Mathematical Learning Years

Preschool Years. During the earliest years, conceptual knowledge and procedural knowledge are closely related. In fact, Sinclair and Sinclair (this volume) suggest that at this point concepts and procedures are so closely related that they become nearly indissociable. Before children enter school, their mathematical knowledge is limited mostly to knowledge about counting (Ginsburg, 1977). How conceptual and procedural knowledge interact as children learn to count is a matter of current debate. Some investigators believe that the development of concepts or principles of counting precedes the acquisition of skills needed to count accurately (Gelman, 1982; Gelman & Gallistel, 1978; Gelman & Meek, 1983, this volume). Children know more conceptually than they are able to demonstrate procedurally. Other researchers believe that some counting skills are acquired initially as rote procedures and later become informed by conceptual knowledge (Baroody, 1984; Baroody & Ginsburg, this volume; Fuson & Hall, 1983; Fuson, Richards, & Briars, 1982). Either way, concepts and procedures are closely intertwined as children learn to count.

During the preschool years, children also are learning about numerical print (Sinclair & Sinclair, 1984, this volume). They acquire meanings for numerical symbols from environmental print and gradually refine and enrich their meanings to correspond to adult conventions. The important point is that the process seems to involve concepts *and* procedures that are closely connected.

The connections between conceptual and procedural knowledge still are in place as children enter school. By this time children have learned to use their counting abilities to solve simple addition and subtraction problems, if they are presented in words rather than symbols. The kinds of strategies they use to solve the problems is the clue to the link between their procedures and their conceptualization of the problem (Carpenter, this volume; Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1984; De Corte & Verschaffel, 1984). For example, if concrete counters are available, almost all beginning school children will solve a missing-addend story by adding on, and a take-away story by taking

away. That is, the child's counting strategy matches the semantic structure of the story. It is as if the child's conceptual knowledge of the situation guides the selection of a solution procedure.

Additional weight for the argument that conceptual and procedural knowledge are closely related at this point is provided by two models proposed to account for children's performance on addition and subtraction word problems (Briars & Larkin, 1984; Riley, Greeno, & Heller, 1983). Although there are some important differences between them, both models assume that increases in procedural skill are tied to advances in conceptual knowledge. Improvements in performance are related to improvements in understanding, not merely to increases in memorized procedures. Students have not yet learned algorithms that take them beyond their level of conceptual knowledge. At this point in their mathematical careers, students do not get right answers for problems they do not understand.

School Years. Classroom instruction in arithmetic introduces students to the formal symbolic language of mathematics. If students connect the symbols with conceptually based referents, the symbols acquire meaning and become powerful tools for recording and communicating mathematical events. Unfortunately, many students seem to learn symbols as meaningless marks on paper (Hiebert, 1984a). The symbols are separated from the conceptual knowledge they are supposed to represent. For instance, after instruction on writing number sentences (equations), many first graders still do not see the connection between the story problems and the number sentences that represent them (Carpenter, Hiebert & Moser, 1983; Lindvall & Ibarra, 1980; Matthews, 1983). They view the process of solving a story problem and solving the related number sentence as two separate, independent events. Their conceptual knowledge of addition and subtraction accessed through story problems is not connected with the symbolism of arithmetic.

For many children, the effect of initial instruction on arithmetic symbols is to pry apart conceptual and procedural knowledge and send them in different directions. Up to this point both types of knowledge seem to develop in close synchrony, continually informing each other. But with the introduction of written symbols whose meanings are not well established, the dynamic interaction is broken. Probably due to the emphases of conventional instruction (see chapters in this volume by Baroody and Ginsburg, by Carpenter, by VanLehn, and by Schoenfeld), the focus of attention shifts to the procedural level. New symbols are introduced, and rules for manipulating the symbols are heavily emphasized. Students learn to execute the rules and algorithms according to syntactic constraints, and the rules produce current answers if they are fed the problems in familiar form. But the rules are tied only to the symbols, not to the conceptual knowledge that provides their rationale.

By the time students are in third and fourth grade, they have acquired a large array of symbol manipulation rules. In general, the rules are more sensitive to

syntactic constraints than to conceptual underpinnings. Even if conceptual understandings exist, they have little effect on the selection or execution of procedural skills (Davis & McKnight, 1980; Resnick, 1982). The nearly exclusive reliance on procedural knowledge is evidenced in two ways. First, most of students' behavior on arithmetic computation tasks can be described in purely syntactic terms (Brown & VanLehn, 1982; VanLehn, 1983, this volume). Second, students' computation errors reflect attempts to repair broken procedures according to syntactic constraints rather than semantic or conceptual considerations (VanLehn, this volume).

As students move through elementary and junior high school, conceptual knowledge and procedural knowledge continue to develop along separate tracks. The focus of instruction remains on procedural knowledge, and students acquire new rules for new symbol systems such as common and decimal fractions (Hiebert & Wearne, this volume). As with whole numbers, there are few connections apparent between procedures and concepts. For example, students' behavior on decimal computation problems can be modeled well by appealing only to procedural, symbol manipulation rules (Hiebert & Wearne, 1985). Also, the errors students make often are produced by rules that mildly distort the syntactic guidelines but greatly violate the relevant concepts. As an illustration, an earlier National Assessment of Educational Progress confirms that many students add common fractions by adding numerators and adding denominators (Post, 1981). From a procedural point of view, it is easy to explain why students do this, but conceptually it makes no sense.

Reliance on syntactic symbol manipulation rules continues into high school. Students' behavior on algebra tasks can be characterized by identifying a list of syntactic rules together with distortions of the rules triggered by surface features of the tasks (Matz, 1980). Notably absent from the use of algebraic symbols is a link with conceptual content (Kaput, 1982; Resnick & Clement, 1980). In geometry, the separation between types of knowledge can be characterized somewhat differently (Schoenfeld, this volume), but the theme remains the same. Students learn rules for solving specific kinds of problems, and the rules remain isolated from students' conceptual knowledge of the subject.

In spite of the fact that students' procedural rules often become flawed, the rules can take them well beyond their level of conceptual understanding. Students are able to get correct answers for many problems they do not understand. Studies have shown that students from elementary school through college can perform successfully on routine paper and pencil problems but lack essential, underlying conceptual knowledge. This happens in mathematics (e.g., Erlwanger, 1975; Resnick & Clement, 1980), in science (see Gentner & Stevens, 1983), and undoubtedly in other disciplines as well.

The problem with learning procedures without concepts is that the procedures become likely victims of all the maladies identified earlier. Procedures that lack connections with conceptual knowledge may deteriorate quickly and are not

reconstructable; they may be only partially remembered and combined with other subprocedures in inappropriate ways; they often are bound to the specific context in which they were learned and do not transfer easily to new situations; and they can be applied inappropriately without the benefit of a validating critic to check the reasonableness of the outcome. Hence, although routinized procedural skills are essential for efficient problem solving, related conceptual knowledge is needed to give procedures stability and effectiveness. "Without these meanings to hold skills and ideas together in an intelligible, unified system, pupils in our schools for too long a time have 'mastered' skills which they do not understand, which they can use only in situations closely paralleling those of learning, and which they must soon forget" (Brownell, 1947, p. 260).

Formal mathematics instruction seems to do a better job of teaching procedures than concepts or relationships between them. There are undoubtedly many reasons for this. Chapters in this volume by Baroody and Ginsburg, by Carpenter, by Schoenfeld, and by VanLehn discuss some of these, and the literature on classroom processes and teacher decision making identifies others (Brophy, 1982; Doyle, 1983; Good, 1984). An examination of instructional processes lies beyond the scope of this chapter. Nevertheless, it should be recognized that the preeminence of procedures over concepts and the lack of relationships between the two can be explained in part by the nature of formal school instruction.

THE MISSION OF THE BOOK

"The relationship between computational skill and mathematical understanding is one of the oldest concerns in the psychology of mathematics" (Resnick & Ford, 1981, p. 246). Relationships between procedural and conceptual knowledge have had a long history for two reasons. First, the issue has been recognized as an important one by many psychologists and educators through the years. It is important because it seems to hold the key to many learning processes and problems. If we understood more about the acquisition of these kinds of knowledge and the interplay between them in mathematical performance, we surely could unlock some doors that have until now hidden significant learning problems in mathematics. Second, the issue has been extremely difficult to resolve. Relationships between conceptual and procedural knowledge change over time and are influenced by many forces, both internal and external to the learner. Describing such complex relationships is a monumental task.

Why should we think that this book will succeed in describing the relationships between conceptual and procedural knowledge? Certainly, it will not succeed in explicating all of the issues surrounding the conceptual and procedural knowledge debate. We simply don't know everything needed to finish the discussion. But we do believe that the remaining chapters take us significantly

further in our understanding of conceptual and procedural knowledge than we have come so far.

We are optimistic for two reasons. One is that there is a clearer understanding of the questions that need to be asked, a clearer focus on the issues that will yield useful insights. Building on past work, it now is evident that it is the *relationships* between conceptual and procedural knowledge that hold the key. The skills and understanding issue is important, to be sure, but not because instruction should choose between them. Rather, skills and understandings are important because they signal two kinds of knowledge that play crucial, interactive roles in the development of mathematical competence. It is understanding the relationships between these two forms of knowledge that will provide the real payoff (Glaser, 1979).

Our second reason for optimism is the convergence in focus of several lines of research. Two dimensions of this joining of forces are especially significant. One is the coming together of two previously separate paradigms of psychology research: Gestalt psychology, with an emphasis on concept learning and development, and behavioral psychology, with its emphasis on skilled performance. Studying the *relationship* between concepts and procedures focuses these two perspectives on the same issues. A second form of convergence is that of two disciplines: mathematics education and cognitive psychology. The nature of the relationships between conceptual and procedural knowledge is receiving concentrated attention by researchers from both fields, using a nearly common language. The broad field of cognitive science provides much common ground that facilitates and encourages serious communication and collaborative efforts among those from various research traditions. Past experience suggests that joint efforts by mathematics educators and psychologists are especially productive in providing insights in children's learning of mathematics (Hiebert, 1984b). This book represents such an effort.

REFERENCES

- Ackerman, B. P. (1985). Children's retrieval deficit. In C. J. Brainerd & M. Pressley, *Basic processes in memory development: Progress in cognitive development research* (pp. 1-46). New York: Springer-Verlag.
- Anderson, J. R. (1983). *The architecture of cognition*. Cambridge, MA: Harvard University Press.
- Ausubel, D. (1967). *Learning theory and classroom practice*. Toronto: Ontario Institute for Studies in Education.
- Baddeley, A. D. (1976). *The psychology of memory*. New York: Basic books.
- Baroody, A. J. (1984). More precisely defining and measuring the order-irrelevance principle. *Journal of Experimental Child Psychology*, 38, 33-41.
- Briars, D. J., & Larkin, J. H. (1984). An integrated model of skills in solving elementary word problems. *Cognition and Instruction*, 1, 245-296.
- Brophy, J. E. (1982). How teachers influence what is taught and learned in classrooms. *Elementary School Journal*, 83, 1-13.

- Brown, J. S., & VanLehn, K. (1982). Towards a generative theory of "bugs." In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 117-135). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Brownell, W. A. (1935). Psychological considerations in the learning and teaching of arithmetic. In *The teaching of arithmetic. Tenth yearbook of the National Council of Teachers of Mathematics*. New York: Teachers College, Columbia University.
- Brownell, W. A. (1941). Arithmetic in grades I and II. *Duke University Research Studies in Education* (No. 6). Durham, NC: Duke University Press.
- Brownell, W. A. (1947). The place of meaning in the teaching of arithmetic. *Elementary School Journal*, 47, 256-265.
- Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review*, 31, 21-32.
- Bruner, J. S. (1960). *The process of education*. New York: Vintage Books.
- Bruner, J. S. (1973). *Beyond the information given*. New York: Norton.
- Byers, V., & Erlwanger, S. (1984). Content and form in mathematics. *Educational Studies in Mathematics*, 15, 259-275.
- Carpenter, T. P., Hiebert, J., & Moser, J. M. (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*, 12, 27-39.
- Carpenter, T. P., Hiebert, J., & Moser, J. M. (1983). The effect of instruction on children's solutions of addition and subtraction word problems. *Educational Studies in Mathematics*, 14, 55-72.
- Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 15, 179-202.
- Carragher, T. N., & Schliemann, A. D. (1985). Computation routines prescribed by schools: Help or hindrance? *Journal for Research in Mathematics Education*, 16, 37-44.
- Case, R. (1985). *Intellectual development: Birth to adulthood*. New York: Academic Press.
- Chase, W. G., & Simon, H. A. (1973). The mind's eye in chess. In W. G. Chase (Ed.), *Visual information processing*. New York: Academic Press.
- Chi, M. (1978). Knowledge structures and memory development. In R. Siegler (Ed.), *Children's thinking: What develops?* (pp. 73-96). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Chi, M., Glaser, P., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science*, 5, 121-152.
- Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Norwood, NJ: Ablex.
- Davis, R. B., & McKnight, C. (1980). The influence of semantic content on algorithmic behavior. *Journal of Mathematical Behavior*, 3, (1), 39-87.
- De Corte, E., & Verschaffel, L. (1984). First graders' solution strategies of addition and subtraction word problems. In J. M. Moser (Ed.), *Proceedings of the sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 15-20). Madison: Wisconsin Center for Education Research.
- Dewey, J. (1910). *How we think*. Boston: Heath.
- Doyle, W. (1983). Academic work. *Review of Educational Research*, 53, 159-199.
- Erlwanger, S. H. (1975). Case studies of children's conceptions of mathematics—Part I. *Journal of Children's Mathematical Behavior*, 1 (3), 157-183.
- Fuson, K. C., & Hall, J. W. (1983). The acquisition of early number word meanings: A conceptual analysis and review. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 49-107). New York: Academic Press, 1983.
- Fuson, K. C., Richards, J., & Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In C. J. Brainerd (Ed.), *Children's logical and mathematical cognition* (pp. 33-92). New York: Springer-Verlag.
- Gagné, R. M. (1977). *The conditions of learning* (3rd ed.). New York: Holt, Rinehart, & Winston.
- Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.
- Gelman, R. (1982). Basic numerical abilities. In R. J. Sternberg (Ed.), *Advances in psychology of human intelligence* (Vol. 1) (pp. 181-205). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Gelman, R., & Meck, E. (1983). Preschoolers' counting: Principles before skill. *Cognition*, 13, 343-359.
- Gentner, D., & Stevens, A. L. (Eds.). (1983). *Mental models*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Ginsburg, H. (1977). *Children's arithmetic: The learning process*. New York: Van Nostrand.
- Glaser, R. (1979). Trends and research questions in psychological research on learning and schooling. *Educational Researcher*, 8 (10), 6-13.
- Good, T. L. (1984, April). *Recent studies of teaching: Implications for research and policy in mathematics education*. Invited address to the Special Interest Group for Research in Mathematics Education at the annual meeting of the American Educational Research Association, New Orleans.
- Greeno, J. G. (1980). Analysis of understanding in problem solving. In R. H. Kluwe & H. Spada (Eds.), *Developmental models of thinking* (pp. 199-212). New York: Academic Press.
- Greeno, J. G. (1983a). Conceptual entities. In D. Gentner & A. L. Stevens (Eds.), *Mental models* (pp. 227-252). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Greeno, J. G. (1983b). Forms of understanding in mathematical problem solving. In S. G. Paris, G. M. Olson, & H. W. Stevenson (Eds.), *Learning and motivation in the classroom* (pp. 83-111). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Hiebert, J. (1984a). Children's mathematics learning: The struggle to link form and understanding. *Elementary School Journal*, 84, 497-513.
- Hiebert, J. (1984b). Complementary perspectives. [Review of *Acquisition of mathematics concepts and processes, Children's logical and mathematical cognition: Progress in cognitive development research, and The development of mathematical thinking*]. *Journal for Research in Mathematics Education*, 15, 229-234.
- Hiebert, J., & Weare, D. (1985). A model of students' decimal computation procedures. *Cognition and Instruction*, 2, 175-205.
- Hilgard, E. R. (1957). *Introduction to psychology* (2nd ed.). New York: Harcourt Brace.
- Jones, P. S. (Ed.). (1970). *A history of mathematics education in the United States and Canada: Thirty-second yearbook*. Washington, DC: National Council of Teachers of Mathematics.
- Kaput, J. (1982, March). *Intuitive attempts at algebraic representation of quantitative relationships*. Paper presented at the annual meeting of the American Educational Research Association, New York.
- Kotovsky, K., Hayes, J. R., & Simon, H. A. (1985). Why are some problems hard? Evidence from Tower of Hanoi. *Cognitive Psychology*, 17, 248-294.
- Larkin, J. H. (1983). The role of problem representation in physics. In D. Gentner & A. L. Stevens (Eds.), *Mental models* (pp. 75-98). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Larkin, J., McDermott, J., Simon, D. P., & Simon, H. A. (1980). Expert and novice performers in solving physics problems. *Science*, 208, 1335-1342.
- Lave, J., Murtaugh, M., & de La Rocha, O. (1984). The dialectical construction of arithmetic in grocery shopping. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 67-94). Cambridge, MA: Harvard University.
- Lawler, R. W. (1981). The progressive construction of mind. *Cognitive Science*, 5, 1-30.
- Lesh, R., Landau, M., & Hamilton, E. (1983). Conceptual models and applied mathematical problem-solving research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 263-343). New York: Academic Press.

- Lindvall, C. M., & Ibarra, C. G. (1980). Incorrect procedures used by primary grade pupils in solving open addition and subtraction sentences. *Journal for Research in Mathematics Education*, 11, 50-62.
- Mathews, J. (1983). A subtraction experiment with six and seven year old children. *Educational Studies in Mathematics*, 14, 139-154.
- Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior*, 3 (1), 93-166.
- McLaughlin, K. L. (1935). Number ability in preschool children. *Childhood Education*, 11, 348-353.
- McLellan, J. A., & Dewey, J. (1895). *The psychology of number and its applications to methods of teaching arithmetic*. New York: D. Appleton.
- Nemais, N., Herscovics, N., & Bergeron, J. C. (1984). The skills-understanding dilemma in mathematics education. In J. M. Moser (Ed.), *Proceedings of the sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 229-235). Madison, WI: Wisconsin Center for Education Research.
- Newell, A., & Simon, H. A. (1972). *Human problem solving*. Englewood Cliffs, NJ: Prentice-Hall.
- Norman, D. A., & Rumelhart, D. E. (1975). *Explorations in cognition*. San Francisco: Freeman.
- Piaget, J. (1960). *The psychology of intelligence*. Totowa, NJ: Littlefield, Adams.
- Piaget, J. (1978). *Success and understanding*. Cambridge, MA: Harvard University Press.
- Posner, G. J., Strike, K. A., Hewson, P. W., & Gertzog, W. A. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. *Science Education*, 66, 211-227.
- Post, T. R. (1981). Fractions: Results and implications from National Assessment. *Arithmetic Teacher*, 28 (9), 26-31.
- Pressley, M. (1982). Elaboration and memory development. *Child Development*, 53, 296-309.
- Resnick, L. B. (1982). Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 136-155). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Resnick, L. B., & Ford, W. W. (1981). *The psychology of mathematics for instruction*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153-196). New York: Academic Press.
- Rohwer, W. D., Jr. (1973). Elaboration and learning in childhood and adolescence. In H. W. Reese (Ed.), *Advances in child development and behavior* (Vol. 8) (pp. 1-57). New York: Academic Press.
- Rosnick, P., & Clement, J. (1980). Learning without understanding: The effect of tutoring strategies on algebra misconceptions. *Journal of Mathematical Behavior*, 3 (1), 3-24.
- Schfeffer, I. (1965). *Conditions of knowledge: An introduction to epistemology and education*. Chicago: University of Chicago Press.
- Schoenfeld, A. H. (1983). Episodes and executive decisions in mathematical problem-solving. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 345-395). New York: Academic Press.
- Shulman, L. S. (1970). Psychology and mathematics education. In E. G. Begle (Ed.), *Mathematics education: The sixty-ninth yearbook of the National Society for the Study of Education* (pp. 23-71). Chicago: University of Chicago Press.
- Silver, E. A. (1982). Knowledge organization and mathematical problem solving. In F. K. Lester & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research* (pp. 15-25). Philadelphia: Franklin Institute Press.
- Sinclair, A., & Sinclair, H. (1984). Preschool children's interpretation of written numbers. *Human Learning*, 3, 173-184.

- Skepp, R. R. (1971). *The psychology of learning mathematics*. Middlesex, England: Penguin.
- Skepp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 1-7.
- Struk, D. J. (1967). *A concise history of mathematics*. New York: Dover.
- Thorndike, E. L. (1922). *The psychology of arithmetic*. New York: Macmillan.
- Tulving, E. (1983). *Elements of episodic memory*. New York: Oxford University Press.
- Van Engen, H. (1949). An analysis of meaning in arithmetic. *Elementary School Journal*, 49, 321-329, 395-400.
- Van Engen, H. (1953). The formation of concepts. In H. F. Fehr (Ed.), *The learning of mathematics: Its theory and practice. Twenty-first Yearbook of the National Council of Teachers of Mathematics* (pp. 69-98). Washington, DC: NCTM.
- VanLehn, K. (1983). On the representation of procedures in repair theory. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 201-252). New York: Academic Press.