Problem set 2.

1. $$|H - E\mathbb{1}| = \begin{vmatrix} \alpha - E & \beta & 0 \\ \beta & \alpha - E & \beta \\ 0 & \beta & \alpha - E \end{vmatrix} = 0$$

$$(\alpha - E)^3 - 2\beta^2(\alpha - E) = 0$$

or $$(\alpha - E)\left\{ (\alpha - E)^2 - 2\beta^2 \right\} = 0$$

$$\begin{cases} E = \alpha, & (\alpha - E)^2 = 2\beta^2 \Rightarrow \alpha - E = \pm\sqrt{2}\beta \\ E = \alpha \pm \sqrt{2}\beta \end{cases}$$

Eigenvectors

$$\begin{aligned} (\alpha - E)c_1 + \beta c_2 &= 0 \\ \beta c_1 + (\alpha - E)c_2 + \beta c_3 &= 0 \\ 0 \quad \beta c_2 + (\alpha - E)c_3 &= 0 \end{aligned}$$

First $\alpha = E$, $\quad \beta c_2 = 0$

$$\beta[c_1 + c_3] = 0 \Rightarrow c_1 = -c_3$$

$$\psi_1 \cong \phi_1 - \phi_3 \Rightarrow \psi_1 = \frac{1}{\sqrt{2}}(\phi_1 - \phi_3)$$

next, $E = \alpha + \sqrt{2}\beta$

$$-\sqrt{2}\beta c_1 + \beta c_2 = 0 \Rightarrow c_2 = \sqrt{2}c_1$$

$$\beta c_2 - \sqrt{2}c_3 = 0 \Rightarrow c_2 = \sqrt{2}c_3$$

$$\therefore c_1 = c_3$$

and, $\psi_2 \sim \phi_1 + \phi_3 + \sqrt{2}\phi_2$

normalize

$$\psi_2 = \frac{1}{2}[\phi_1 + \sqrt{2}\phi_2 + \phi_3]$$

next, $E = \alpha - \sqrt{2}\beta$

$$\sqrt{2}\beta c_1 + \beta c_2 = 0 \Rightarrow c_2 = -\sqrt{2}c_1 \text{ and}$$

$$\psi_3 = \frac{1}{2}[\phi_1 - \sqrt{2}\phi_2 + \phi_3]$$

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}; S^{-1} = S^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

and after matrix multiplication

$$S^{-1}HS = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha+\sqrt{2}\beta & 0 \\ 0 & 0 & \alpha-\sqrt{2}\beta \end{pmatrix} \text{ amazing!}$$

2. Perturbed particle in a box

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L)$$

$$E_n^{(0)} = h^2 n^2 / 8mL^2$$

$$E_n^{(1)} = \langle n | H^{(1)} | n \rangle$$

$$E_n^{(2)} = \sum_{m \neq n} \frac{\langle n | H^{(1)} | m \rangle \langle m | H^{(1)} | n \rangle}{E_n^0 - E_m^0}$$

$$\psi_n = \psi_n^{(0)} + \psi_n^{(1)}, \quad \psi_n^{(1)} = -\sum_{m \neq n} |m\rangle \frac{\langle m | H^{(1)} | m \rangle}{E_m^{(0)} - E_n^{(0)}}$$

Since

$$H^{(1)} = u(x/L)$$

$$E_n^{(1)} = \langle n | H^{(1)} | n \rangle = \frac{1}{2} u$$

$$\langle n | H^{(1)} | m \rangle = \frac{4nm}{\pi^2 (n-m)^2 (n+m)^2} \left(\cos(n\pi)\cos(m\pi) - 1 \right)$$

$$E_n^{(0)} - E_m^{(0)} = (n^2 - m^2)\, h^2 / 8mL^2$$

$$E_n^{(2)} = \sum_{m \neq n} \frac{\left(\frac{4nm\,(\cos(n\pi)\cos(m\pi) - 1)}{\pi^2 (n-m)^2 (n+m)^2} \right)^2}{(n^2 - m^2)} \frac{u^2}{(h^2/8mL^2)}$$

etc.

3. Prove that $[L_x, L_y] = i\hbar L_z$

$$L_x = -i\hbar\left\{ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}, \quad L_y = -i\hbar\left\{ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}$$

$$L_x L_y f = (-i\hbar)^2\left\{ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}\left\{ z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right\}$$

$$= (-i\hbar)^2\left\{ y\frac{\partial f}{\partial x} + yz\frac{\partial^2 f}{\partial x \partial z} - z^2\frac{\partial^2 f}{\partial x \partial y} - xy\frac{\partial^2 f}{\partial z^2} + xz\frac{\partial^2 f}{\partial y \partial z}\right\}$$

$$L_y L_x f = (-i\hbar)^2\left\{ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}\left\{ y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right\}$$

$$= (-i\hbar)^2\left\{ yz\frac{\partial^2 f}{\partial x \partial z} - zz\frac{\partial^2 f}{\partial x \partial y} - xy\frac{\partial^2 f}{\partial z^2} + xz\frac{\partial^2 f}{\partial y \partial z} + x\frac{\partial f}{\partial y}\right\}$$

subtract od cancel terms

$$(L_x L_y - L_y L_x)f = (-i\hbar)^2\left\{ y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right\}$$

$$= (-i\hbar)\underbrace{(-i\hbar)\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)}_{= -L_z}$$

$\therefore\ [L_x, L_y]f = i\hbar L_z f$ for arbitrary f.