QM II Lecture # 10, week 5 ①

Phys 652 ~~[illegible]~~

Rotations in the spin space

Consider an arbitrary spin state $|\alpha\rangle$ in the case of $S=\frac{1}{2}$ particle (i.e. the basis is $\{|+\rangle, |-\rangle\}$)

$$|\pm\rangle \equiv |\underset{\uparrow\atop s}{\tfrac{1}{2}}, \underset{\uparrow\atop m_s}{\pm\tfrac{1}{2}}\rangle$$

$$|+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} \; ; \; |-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix} \; ; \; |\alpha\rangle \doteq \begin{pmatrix}\langle +|\alpha\rangle \\ \langle -|\alpha\rangle\end{pmatrix}$$

2-component spinor

How does $|\alpha\rangle$ change upon rotation?

$\Downarrow$

apply rotation operator

$\Downarrow$

$$\underset{\substack{\uparrow \\ \text{spin space}}}{{}^{(s)}}\mathcal{D}(\vec{n},\varphi) = e^{-\frac{i}{\hbar}\varphi\vec{S}\cdot\vec{n}} \underset{\substack{\uparrow \\ s=\frac{1}{2}}}{\doteq} \underbrace{e^{-i\frac{\varphi}{2}\hat{\vec{\sigma}}\cdot\vec{n}}}_{\substack{2\times 2 \text{ matrix} \\ \text{representation of} \\ {}^{(s)}\mathcal{D}(\vec{n},\varphi)}} =$$

Pauli matrices

$$\underset{\substack{\uparrow \\ \text{HW! (\#7)}}}{=} I\cos\frac{\varphi}{2} - i\,(\hat{\vec{\sigma}}\cdot\vec{n})\sin\frac{\varphi}{2} =$$

$$\hat{\vec{\sigma}}\cdot\vec{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z = \begin{pmatrix}0&1\\1&0\end{pmatrix} n_x + \begin{pmatrix}0&-i\\i&0\end{pmatrix} n_y +$$

$$+\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} n_z = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\varphi}{2} & 0 \\ 0 & \cos\frac{\varphi}{2}\end{pmatrix} - i\sin\frac{\varphi}{2}\begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z\end{pmatrix} =$$

$$= \begin{pmatrix} \cos\frac{\varphi}{2} - i n_z \sin\frac{\varphi}{2} & -i\sin\frac{\varphi}{2}(n_x - i n_y) \\ -i\sin\frac{\varphi}{2}(n_x + i n_y) & \cos\frac{\varphi}{2} + i n_z \sin\frac{\varphi}{2}\end{pmatrix}$$

$$\Downarrow$$

$$\underset{1/2}{\overset{(s)}{\mathcal{D}}}(\vec{n}, \varphi)\,|\alpha\rangle \doteq e^{-i\frac{\varphi}{2}\hat{\vec{\sigma}}\cdot\vec{n}} \begin{pmatrix}\langle +|\alpha\rangle \\ \langle -|\alpha\rangle\end{pmatrix}$$

Consider ${}^{1/2}\mathcal{D}(\vec{n}, \varphi = 2\pi) \doteq \begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}$ for any $\vec{n}$

$$\Downarrow$$

$${}^{1/2}\mathcal{D}(\vec{n}, 2\pi)\,|\alpha\rangle = -|\alpha\rangle$$

$\Downarrow$ consequence of the half-integral value of spin!

This is in contrast to

${}^{(\vec{r})}\mathcal{D}(\vec{n}, 2\pi)\,|\psi\rangle = |\psi\rangle$ in the coordinate space

So, matrix representing a 2π-rotation (3)
in a coordinate space is the identity matrix I
while that in a spin-1/2 space is $-I$, and
the rotation by <u>4π</u> is represented by I.
Recall that rotations in a coordinate space
belong to $SO(3)$. The rotations in spin space
belong to $SU(2)$

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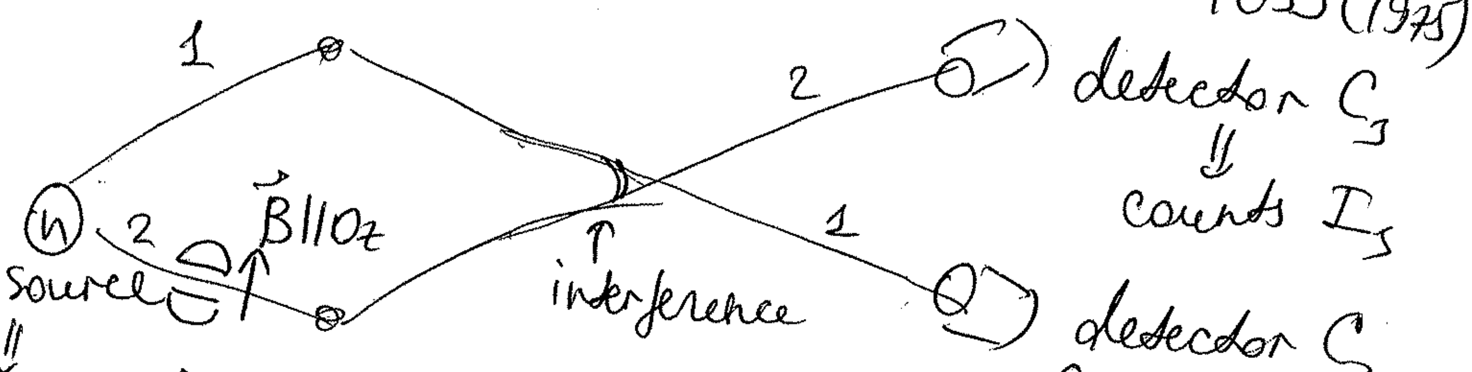
special ← S, unitary ← U

$|\det U| = 1$ ← unimodular

Can we show ${}^{(S)}_{1/2}\mathcal{D}(\vec{n}, 2\pi)|\alpha\rangle = -|\alpha\rangle$
experimentally?

yes !!

neutron interferometry, Werner et al., PRL 35
1053 (1975)



$|\alpha; t=0\rangle$

Say, $|\alpha; 0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ $\quad (S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle)$

Then, $|\alpha; t\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha, 0\rangle \Rightarrow$

Path 1 : $H = \frac{p^2}{2M}$ → doesn't act on spin variables

Path 2 : $H = \frac{p^2}{2M} - \vec{\mu}\cdot\vec{B}$

$-\vec{\mu}\cdot\vec{B} \underset{\vec{B} \parallel Oz}{=} -\frac{g_n e}{m_p c} S_z B$, where $g_n \approx -1.91$ ← neutron magnetic moment in nuclear magnetons

Then $|\alpha; t\rangle \underset{\text{path 2}}{=} e^{-\frac{i}{\hbar}\left(-\frac{g_n e}{m_p c}\right) S_z B t} |\alpha; 0\rangle =$

(neglect $\frac{p^2}{2M}$ part) since it doesn't act on $|\alpha\rangle$

$S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

$= e^{\frac{i}{\hbar}\omega S_z t} |\alpha; 0\rangle = \underline{\frac{1}{\sqrt{2}} e^{i\frac{\omega}{2}t}|+\rangle + \frac{1}{\sqrt{2}} \cdot e^{-i\frac{\omega}{2}t}|-\rangle}$

$\frac{g_n e}{m_p c} B \equiv \omega$ ← spin-precession frequency

change of the spin state in time in path 2

~~[illegible]~~

~~[illegible]~~

~~since $H = \frac{p^2}{2M}$ doesn't change spin state~~

Th

Interference $\Rightarrow$ combine $\underbrace{|\alpha;0\rangle}_{\text{path 1}}$ and $|\alpha$

$\Downarrow$ path

measure (since $H = \frac{p^2}{2m}$ doesn't change the spin state

$$\left| |\alpha;0\rangle + |\alpha;t\rangle \right|^2 \sim 2 + 2\,\mathrm{Re}\,\langle\alpha;0|\alpha;t$$

$$= 2\left(1 + \cos\frac{\omega}{2}t\right)$$

$$\frac{1}{2}e^{i\frac{\omega}{2}t} + \frac{1}{2}e$$

$$= \cos\frac{\omega}{2}t$$

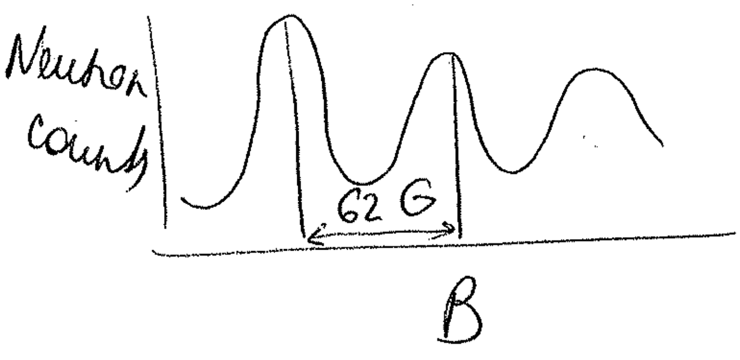
$\Downarrow$

Since $\omega \sim B$

$\Downarrow$

at fixed t, vary B $\Rightarrow$ observe oscillations

$\Downarrow$ Fig. 3



$\Rightarrow$ period corresponds to a complete perio of spin precession, i.e. 4 as predicted!!

HW: read the PRL & Sakurai 3.2