Solutions of HW #3

Problem #1

1s state implies that $n=1$, $l=m=0$ $\Rightarrow$

$$\Psi_{1s}(r,\theta,\varphi) = \underbrace{\frac{2}{a_0^{3/2}} e^{-r/a_0}}_{R_{10}(r)} \underbrace{\frac{1}{\sqrt{4\pi}}}_{Y_0^0(\theta,\varphi)}$$

Then, the probability to find the electron in the region $r_1 \le r \le r_2$ is:

$$P = \int_{r_1}^{r_2} |R_{10}(r)|^2 r^2 dr \underbrace{\int_0^{\pi}\int_0^{2\pi} |Y_0^0(\theta,\varphi)|^2 \sin\theta d\theta d\varphi}_{=1}$$

$$= \frac{4}{a_0^3}\int_{r_1}^{r_2} e^{-2r/a_0} r^2 dr \underset{\text{by parts}}{=} \frac{4}{a_0^3}\frac{-a_0}{2}\int_{r_1}^{r_2} r^2 de^{-2r/a_0} =$$

$$= -\frac{2}{a_0^2}\left[r^2 e^{-2r/a_0}\Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} e^{-2r/a_0} \cdot 2r dr \right] =$$

$$= -\frac{2}{a_0^2}\left[r^2 e^{-2r/a_0}\Big|_{r_1}^{r_2} + 2\cdot\frac{a_0}{2}\int_{r_1}^{r_2} r\, de^{-2r/a_0} \right] =$$

$$= -\frac{2}{a_0^2}\left[r^2 e^{-2r/a_0}\Big|_{r_1}^{r_2} + a_0 \left(r e^{-2r/a_0}\Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} e^{-2r/a_0}\, dr \right)\right] = -\frac{2}{a_0^2}\left[\left(r^2 + a_0 r + \frac{a_0^2}{2}\right) e^{-2r/a_0}\right]_{r_1}^{r_2}$$

Check: $r_1 = 0$, $r_2 = \infty \Rightarrow$

$$P = -\frac{2}{a_0^2}\left[0 - \frac{a_0^2}{2}\right] = 1 \quad \checkmark$$

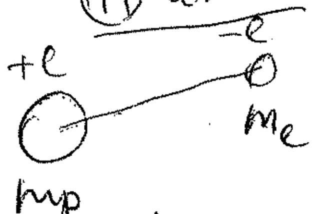
$r_1 = 0$, $r_2 = \frac{a_0}{2} \Rightarrow$

$$P = -\frac{2}{a_0^2}\left[\left(\frac{a_0^2}{4} + \frac{a_0^2}{2} + \frac{a_0^2}{2}\right) e^{-1} - \frac{a_0^2}{2}\right] =$$

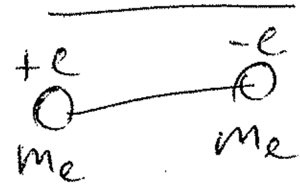
$$= 1 - \frac{5}{2e} \approx 0.08$$

Problem #2

(a) H atom



Positronium P



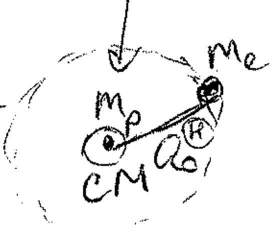
$\Downarrow$

$m_p \gg m_e$

$\Downarrow$

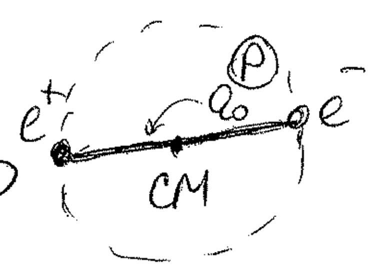
$\mu \approx m_e \Rightarrow$

center of mass (CM)



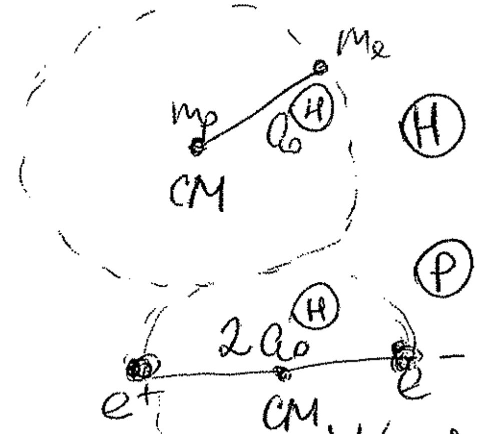
$\Downarrow$

$\mu \approx \frac{m_e}{2} \Rightarrow$



$a_0^{(H)}$ ← hydrogen atom

$$a_0^{(H)} = \frac{\hbar^2}{\mu e^2} \approx \frac{\hbar^2}{m_e e^2}$$



← positronium

$$a_0^{(P)} = \frac{\hbar^2}{\mu e^2} = \frac{2\hbar^2}{m_e e^2} = 2 a_0^{(H)}$$

Does it mean that the radius of positronium is twice that of the (H)-atom? ⇒ No!

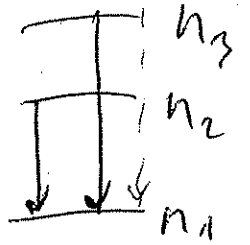
The Bohr radius a_0 gives an idea of the extension of the wave functions associated with the "relative" particle (Lecture #1) whose position $\vec{r} = \vec{r}_1 - \vec{r}_2$ is related to the distance between the two particles and not to the distance between them and the center of mass!

⇓

So, (H) and (P) are of equal size!

(b) Spectra in the visible wavelength region ⇒

Balmer series ⇒ $n_1 = 2$
and $n_2 = 3$
$n_3 = 4$



⇓

$$\frac{hc}{\lambda} = E_I\left(\frac{1}{n_1^2} - \frac{1}{n_{2,3,\dots}^2}\right) \Rightarrow$$

(4)

Consider transitions between the same levels n_1 and n_2 in H and P $\Rightarrow$

$$\lambda_{H} = \frac{hc}{E_I^{H}\left(\frac{1}{n_1^2} - \frac{1}{n_{2,3,\ldots}^2}\right)} \Rightarrow$$

$$\lambda_{P} = \frac{hc}{E_I^{P}\left(\frac{1}{n_1^2} - \frac{1}{n_{2,3,\ldots}^2}\right)}$$

$$\frac{\lambda_H}{\lambda_P} = \frac{E_I^{P}}{E_I^{H}} = \frac{1}{2} \Rightarrow \underline{\lambda_P = 2\lambda_H}$$

$$E_I = \frac{\mu e^4}{2\hbar^2}$$

expect absorption at 1312.4, 972.2, 868.0, and 820.2 nm $\Rightarrow$ near-infrared wavelengths

Problem #3

(a) He^+: $a_0(Z) \overset{\mu_H \approx \mu_{He^+}}{\approx} \frac{a_0^{H}}{Z} = \frac{a_0^{H}}{2} \Rightarrow$ smaller!

(b) μ-atom: $a_0 \approx \frac{\hbar^2}{m_\mu e^2} \approx \frac{a_0^{H}}{200} \Rightarrow$ much smaller!

$$\mu = \frac{m_p m_\mu}{m_p + m_\mu}, \quad \frac{m_p}{m_\mu} > 9 \Rightarrow \mu \approx m_\mu$$

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(3.17 in gray) red edition

Problem #3 (Sakurai 3.15)

$\psi(\vec{x}) = (x+y+3z) f(r)$

(a) rewrite $\psi(\vec{x})$ in terms of Y_e^m $\Rightarrow$

$\sin\theta e^{i\varphi} = -\sqrt{\frac{8\pi}{3}} Y_1^1$

$\sin\theta e^{-i\varphi} = \sqrt{\frac{8\pi}{3}} Y_1^{-1}$ $\Longrightarrow$

$\cos\theta = \sqrt{\frac{4\pi}{3}} Y_1^0$

$\frac{x}{r} = \sin\theta\cos\varphi = \sqrt{\frac{8\pi}{3}} \cdot \frac{1}{2} \cdot (Y_1^{-1} - Y_1^1)$;

$\frac{y}{r} = \sin\theta \sin\varphi = -\sqrt{\frac{8\pi}{3}} \cdot \frac{1}{2i} \cdot (Y_1^1 + Y_1^{-1})$

$\Downarrow$

$$\psi(\vec{x}) = r f(r) \cdot \left[\sqrt{\frac{2\pi}{3}} (1+i) Y_1^{-1} + \sqrt{\frac{2\pi}{3}} (i-1) Y_1^1 + 2\sqrt{3\pi} Y_1^0 \right]$$

Obviously, $\vec{L}^2 \psi = \hbar^2 \cdot \underset{l}{1} \cdot (\underset{l}{1}+1) \underbrace{\psi}_{= \sum_m C_m Y_1^m}$ ← eigenfuncti

$\underline{l = 1}$

(b) $P_{m=\pm 1} = \frac{\frac{2\pi}{3} \cdot 2}{\frac{2\pi}{3} \cdot 2 + \frac{2\pi}{3} \cdot 2 + 4 \cdot 3\pi} = \frac{1}{11}$; $P_{m=0} = \frac{4 \cdot 3\pi}{\frac{44\pi}{3}} = \frac{9}{11}$

(c) $H\Psi(\vec{x}) = E\Psi(\vec{x})$

From (a) $\Rightarrow$ $\Psi(\vec{x}) = \underbrace{r f(r)}_{R(r)} \sum_{m=0,\pm 1} C_m Y_1^m(\theta,\varphi)$

$u(r) = R(r)\cdot r = r^2 f(r)$

$\Updownarrow$ Lecture #5

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + \underbrace{V_{eff}(r)}_{=V(r)+\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\overset{\ell=1}{=}V(r)+\frac{\hbar^2}{\mu r^2}} u(r) = E\,u(r)$$

$$-\frac{\hbar^2}{2\mu}\underbrace{\frac{d^2}{dr^2}\left(r^2 f(r)\right)}_{=\cancel{2f(r)}+4rf'(r)+r^2f''(r)} + \left[V(r) + \cancel{\frac{\hbar^2}{\mu r^2}} - E\right] r^2 f(r) = 0$$

$$-\frac{\hbar^2}{2\mu}\left(4rf'(r) + r^2 f''(r)\right) + \left[V(r) - E\right] r^2 f(r) = 0$$

$$V(r) = E + \frac{\hbar^2}{2\mu}\left(\frac{4}{r}\frac{f'(r)}{f(r)} + \frac{f''(r)}{f(r)}\right)$$