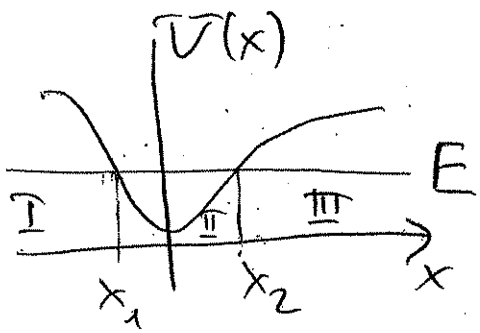
QM I
Phys 651

Lecture #25

The WKB method

Recall:



$$\psi_I^{WKB}(x) = \frac{C_1}{\sqrt[4]{2m(V(x)-E)}} \cdot \exp\left[-\frac{1}{\hbar}\int_x^{x_1}\sqrt{2m(V(x')-E)}\,dx'\right],$$

$x < x_1$

$$\psi_{II}^{WKB}(x) = \frac{C_2}{\sqrt[4]{2m(E-V(x))}}\exp\left[\frac{i}{\hbar}\int_x \sqrt{2m(E-V(x'))}\,dx'\right] +$$

$$+\frac{\tilde{C}_2}{\sqrt[4]{2m(E-V(x))}}\exp\left[-\frac{i}{\hbar}\int_x \sqrt{2m(E-V(x'))}\,dx'\right], \quad x_1 < x < x_2$$

$$\psi_{III}^{WKB}(x) = \frac{C_3}{\sqrt[4]{2m(V(x)-E)}}\exp\left[-\frac{1}{\hbar}\int_{x_2}^{x}\sqrt{2m(V(x)-E)}\,dx'\right], \quad x > x_2$$

Near x_1, x_2: $-\frac{\hbar^2}{2m}\psi'' + \left.\frac{dV}{dx}\right|_{x_{1,2}}(x-x_{1,2})\psi = 0$

$\Rightarrow$ Airy functions

Then $\Rightarrow$ match boundary conditions

Summary of WKB;

1) If there are <u>no</u> rigid walls (i.e. no $V \to \infty$) $\Rightarrow$ allowed energies for bound states can be found from

$$\oint \underbrace{\sqrt{2m(E_n - V(x'))}}_{= p(x')}\, dx' = 2\int_{x_1}^{x_2} p(x')\, dx' = h\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, 3$$

x_1, x_2: turning points

$$\int_{x_1}^{x_2} \sqrt{2m(E_n - V(x))}\, dx = \pi\hbar\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, 3 \qquad (26.1)$$

similar to "old quantum theory" due to Sommerfeld-Wilson $\Rightarrow \oint p\, dq = nh$

2) If there is <u>one</u> rigid wall $\Rightarrow$

$$\int_{x_1}^{x_2} \sqrt{2m(E_n - V(x))}\, dx = \pi\hbar\left(n + \frac{3}{4}\right), \quad n = 0, 1, 2, 3 \qquad (26.2)$$

3) If there are <u>two</u> rigid walls $\Rightarrow$

$$\int_{x_1}^{x_2} \sqrt{2m(E_n - V(x))}\, dx = \pi\hbar\,(n + 1), \quad n = 0, 1, 2, \ldots \qquad (26.3)$$

Note: (26.1) - (26.3) are identical in the limit of large n's, which is where the semiclassical approximation is most accurate.

Examples

1) Use the WKB method to estimate the energy levels of a 1D harmonic oscillator.

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Find the turning points: $E = \frac{m\omega^2 x_{1,2}^2}{2} \Rightarrow$

$$x_{1,2} = \pm\sqrt{\frac{2E}{m\omega^2}}$$

Use (26.1) $\Rightarrow$ $\int_{x_1}^{x_2} \sqrt{2m\left(E - \frac{m\omega^2 x^2}{2}\right)}\, dx = \pi\hbar\left(n + \frac{1}{2}\right)$

$$\overset{\parallel}{2\int_0^{\sqrt{\frac{2E}{m\omega^2}}} \sqrt{2m\left(E - \frac{m\omega^2 x^2}{2}\right)}\, dx} = 2\cdot\sqrt{2mE}\,\cdot$$

$$\int_0^{\sqrt{\frac{2E}{m\omega^2}}} \sqrt{\underbrace{1 - \frac{m\omega^2}{2\cdot 2mE}x^2}_{\cos^2\theta}}\, dx = \frac{\pi E \cdot \cancel{2m\omega}}{\cancel{2m\omega^2}} \Rightarrow$$

$$\underline{E_n^{WKB} = \hbar\omega\left(n + \frac{1}{2}\right)} \Leftrightarrow \text{yields an } \underline{\text{exact}} \text{ answer!}$$

2) Use the WKB approximation to calculate the energy levels of a spinless particle of mass m moving in a 1D box with walls at $x=0$ and $x=L$.

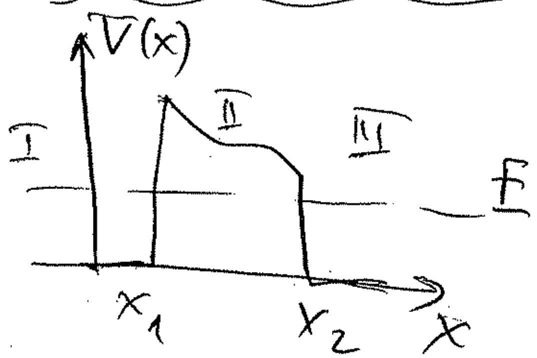
$$\int_0^L \sqrt{2mE}\, dx = L\sqrt{2mE} \underset{(26.3)}{=} \pi\hbar(n+1) \qquad (n=0,1,2\ldots)$$

$$\Downarrow$$

$$E_n^{WKB} = \frac{\pi^2\hbar^2 n^2}{2mL^2} \leftarrow 1,2,3,\ldots$$

$\leftarrow$ exact result

Tunneling through a potential barrier



Consider particle incident from left with $p_0 = \sqrt{2mE}$, $E < V$ in II

$$\Downarrow$$

$$\Psi_I(x) = Ae^{i\frac{p_0 x}{\hbar}} + Be^{-i\frac{p_0 x}{\hbar}}$$

$$\Psi_{III} = Ce^{\frac{ip_0 x}{\hbar}}$$

Ψ_{II} - ? $\Rightarrow$ use WKB $\Rightarrow$

$$\Psi_{II} = \frac{D}{\sqrt{2m(V(x)-E)}} \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x}\sqrt{2m(V(x')-E)}\,dx'\right] +$$

$$+ \frac{\tilde{D}}{\sqrt{2m(V(x)-E)}} \exp\left[\frac{1}{\hbar}\int_{x_1}^{x}\sqrt{2m(V(x')-E)}\,dx'\right]$$

Transmission $\Rightarrow$ $T = \dfrac{v_{trans}}{v_{inc}} \dfrac{|\psi_{III}(x)|^2}{|\psi_I(x)|^2} = \dfrac{|C|^2}{|A|^2}$

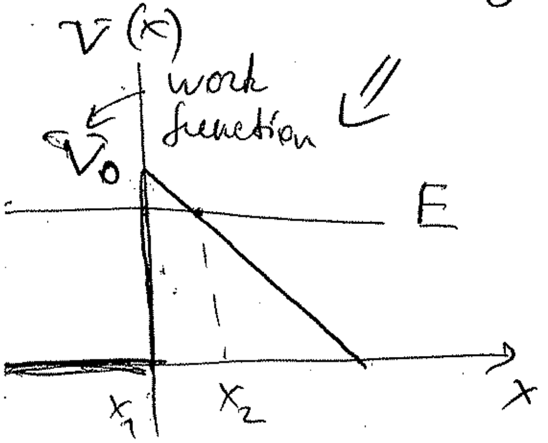
For a <u>wide</u> barrier ($\tilde{D} \simeq 0$) $\Rightarrow$

$$T \sim e^{-2\gamma}, \quad \gamma = \frac{1}{\hbar}\int_{x_1}^{x_2} \sqrt{2m(V(x)-E)}\,dx$$

useful for quick estimates

<u>Example</u> Cold emission

Use WKB to estimate the transmission coefficient of a particle of mass m and energy E moving in a following $V(x)$:



$$V(x) = \begin{cases} 0, & x<0 \\ V_0 - \underbrace{\lambda}_{e\mathcal{E} \leftarrow \text{external electric field}} x, & x>0 \end{cases}$$

$x_1 = 0$

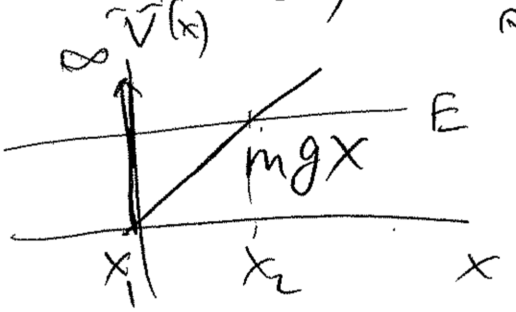
$V_0 - \lambda x_2 = E \Rightarrow x_2 = \dfrac{V_0 - E}{\lambda}$

$$\gamma = \frac{1}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\,dx = \frac{1}{\hbar}\int_0^{\frac{V_0-E}{\lambda}}\sqrt{2m}\cdot\sqrt{V_0-\lambda x-E}\,dx = \underline{\frac{2\sqrt{2m}}{3\hbar\lambda}(V_0-E)^{3/2}}, \quad T \sim e^{-2\gamma}$$

Example from Sakurai (p. 108) $\Rightarrow$

Find energy levels of a particle in $V(x) \Rightarrow$

$$V(x) = \begin{cases} mgx, & x>0 \\ \infty, & x<0 \end{cases} \Rightarrow$$



turning points: $x_1 = 0$

$x_2 = \dfrac{E}{mg}$

Use (26.2) $\Rightarrow$

$$\int_0^{E/mg} \sqrt{2m(E-mgx)}\,dx = \sqrt{2mE}\int_0^{E/mg}\sqrt{1-\underbrace{\frac{mg}{E}x}_{y}}\,dx =$$

$$dy = \frac{mg}{E}dx$$

$$= \sqrt{2mE}\,\frac{E}{mg}\underbrace{\int_0^1\sqrt{1-y}\,dy} = \frac{E^{3/2}\sqrt{2}}{\sqrt{m}\,g}\cdot\frac{2}{3} = \pi\hbar\left(n+\frac{3}{4}\right)$$

$$\frac{(1-y)^{3/2}}{3/2}\Bigg|_1^0 = \frac{2}{3}$$

$$E = \left[\frac{3}{2}\,\frac{g}{\sqrt{2}}\,\sqrt{m}\,\pi\hbar\left(n+\frac{3}{4}\right)\right]^{2/3} = \left(mg^2\hbar^2\right)^{1/3}\cdot\left(\frac{3\left(n+\frac{3}{4}\right)\pi}{2}\right)^{2/3}$$

$n = 0, 1, 2, 3, \ldots$