QM
Phys651

Lecture # 16

Time evolution of expectation values

Consider an observable A and its expectation value in the (normalized) state $|\Psi\rangle$:

$$\langle A \rangle = \langle \Psi | A | \Psi \rangle$$

$$\frac{d}{dt}\langle A \rangle = \left\langle \frac{\partial \Psi}{\partial t} \right| A \left| \Psi \right\rangle + \left\langle \Psi \right| \frac{\partial A}{\partial t} \left| \Psi \right\rangle + \left\langle \Psi \right| A \left| \frac{\partial \Psi}{\partial t} \right\rangle = \langle (i\hbar)^{-1} H\Psi | A | \Psi \rangle +$$

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

$$+ \left\langle \Psi \right| \frac{\partial A}{\partial t} \left| \Psi \right\rangle + \langle \Psi | A | (i\hbar)^{-1} H \Psi \rangle =$$

$$= \left\langle \Psi \right| \frac{\partial A}{\partial t} \left| \Psi \right\rangle - (i\hbar)^{-1} \langle \Psi | \underbrace{H}_{= H^+} A | \Psi \rangle + \langle \Psi | A H | \Psi \rangle \cdot (+i\hbar)^{-1} = \left\langle \Psi \right| \frac{\partial A}{\partial t} \left| \Psi \right\rangle + (i\hbar)^{-1} \langle \Psi | [A, H] | \Psi \rangle$$

$$= \left\langle \frac{\partial A}{\partial t} \right\rangle - \frac{i}{\hbar} \langle [A, H] \rangle \implies$$

If A does not depend explicitly on time $\Rightarrow$ (2)

$$\frac{d}{dt}\langle A\rangle = -\frac{i}{\hbar}\langle [A,H]\rangle \Rightarrow \text{if } [A,H]=0$$

$\Downarrow$

A is a constant of the motion

Example: Let $A = H \neq H(t)$

$\Downarrow$

$$\frac{d}{dt}\langle H\rangle = -\frac{i}{\hbar}\langle \underbrace{[H,H]}_{=0}\rangle = 0 \Rightarrow$$ total energy is a constant of motion

analog of energy conservation in classical mechanics!

Note:

Consider an arbitrary observable B such as $[B,H]\neq 0$ (generally)

Now find the expectation value of B in the state $|\varphi_k\rangle$ (at $t=0$), which is an eigenstate of H:

$$H|\varphi_k\rangle = E_k|\varphi_k\rangle$$

$$\langle\varphi_k|B|\varphi_k(t)\rangle\Big|_t = \langle\varphi_k|e^{\frac{iE_k t}{\hbar}}Be^{-\frac{i}{\hbar}E_k t}|\varphi_k\rangle =$$

$$|\varphi_k(t)\rangle = |\varphi_k(0)\rangle e^{-\frac{i}{\hbar}E_k t}$$

$$= \langle\varphi_k|B|\varphi_k\rangle \Rightarrow$$ the expect. value with respect to the energy eigenstate is time-independent

$\langle [B,H]\rangle = 0$ in the stationary st... $\Leftarrow$

$\Rightarrow$ energy eigenstate is a stationary state

For an arbitrary state $|\alpha, t_0=0\rangle = \sum_n C_n |\varphi_n\rangle$

$$\langle B \rangle = \Big[\sum_n \sum_m C_m^*(0) e^{\frac{i}{\hbar}E_m t} \langle \varphi_m | B | \varphi_n \rangle$$

$$|\alpha, t_0=0; t\rangle = \sum_n C_n(0) e^{-\frac{i}{\hbar}E_n t} |\varphi_n\rangle$$

$$\langle \alpha, t_0=0; t| = \sum_m C_m^*(0) e^{\frac{i}{\hbar}E_m t} \langle \varphi_m|$$

$$\cdot\, C_n(0) e^{-\frac{i}{\hbar}E_n t}\Big] = \sum_{m,n} C_m^*(0) C_n(0) \langle \varphi_m | B | \varphi_n \rangle \cdot e^{-\frac{i}{\hbar}(E_n - E_m)t}$$

in a general case of a nonstationary state $\Rightarrow$ $\langle B \rangle$ is a function of time

a collection of terms oscillating with Bohr frequencies $\omega_{nm} = \dfrac{E_n - E_m}{\hbar}$ $\Leftarrow$

The Virial theorem

Consider a particle of mass m moving in a potential $V(\vec{r})$ $\Rightarrow$ $H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$

Consider a time-independent operator $A = \vec{r}\cdot\vec{p}$

Then, for the stationary states of H $\Rightarrow$

$$\frac{d}{dt}\langle A \rangle = \frac{d}{dt}\langle \varphi_E | A | \varphi_E \rangle = 0 \quad \Rightarrow$$

$$H|\varphi_E\rangle = E|\varphi_E\rangle \qquad \langle [A, H] \rangle = 0$$

Let's find $[A, H] = [\vec{r}\cdot\vec{p}, H] = [\vec{r}\cdot\vec{p}, \frac{\vec{p}^{\,2}}{2m} + V]$

$$= \left[xp_x + yp_y + zp_z, \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V(x,y,z) \right] =$$

$$= 2i\hbar T - i\hbar \vec{r}\cdot\vec{\nabla}V, \text{ where } T = \frac{\vec{p}^{\,2}}{2m}$$

$\leftarrow$ kinetic energy operator

$\uparrow$ show at Homework!

So, for a stationary state $\Rightarrow$ $2\cancel{i\hbar}\langle T \rangle = \cancel{i\hbar}\cdot\langle \vec{r}\cdot\vec{\nabla}V \rangle$

$$\boxed{2\langle T \rangle = \langle \vec{r}\cdot\vec{\nabla}V \rangle}$$

virial theorem

Comp. exam – September 2005 problem

Consider a quantum system in 1D with a time-independent potential $V(x)$. The system is described by a wave function $\Psi(x,t)$, which doesn't have to be an eigenstate. Consider the expectation value $\langle xp \rangle (t)$. Derive a relation between $\frac{d}{dt}\langle xp\rangle(t)$, $\langle T \rangle$ and a term that depends on $V(x)$.

$$\frac{d}{dt}\langle xp \rangle = -\frac{i}{\hbar}\langle [xp, H] \rangle$$

$$H = \frac{p^2}{2m} + V(x)$$

$$[xp, H] = x[p, H] + [x, H]p$$

- $[p, H] = \left[p, \frac{p^2}{2m} + V(x)\right] = [p, V(x)] = ?$

$$\left[-i\hbar\frac{d}{dx}, V(x)\right] f(x) = -i\hbar\frac{d}{dx}(Vf) + V(x)\cdot i\hbar\frac{df}{dx} =$$

$$= -i\hbar(V'f + Vf') + i\hbar Vf' = -i\hbar\frac{dV}{dx}f(x) \Rightarrow$$

$$[p, V(x)] = -i\hbar\frac{dV}{dx}$$

⑤

- $[x, H] = [x, \frac{p^2}{2m} + V(x)] = [x, \frac{p^2}{2m}] =$

$$= \frac{1}{2m}\left(\underbrace{[x,p]}_{i\hbar}p + p\underbrace{[x,p]}_{i\hbar}\right) = \frac{i\hbar}{m}p$$

So, $[xp, H] = x\cdot\left(-i\hbar\frac{dV}{dx}\right) + \frac{i\hbar}{m}p^2$

Then, $\frac{d}{dt}\langle xp\rangle = \langle [xp, H]\rangle \cdot \frac{-i}{\hbar} =$

$$= \underbrace{\left\langle \frac{p^2}{m}\right\rangle}_{2\langle T\rangle} - \left\langle x\frac{dV}{dx}\right\rangle$$

Note: if $\Psi(x,t)$ which describes the state of the system were an eigenstate of $H \Rightarrow$ then $\frac{d}{dt}\langle xp\rangle = 0$

Consider a potential $V(x) = V_n x^n$ and assume that the system is in an eigenstate j with energy E_j. Find the expectation value of the potential in this state $V_j = \langle V\rangle_j$

Since we are in an eigenstate $\Rightarrow$

$$\frac{d}{dt}\langle xp\rangle = 0 \Rightarrow 2\langle T\rangle_j - \langle x\frac{dV}{dx}\rangle_j = 0$$

$$2\langle T\rangle_j = \langle x \cdot n V_n x^{n-1}\rangle_j = \langle n V_n x^n\rangle_j =$$

$$= \langle n V(x)\rangle_j$$

Since $E_j = \langle T\rangle_j + \langle V\rangle_j \Rightarrow$

$$2(E_j - \langle V\rangle_j) = n\langle V_{\cancel{j}}\rangle_j \Rightarrow$$

$$\boxed{\langle V\rangle_j = \frac{2E_j}{n+2}}$$

Comp. exam problem in 10 min $\Rightarrow$ solved!

Comprehensive exam, September 2005

Problem 7.

Consider a quantum system in one dimension, with a time independent potential $V(x)$. The system is described by a wave function $\psi(x,t)$, which does not have to be an eigenstate. Consider the expectation value of the product of position and momentum for this system, i.e.. $< xp > (t)$, as a function of time. The quantum virial theorem relates the time derivative of this quantity, $\frac{d}{dt} < xp > (t)$, to expectation values of the kinetic energy and a term which depends on the potential. Derive such a relation.

Consider a potential $V(x) = V_n x^n$, and assume that the system is in an eigenstate j with energy E_j. Show that in this case the expectation value of the potential is given by $\frac{2}{n+2}E_j$. One may assume that n is a positive, even number and that $V_n > 0$.

Problem 8.

One way to attempt nuclear fusion is to use magnetic confinement to raise the pressure of a hot plasma. Consider a modest model of this process, consisting of only a very thin-walled, hollow conducting tube of radius R through which current I is driven. When I is sufficiently large, the tube can be crushed.

1. For a tube oriented along $\hat{z}$, find $\vec{B}(\rho, \phi, z)$ inside and outside the tube.
2. Determine the inward pressure on the tube. One approach to this problem is to orient the tube along $\hat{z}$ and calculate the total force in the $\hat{x}$ direction on one side of the tube.
3. How does the pressure change as the tube collapses?