Lecture #11

Translation in space

Consider an initial state $|\vec{x}'\rangle$, which we want to transform into $|\vec{x}' + \underbrace{d\vec{x}'}_{\text{infinitesimal}}\rangle \Rightarrow$

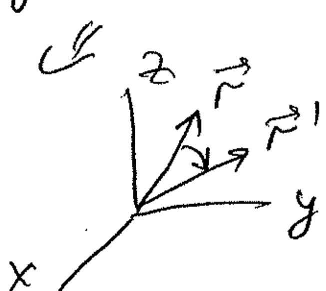
$$\hat{U}_{d\vec{x}'} = \hat{I} - i\, d\vec{x}' \cdot \hat{\vec{G}}, \quad \hat{U}_{d\vec{x}} |\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle$$

$$\vec{G} = \frac{\vec{P}}{\hbar} \leftarrow \text{momentum operator}$$

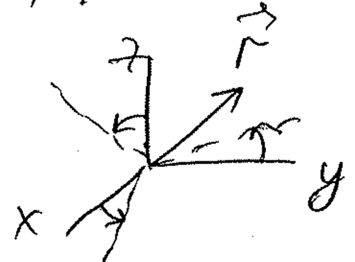
Obviously, $|\vec{x}'\rangle$ is <u>not</u> an eigenket of $\hat{U}_{d\vec{x}'}$

<u>Note:</u> Transformation of a state vector itself or operators is an <u>active</u> transformation, whereas transformation of a coordinate system is a <u>passive</u> transformation.

<u>Example</u>



$\vec{r} \to \vec{r}' \Rightarrow$ active
(x y z fixed)



$\vec{r}$ fixed
$x\,y\,z \to x'\,y'\,z' \Rightarrow$ passive

Here we will stick with <u>active</u> transformation

Consider successive translations $\Rightarrow$

$$\hat{U}_{d\vec{x}''}\,\hat{U}_{d\vec{x}'} = \left(\hat{I} - \frac{i}{\hbar}\vec{P}\cdot d\vec{x}''\right)\left(\hat{I} - \frac{i}{\hbar}\vec{P}\cdot d\vec{x}'\right) =$$

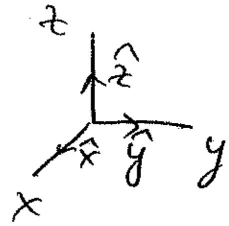
$$\underset{\uparrow}{=} \hat{I} - \frac{i}{\hbar}\vec{P}\cdot\left(d\vec{x}' + d\vec{x}''\right) = \hat{U}_{d\vec{x}'+d\vec{x}''}$$

neglect terms with $d\vec{x}'\cdot d\vec{x}''$

Do successive transformations commute? $\Rightarrow$

Ex. $\underbrace{\hat{U}_{d\vec{x}=dx}}_{=\Delta x\cdot\hat{x}}\ \hat{U}_{\underset{=\Delta y\cdot\hat{y}}{dy}} = \hat{U}_{dy}\,\hat{U}_{dx}$?

$\uparrow$ unit vector along x



$$[\underbrace{\hat{U}_{dx}}_{=\hat{I}-\frac{i}{\hbar}P_x\cdot\Delta x},\ \underbrace{\hat{U}_{dy}}_{=\hat{I}-\frac{i}{\hbar}P_y\Delta y}] = \overbrace{[\hat{I},\hat{I}]}^{=0} + \left(\frac{i}{\hbar}\right)^2[P_x\Delta x, P_y\Delta y] - \frac{i}{\hbar}\underbrace{[P_x\Delta x, \hat{I}]}_{=0} - \dots =$$

$$= -\frac{1}{\hbar^2}\underbrace{[P_x, P_y]}_{=0}\Delta x\Delta y = 0$$

$\leftarrow [P_i, P_j] = 0$

$\Rightarrow$

So, the fact that successive transformations commute is directly related to the fact that $[P_i, P_j] = 0$! $\Rightarrow$ comes from the fact that translation group is Abelian.

It's not the case for rotations, since $\hat{G} \sim \vec{J}$,
angular momentum

and $[J_i, J_j] = i\hbar J_k \neq 0$

$\Downarrow$

rotations about different axes do not commute!

$\Downarrow$

Phys 652

Since $[P_i, P_j] = 0 \Rightarrow P_x, P_y, P_z$ are mutually compatible observables

$\Downarrow$

form $|\vec{p}\,'\rangle = |p_x', p_y', p_z'\rangle$
(common basis)

$\Downarrow$

$$P_{\substack{x\\y\\z}} |\vec{p}\,'\rangle = p_{\substack{x\\y\\z}} |\vec{p}\,'\rangle$$

Is $|\vec{p}\,'\rangle$ an eigenket of $\hat{U}_{d\vec{x}'}$? $\Rightarrow$

$$\hat{U}_{d\vec{x}'} |\vec{p}\,'\rangle = (\hat{I} - \tfrac{i}{\hbar} \vec{P} \cdot d\vec{x}') |\vec{p}\,'\rangle =$$

$\vec{P}|\vec{p}\,'\rangle = \vec{p}\,|\vec{p}\,'\rangle$

$= (1 - \frac{i}{\hbar}\vec{p}\cdot d\vec{x}')|\vec{p}'\rangle \Rightarrow$

$|\vec{p}'\rangle$ is an eigenket of $\hat{\mathcal{T}}_{d\vec{x}'}$

Wave functions in position and momentum space

1) Wave function in position space

$$\psi_\alpha(x') = \langle x'|\alpha\rangle$$

2) Express $\langle\beta|\alpha\rangle$ via wave functions:

$$\langle\beta|\alpha\rangle = \int dx' \langle\beta|x'\rangle\langle x'|\alpha\rangle =$$

$$= \int dx' \psi_\beta^*(x')\psi_\alpha(x')$$

3) Expansion in terms of eigenfunctions of $A \Rightarrow$

$\psi_\alpha(x') = \sum_n c_n \varphi_n(x)$, where $A\varphi_n(x) = a_n\varphi_n(x)$

4) $\langle\beta|A|\alpha\rangle = \int dx' \int dx'' \langle\beta|x'\rangle\langle x'|A|x''\rangle \cdot$

$\cdot \langle x''|\alpha\rangle = \int dx' \int dx'' \psi_\beta^*(x') \underbrace{\langle x'|A|x''\rangle}_{\text{matrix element}}$

$\cdot \psi_\alpha(x'')$

Example: $A = X^2$, $\langle x' | X^2 | x'' \rangle$?

$$\langle x' | X^2 | x'' \rangle = \langle x' | x''^2 | x'' \rangle =$$

$$= x''^2 \delta(x' - x'') = x'^2 \delta(x' - x'')$$

Then, $\langle \beta | X^2 | \alpha \rangle = \int dx' \int dx'' \, \psi_\beta^*(x') \cdot$

$$\cdot \, x'^2 \delta(x' - x'') \psi_\alpha(x'') = \int dx' \, \psi_\beta^*(x') x'^2 \psi_\alpha(x')$$

In general, $\langle \beta | \underbrace{f(X)}_{\text{function of the operador } X} | \alpha \rangle =$

$$= \int dx' \, \psi_\beta^*(x') \cdot \underbrace{f(x')}_{\text{function of the eigenvalue}} \psi_\alpha(x')$$