QM
Phys 651

Lecture #10

Infinitesimal and finite unitary transformati

Recall: $\hat{U}^+ = \hat{U}^{-1}$

If $|\Psi'\rangle = \hat{U}|\Psi\rangle \Rightarrow \hat{A}' = \hat{U}\hat{A}\hat{U}^+$

$(|\Psi\rangle = \hat{U}^+|\Psi'\rangle)$ $\quad \hat{A} = \hat{U}^+\hat{A}'\hat{U}$

unitary equivalent observables

Consider $\hat{U}_\varepsilon(G) = \hat{I} + i\varepsilon\hat{G}$

identity operator — infinitely small real number — generator of the infinitesimal transformatio

Under what conditions is $\hat{U}_\varepsilon(G)$ a unitary transformatio

$$\hat{U}_\varepsilon \hat{U}_\varepsilon^+ = (\hat{I} + i\varepsilon\hat{G})(\hat{I} - i\varepsilon\hat{G}^+) = \hat{I} + i\varepsilon(\hat{G} - \hat{G}^+)$$

$$+ \underbrace{\varepsilon^2 \hat{G}\hat{G}^+}_{O(\varepsilon^2)} \approx \hat{I} + i\varepsilon(\hat{G} - \hat{G}^+) = \hat{I}$$

should be for a unitary operator

$\hat{G} = \hat{G}^+$ ← Hermitian!

Then, the transformation of a state vector $|\Psi\rangle$ is: $|\Psi'\rangle = (\hat{I} + i\varepsilon\hat{G})|\Psi\rangle =$

$$= |\Psi\rangle + i\varepsilon\hat{G}|\Psi\rangle$$

The transformation of an operator $\hat{A}$ is:

$$\hat{A}' = \hat{U}\hat{A}\hat{U}^+ = (\hat{I} + i\varepsilon\hat{G})\hat{A}(\hat{I} - i\varepsilon\hat{G}) =$$

$$= \hat{A} + i\varepsilon[\hat{G},\hat{A}] + O(\varepsilon^2) \approx \underline{\hat{A} + i\varepsilon[\hat{G},\hat{A}]}$$

If $[\hat{G},\hat{A}] = 0 \Rightarrow \underline{\hat{A}' = \hat{A}}$

Now construct a <u>finite</u> unitary transformation as a succession of infinitesimal transformations in steps of ε. Introduce a finite parameter α, so that $\varepsilon = \frac{\alpha}{N}$ ← real number

↖ large integer

$$\hat{U}_\alpha(\hat{G}) = \underbrace{\hat{U}_\varepsilon(\hat{G})\,\hat{U}_\varepsilon(G)\ldots\hat{U}_\varepsilon(G)}_{N \text{ times}} =$$

$$= \lim_{N\to\infty} \prod_{K=1}^{N} \left(1 + i\frac{\alpha}{N}\hat{G}\right) = \lim_{N\to\infty}\left(1 + i\frac{\alpha}{N}\hat{G}\right)^N = e^{i\alpha\hat{G}}$$

recall from math

$\lim (1+\frac{a}{n})^n = e^a$

Is $\hat{U}_\alpha(\hat{G})$ unitary? $\Rightarrow$ $(e^{i\alpha\hat{G}})^\dagger = e^{-i\alpha\hat{G}} = (e^{i\alpha\hat{G}})^{-1}$ ✓

(Since α is real and $\hat{G}$ is Hermitian)

The transformation of $\hat{A}$:

$$\hat{A}' = \hat{U}\hat{A}\hat{U}^\dagger = e^{i\alpha\hat{G}}\hat{A}e^{-i\alpha\hat{G}} = \hat{A} + i\alpha[\hat{G},\hat{A}] + \frac{(i\alpha)^2}{2!}[\hat{G},[\hat{G},\hat{A}]] + \ldots$$

recall HW # ~~5~~, Problem # ~~3~~

Similar to the case of infinitesimal transformations,

if $[\hat{G},\hat{A}] = 0 \Rightarrow \underline{\hat{A}' = \hat{A}}$

Applications of unitary transformations

- Time translations: $\hat{G} = -\frac{\hat{H}}{\hbar}$ ← Hamiltonian

$$\underbrace{\hat{U}_{\delta t}}_{\text{infinitesimal increase in time}}(\hat{H}) = \hat{I} + \frac{i}{\hbar}(-\hat{H})\delta t \Rightarrow$$

$$\hat{U}_{\delta t}|\Psi(t)\rangle = (\hat{I} - \frac{i}{\hbar}\delta t\,\hat{H})\underbrace{|\Psi(t)\rangle}_{} =$$

Schrödinger equation → $i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t}$

$$= |\Psi(t)\rangle - \frac{i}{\cancel{\hbar}}\delta t \cdot i\cancel{\hbar}\frac{\partial|\Psi(t)\rangle}{\partial t} \approx$$

$$\approx |\Psi(t+\delta t)\rangle$$

Taylor series

$\Rightarrow$ $\hat{H}$ is a generator of time translation!

- Spatial translations (e.g. consider translation in x-direction)

$$\hat{G} = \frac{\hat{P}_x}{\hbar} \leftarrow \text{momentum operator}$$

$$\hat{U}_\varepsilon(\hat{P}_x) = \hat{I} + \frac{i}{\hbar}\varepsilon\hat{P}_x \Rightarrow$$

$$\hat{U}_\varepsilon(\hat{P}_x)\underbrace{\Psi(x)}_{\text{or } |x\rangle} = \underbrace{\Psi(x)}_{\text{or } |x\rangle} + \frac{i}{\hbar}\varepsilon\left(-i\hbar\frac{\partial}{\partial x}\right)\underbrace{\Psi(x)}_{\text{or } |x\rangle} =$$

$$= \Psi(x) + \varepsilon\frac{\partial\Psi(x)}{\partial x} \approx \overbrace{\Psi(x+\varepsilon)}^{\text{or } |x+\varepsilon\rangle} \Rightarrow \hat{P}_x \text{ is a generator of translation along } x$$

How does the position operator $\hat{X}$ transform? $\Rightarrow$

$$\hat{X}' = \left(\hat{I} + \frac{i}{\hbar}\varepsilon\hat{P}_x\right)\hat{X}\left(\hat{I} - \frac{i}{\hbar}\varepsilon\hat{P}_x\right) \overset{\text{neglect } O(\varepsilon^2)}{=}$$

$$= \hat{X} + \frac{i}{\hbar}\varepsilon\underbrace{[\hat{P}_x, \hat{X}]}_{= -i\hbar} = \hat{X} + \varepsilon$$

- Rotations : $\hat{G} = \frac{\hat{J}_z}{\hbar}$ ← angular momentum operator
  (e.g. rotation around z-axis)

$$\hat{U}_{d\varphi} = \hat{I} + \frac{i}{\hbar} d\varphi \hat{J}_z$$

will talk about it a lot in Phys 652

What does condition $[\hat{G}, \hat{A}] = 0$ mean physically? $\Rightarrow$ $\hat{A}' = \hat{A}$

Consider $\hat{G} = \frac{\hat{P}_x}{\hbar}$; $\hat{A} = Y$

$$[\hat{G}, \hat{A}] = \frac{1}{\hbar}[\hat{P}_x, Y] = 0 \Rightarrow Y' = Y$$

Y is conserved upon translation along X

Without going too much into the details at this point, the condition $[\hat{G}, \hat{A}] = 0$ is directly related to symmetries towards properties defined by a type of $\hat{G}$ (i.e. time translation, spatial translation, rotation ...) and conservation laws.

Position, momentum, and translation

So far we have been dealing with discrete eigenvalue spectra $\Rightarrow A|\varphi_n\rangle = a_n|\varphi_n\rangle$

How do we generalize our approach to a continuous spectrum of eigenvalues (e.g. if we measure z-component of momentum, p_z, we can get any real number from $-\infty$ to $+\infty$) $\Rightarrow$

Discrete		Continuous
$A\|\varphi_n\rangle = a_n\|\varphi_n\rangle$	eigenvalue equation	$B\|\xi'\rangle = b\|\xi'\rangle$
		B: operator; b: number (continuous variable); $\|\xi'\rangle$: eigenve[ctor]
$\langle\varphi_n\|\varphi_m\rangle = \delta_{nm}$	orthonormality	$\langle\xi'\|\xi''\rangle = \delta(\xi'-\xi'')$

Dirac δ-function

$$\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ik(x-x_0)};$$

$$\int_{-\infty}^{+\infty} dx\, \delta(x-x_0) f(x) = f(x_0)$$

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = I \qquad \int d\xi' |\xi'\rangle\langle\xi'| = I$$

completeness

$$|\psi\rangle = \sum_n c_n |\varphi_n\rangle = \sum_n |\varphi_n\rangle\langle\varphi_n|\psi\rangle \qquad |\psi\rangle = \int d\xi' |\xi'\rangle\langle\xi'|\psi\rangle$$

expansion in terms of basis vectors

$$\sum_n |c_n|^2 = \sum_n |\langle\varphi_n|\psi\rangle|^2 = 1, \qquad \int d\xi' |\langle\xi'|\psi\rangle|^2 = 1$$

orthonormality + completeness

$$\langle\chi|\psi\rangle = \sum_n \langle\chi|\varphi_n\rangle \cdot \langle\varphi_n|\psi\rangle \qquad \langle\chi|\psi\rangle = \int d\xi' \langle\chi|\xi'\rangle\langle\xi'|\psi\rangle$$

inner product

$$\langle\varphi_n|A|\varphi_m\rangle = a_m \delta_{mn} \qquad \langle\xi''|B|\xi'\rangle = b\,\delta(\xi''-\xi')$$

How does a process of measurement change the initial state when we deal with continuous spectra?

$\Longrightarrow$

<u>Example</u> Consider the position operator $\hat{X}$ in 1D.

$$\hat{X}\,|x'\rangle = \underset{\text{eigenvalue}}{x'}\underbrace{|x'\rangle}_{\text{eigenket}}$$

Initial state $\underbrace{|\Psi\rangle}_{\text{normalized}} = \underbrace{\int_{-\infty}^{+\infty} dx'\, |x'\rangle\langle x'|}_{I}\Psi\rangle$

In a discrete space

$$|\Psi\rangle \underset{\text{measurement}}{\Longrightarrow} |\varphi_n\rangle$$

Here $\Rightarrow$ <u>can't</u> say $|\Psi\rangle \underset{\text{measurement}}{\Longrightarrow} |x'\rangle$! $\Rightarrow$

$$|\Psi\rangle_{\text{after meas.}} = \int_{x'-\frac{\Delta}{2}}^{x'+\frac{\Delta}{2}} dx'\, |x'\rangle\langle x'|\Psi\rangle$$

integral over narrow range around the eigenvalue x', where Δ is a small parameter

The probability to obtain result x' is $|\langle x'|\Psi\rangle|^2 \underbrace{dx'}_{\Delta}$

Probability to find a particle somewhere between $-\infty$ and $+\infty$ $\Rightarrow \int_{-\infty}^{+\infty} dx' |\langle x'|\Psi\rangle|^2 = 1$

$\underline{\langle x'|\Psi\rangle = \Psi(x')}$ ← wave function for a physical state $|\Psi\rangle$