QM I
Phys 657

Solution Of HW #8

Problem #1

(a) $X_{nm} = \int_{-\infty}^{+\infty} \Psi_n^*(x)\, x\, \Psi_m(x)\, dx =$

$$= \frac{N_n N_m}{\alpha^2} \int_{-\infty}^{+\infty} e^{-\xi^2}\, \xi\, H_n(\xi)\, H_m(\xi)\, d\xi \;\stackrel{(=)}{}$$

$\xi = \alpha x$

Hermite polynomials

The simplest way of dealing with this integral is to use the recursion relations for the Hermite polynomials:

$$H_{n+1}(\xi) - 2\xi H_n(\xi) + 2n H_{n-1}(\xi) = 0$$

$$\Downarrow$$

$$\xi H_n(\xi) = \frac{1}{2} H_{n+1}(\xi) + n H_{n-1}(\xi)$$

Then,

$$(=) \; \frac{N_n N_m}{\alpha^2} \int_{-\infty}^{+\infty} e^{-\xi^2} \left(\frac{1}{2} H_{n+1}(\xi) H_m(\xi) + n H_{n-1}(\xi) \cdot H_m(\xi) \right) d\xi = \frac{N_m N_n}{\alpha^2} \cdot \Big[\frac{1}{2} \delta_{n+1,m} \cdot \sqrt{\pi}\, 2^{n+1} (n+1)!$$

$$+ n\,\delta_{n-1,m}\sqrt{\pi}\,2^{n-1}(n-1)!\Big] =$$

(here we used orthogonality of Hermite polynomial

$$\int_{-\infty}^{+\infty} e^{-\xi^2} H_n^2(\xi)\,d\xi = \sqrt{\pi}\,2^n n!$$

$$\int_{-\infty}^{+\infty} e^{-\xi^2} H_n(\xi) H_m(\xi)\,d\xi = 0,\quad n \neq m$$

$$= \frac{\cancel{\alpha}\,\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}\,2^{n/2}\sqrt{n!}\,2^{m/2}\sqrt{m!}\,\alpha^{\cancel{2}}}\Big[\delta_{n+1,m}\,2^n (n+1)! + n\,\delta_{n-1,m}\,2^{n-1}(n-1)!\Big] \Rightarrow$$

$$N_n = \left(\frac{\alpha}{\sqrt{\pi}\,2^n n!}\right)^{1/2}$$

(normalization constant)

$$m = n+1 \Rightarrow x_{nm} = \frac{1}{\alpha}\cdot\frac{1}{n!\,\cancel{2^{n+1/2}}\sqrt{n+1}}\,\cancel{2^n}(n+1)!$$

$$\left(\alpha = \sqrt{\frac{m\omega}{\hbar}}\right) \quad = \frac{1}{\alpha}\frac{\sqrt{n+1}}{\sqrt{2}}$$

$$m = n-1 \Rightarrow x_{nm} = \frac{1}{\alpha}\frac{\sqrt{2}}{\cancel{2^n}\sqrt{n!}\,\sqrt{(n-1)!}}\,n\,\cancel{2^{n-1}}(n-1)!$$

$$= \frac{1}{\alpha}\frac{\sqrt{n!}}{\sqrt{2}\,\sqrt{(n-1)!}} = \frac{1}{\alpha}\sqrt{\frac{n}{2}}$$

$$m \neq n \pm 1 \Rightarrow x_{nm} = 0$$

Alternative way: use generating functions $\Rightarrow$ consider (similarly to what we did in class to derive orthogonality of H_n, H_m)

$$\int_{-\infty}^{+\infty} e^{-\xi^2}\xi\, G(\xi,s)\,G(\xi,t)\,d\xi =$$

$$= \underbrace{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{s^n t^m}{n!\,m!}\int_{-\infty}^{+\infty} e^{-\xi^2}\xi H_n(\xi)H_m(\xi)\,d\xi}\quad (1)$$

On the other hand, $G(\xi, s) = e^{-s^2+2s\xi} \Rightarrow$

$$\int_{-\infty}^{+\infty} e^{-\xi^2}\xi\, G(\xi,s)\,G(\xi,t)\,d\xi =$$

$$= \int_{-\infty}^{+\infty} e^{-\xi^2}\xi\, e^{-s^2+2s\xi}\, e^{-t^2+2t\xi}\,d\xi \underset{\substack{\text{complete}\\ \text{the square}}}{=}$$

$$= e^{2st}\int_{-\infty}^{+\infty}\xi\, e^{-(\xi-s-t)^2}\,d\xi =$$

$$= e^{2st}\underbrace{\int_{-\infty}^{+\infty}(\xi-s-t)\,e^{-(\xi-s-t)^2}\,d(\xi-s-t)}_{=0 \Leftarrow \int_{-\infty}^{+\infty} e^{-x^2}dx} +$$

$$+ e^{2st}(s+t)\underbrace{\int_{-\infty}^{+\infty} e^{-(\xi-s-t)^2}\,d(\xi-s-t)}_{=\sqrt{\pi}} =$$

$$= \sqrt{\pi}\,(s+t)\,e^{2st} = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{2^n s^n t^n}{n!}(s+t) =$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\alpha}{n!} \left(s^{n+1} t^n + s^n t^{n+1} \right) \quad (2)$$ (4)

Compare (2) and (1) $\Longrightarrow$ arrive at the same answers as through recursion relations

(b) Recall that $X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$

$$\langle n | X | m \rangle = X_{nm} = \sqrt{\frac{\hbar}{2m\omega}} \langle n | a + a^+ | m \rangle =$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \Big(\langle n | \underbrace{a | m \rangle}_{= \sqrt{m}\,|m-1\rangle} + \langle n | \underbrace{a^+ | m \rangle}_{= \sqrt{m+1}\,|m+1\rangle} \Big) =$$

$$= \sqrt{\frac{\hbar}{2 \underset{\uparrow \text{mass}}{m} \omega}} \Big(\sqrt{\underset{\uparrow \text{number}}{m}}\, \underbrace{\langle n | m-1 \rangle}_{= \delta_{n,m-1}} + \sqrt{m+1}\, \underbrace{\langle n | m+1 \rangle}_{= \delta_{n,m+1}} \Big) =$$

$$\text{“}\; n+1$$

$$= \underline{\sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1}\, \delta_{n,m-1} + \sqrt{n}\, \delta_{n,m+1} \right)}$$

So, $X_{nm} = 0$ if $n \neq m \pm 1$

$$= \sqrt{\frac{\hbar}{m\omega}} \sqrt{\frac{n+1}{2}} = \frac{1}{\alpha} \sqrt{\frac{n+1}{2}}, \quad \underline{m = n+1}$$

$$= \frac{1}{\alpha} \sqrt{\frac{n}{2}}, \quad \underline{m = n-1}$$

Same as in (a), but much faster!!

Problem #3

Schrödinger equation (time-independent) in:

(a) coordinate space

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{m\omega^2 x^2}{2}\right)\psi_E(x)=E\psi_E(x) \qquad (1)$$

(b) momentum space

$$\left(\frac{p^2}{2m}+\frac{m\omega^2}{2}\cdot\left(-\hbar^2\frac{d^2}{dp^2}\right)\right)\varphi_E(p)=E\varphi_E(p) \qquad (2)$$

Compare Eqs. (1) & (2) $\Rightarrow$ if you replace

$x \to p$

$m\omega \to \frac{1}{m\omega}$

in Eq. (1) $\Rightarrow$ get Eq. (2)!

Then, since we know solution of Eq. (1), i.e.

$$\psi_E(x)=\sqrt{\frac{\alpha}{\sqrt{\pi}\,2^n n!}}\;e^{-\alpha^2x^2/2}\,H_n(\alpha x),$$

$$\alpha=\sqrt{\frac{m\omega}{\hbar}}$$

$\Downarrow$

Then if we replace $x \to p$, $m\omega \to \frac{1}{m\omega}$ in $\psi(x)$ $\Rightarrow$ get $\varphi(p)$!

So, $\phi_{E}(p) = \frac{1}{(\pi\hbar m\omega)^{1/4}\sqrt{2^n n!}}\, e^{-\frac{p^2}{2\hbar m\omega}}\, H_n\left(\frac{1}{\sqrt{m\omega\hbar}}\, p\right)$

$E \updownarrow n$

For example, for $n=0 \Longrightarrow \phi_0(p) = \frac{1}{(\pi\hbar m\omega)^{1/4}}\, e^{-\frac{p^2}{2\hbar m\omega}}$

$E_0 = \frac{\hbar\omega}{2}$

QMI
Phys 681

Solutions of HW # ~~8~~ (ctd) ~~4~~

Problem # ~~3~~

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$$

$$\langle n | X^4 | n \rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \langle n | (a+a^+)^4 | n \rangle \; = $$

$$(a+a^+)^4 = (a^2 + \underbrace{a^+ a}_{N} + \underbrace{a a^+}_{N+1} + a^{+2})^2 =$$

$$= (a^2 + 2N + 1 + a^{+2})^2 = a^4 + a^{+4} + \underline{(2N+1)^2} +$$

$$+ a^2(2N+1) + a^{+2}(2N+1) + (2N+1)a^2 +$$

$$+ (2N+1)a^{+2} + \underline{a^2 a^{+2}} + \underline{a^{+2} a^2}$$

For $\langle n | \ldots | n \rangle$ objects $\Rightarrow$ only underlined terms ~~are~~ non-zero ~~will produce~~ results

$\Downarrow$

$$= \langle n | (2N+1)^2 + a^2 a^{+2} + a^{+2} a^2 | n \rangle \cdot \left(\frac{\hbar}{2m\omega}\right)^2 =$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \cdot \Big[(2n+1)^2 + \langle n | a \underbrace{a a^+}_{N+1} a^+ | n \rangle +$$

$$+ \langle n | a^+ \overbrace{a^+ a}^{N} a | n \rangle \Big] = \left(\frac{\hbar}{2m\omega}\right)^2 \Big[(2n+1)^2 +$$

$$+\langle n|\underbrace{aa^+}_{N+1}|n\rangle + \langle n|a\underbrace{Na^+}_{=a^+ + a^+N}|n\rangle +$$

$$+\langle n|a^+\underbrace{Na}_{=-a+aN}|n\rangle\Big] = \left(\frac{\hbar}{2m\omega}\right)^2\Big[(2n+1)^2 + n+1 +$$

$$+ n+1 + n(n+1) + n^2 - n\Big] = \left(\frac{\hbar}{2m\omega}\right)^2\Big[4n^2+4n+1+$$

$$+ n^2 + n^2 + 2n + 2\Big] = \left(\frac{\hbar}{2m\omega}\right)^2\Big[6n^2+6n+3\Big]$$

(2.18 in "new" (gray) book)
2.20 in "newest" (blue) book
in "old" (red) book

Problem # 4 (Sakurai 2.17)

$$\langle 0|e^{ikX}|0\rangle = \int_{-\infty}^{+\infty} dx' \langle 0|\underbrace{e^{ikX}|x'\rangle}_{e^{ikx'}|x'\rangle}\underbrace{\langle x'|0\rangle}_{\psi_0(x')} =$$

$$= \int_{-\infty}^{+\infty} dx' e^{ikx'} |\psi_0(x')|^2 =$$

$$= \frac{1}{\sqrt{\pi}\, x_0}\int_{-\infty}^{+\infty} e^{-x'^2/x_0^2} e^{ikx'} dx' = \frac{1}{\sqrt{\pi}\, x_0} e^{-\frac{k^2 x_0^2}{4}} \sqrt{\pi}\, x_0 = e^{-\frac{k^2 x_0^2}{4}}$$

$$\psi_0(x') = \frac{1}{\pi^{1/4}\sqrt{x_0}} e^{-x'^2/2x_0^2} \qquad -\frac{x'^2}{x_0^2} + ikx' = -\left(\frac{x'}{x_0} - \frac{ikx_0}{2}\right)^2 - \frac{k^2 x_0^2}{4}$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

From another side, $\langle 0|\underbrace{X^2}_{\frac{x_0^2}{2}(a+a^+)^2}|0\rangle = \frac{x_0^2}{2}\langle 0|2N+1|0\rangle = \frac{x_0^2}{2}$

So, $e^{-k^2\langle 0|X^2|0\rangle/2} = e^{-\frac{k^2 x_0^2}{4}} = \langle 0|e^{ikX}|0\rangle$

~~the same accuracy, unless t_1 and t_2 are related by $\sin\omega(t_1-t_2)=0 \Rightarrow t_1-t_2=\frac{\pi n}{\omega}$. The same for the momentum.~~

Problem #5

(a) No electric field $\Rightarrow$ $H=\frac{P^2}{2m}+\frac{m\omega^2X^2}{2}$

Apply electric field $\Rightarrow$ force $F=q\mathcal{E}$ $\Rightarrow$ additional potential energy $V_{\mathcal{E}}=-q\mathcal{E}X$

So, new Hamiltonian is

$$H_{(new)}=\frac{P^2}{2m}+\frac{m\omega^2X^2}{2}-q\mathcal{E}X=$$

$$=\frac{P^2}{2M}+\frac{m\omega^2}{2}\underbrace{\left(X-\frac{q\mathcal{E}}{m\omega^2}\right)}_{X'}^2-\frac{q^2\mathcal{E}^2}{2m\omega^2}$$

complete the square

X' ← new variable

$$\underbrace{H_{new}}\Psi(X')=E_{new}\Psi(x')$$

$$\underbrace{H_{old}}_{\frac{P^2}{2m}+\frac{m\omega^2X'^2}{2}}(x\to x')-\underbrace{\frac{q^2\mathcal{E}^2}{2m\omega^2}}_{const}\Rightarrow \cancel{H_{old}(x\to x')\Psi(x')=E_{old}+\frac{q^2\mathcal{E}^2}{2m\omega^2}}$$

$$\left(\underset{(x\to x')}{H_{old}} - \frac{q^2\mathcal{E}^2}{2m\omega^2}\right)\psi(x') = \underbrace{\left(E_{old} - \frac{q^2\mathcal{E}^2}{2m\omega^2}\right)}_{E_{new}}\psi(x')$$

So, $E_{new} = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{q^2\mathcal{E}^2}{2m\omega^2}$

$$\psi_n^{(new)}(x') \underset{\psi_n^{(old)},\ x\to x'}{=} \sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}\, e^{-\alpha^2 x'^2/2} H_n(\underset{\sqrt{\frac{m\omega}{\hbar}}}{\alpha} x')$$

$$x' = x - \frac{q\mathcal{E}}{m\omega^2}$$

(b) $\psi_0^{(with\ \mathcal{E})}(x) \underset{t<0}{=} \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\alpha^2\left(x-\frac{q\mathcal{E}}{m\omega^2}\right)^2/2}$

stationary states

At $t>0 \Rightarrow \psi_n^{(\mathcal{E}=0)}(x) = \frac{\sqrt{\alpha}}{\pi^{1/4}\sqrt{2^n n!}} e^{-\alpha^2x^2/2} H_n(\alpha x) \Rightarrow$

superposition of them (arbitrary state) $\Rightarrow \underset{\mathcal{E}=0}{\psi}(x,t) = \sum_n C_n \psi_n(x) e^{-\frac{i}{\hbar}E_n t}$

$E_n = \hbar\omega(n+\frac{1}{2})$

$\mathcal{P}(\underset{\text{ground state}}{E_0}) = |\langle\psi_0^{\mathcal{E}}|\psi_0^{\mathcal{E}=0}\rangle|^2 = \left|\frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\alpha^2(x-b)^2/2}\right.$

$b = \frac{q\mathcal{E}}{m\omega^2}$

$\left.\cdot\, e^{-\alpha^2x^2/2} dx \cdot e^{-\frac{i}{\hbar}E_0 t}\right|^2 =$

same for any t