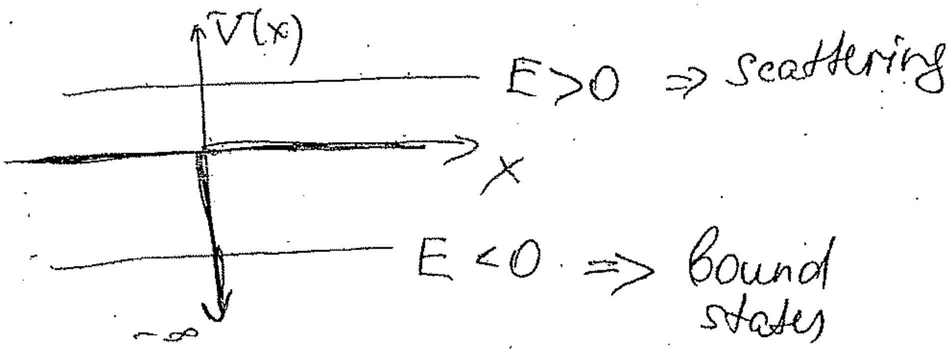
QM I
Phys 651

Solutions of HW

Problem

$$V(x) = -V_0\,\delta(x), \quad V_0 > 0$$



(a) $E < 0$

$$-\frac{\hbar^2}{2m}\Psi''(x) - V_0\,\delta(x)\,\Psi(x) = E\,\Psi(x) \qquad (1)$$

For $x \neq 0 \Rightarrow -\frac{\hbar^2}{2m}\Psi''(x) = E\Psi(x)$ (E<0) $\Rightarrow$

$$\Psi(x) = Ae^{\kappa x} + Be^{-\kappa x}, \quad \kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$$

$\Downarrow$

Physical solution: $\Psi(x) = Ae^{\kappa x}, \quad x<0$

(finite at $x \to \pm\infty$) $\Psi(x) = Be^{-\kappa x}, \quad x>0$

(Since $\Psi(x)$ is continuous at $x=0 \Rightarrow A = B \Rightarrow$

$$\underline{\Psi(x) = Ae^{-\kappa|x|}}$$

Normalization: $\int_{-\infty}^{+\infty} |\Psi(x)|^2 dx = A^2 \left[\int_{-\infty}^{0} e^{2\kappa x} dx + \int_{0}^{\infty} e^{-2\kappa x} dx \right] =$

$$\underset{x \to -x}{=} 2A^2 \underbrace{\int_0^\infty e^{-2\kappa x} dx}_{= \frac{1}{2\kappa}} = \frac{A^2}{\kappa} = 1 \Rightarrow \underline{A = \sqrt{\kappa}}$$

$$\underline{\Psi(x) = \sqrt{\kappa}\, e^{-\kappa |x|}}$$

Now find allowed energy values:

Integrate Eq. (1) from $-\varepsilon$ to ε, $\varepsilon \to 0 \Rightarrow$

$$-\frac{\hbar^2}{2m}\left(\frac{d\Psi}{dx}\Big|_{x=+\varepsilon} - \frac{d\Psi}{dx}\Big|_{x=-\varepsilon} \right) - V_0 \Psi(0) =$$

$$= E \underbrace{\int_{-\varepsilon}^{\varepsilon} \Psi(x) dx}_{\searrow 0} \Rightarrow \underbrace{\frac{d\Psi}{dx}\Big|_{+\varepsilon}}_{= -\kappa A} - \underbrace{\frac{d\Psi}{dx}\Big|_{-\varepsilon}}_{= \kappa A} \underset{\varepsilon \to 0}{\overset{\lim}{=}} -\frac{2mV_0}{\hbar^2} \underbrace{\Psi(0)}_{= A}$$

$$\Downarrow$$

$$\kappa \cancel{A} = \frac{mV_0}{\hbar^2} \cancel{A} \Rightarrow \kappa = \frac{mV_0}{\hbar^2} = \sqrt{\frac{2m|E|}{\hbar^2}}$$

$$\Downarrow$$

$$\underline{E = -\frac{mV_0^2}{2\hbar^2}} \Leftarrow \underline{|E| = \frac{mV_0^2}{2\hbar^2}}$$

$\leftarrow$ 1 bound state $\Rightarrow$

$$\Psi(x) = \sqrt{\frac{m\bar{V}_0}{\hbar^2}}\, \exp\left(-\frac{m\bar{V}_0}{\hbar^2}|x|\right)$$

(b) If $\bar{V}_0 \to \bar{V}_0{}' \Rightarrow \Psi(x) \to \Psi'(x)$

$$\Psi'(x) = \sqrt{\frac{m\bar{V}_0'}{\hbar^2}}\, \exp\left[-\frac{m\bar{V}_0'}{\hbar^2}|x|\right]$$

if it's still bound

$$\mathcal{P} = |\langle \Psi' | \Psi \rangle|^2 = \left(\sqrt{\frac{m\bar{V}_0}{\hbar^2}} \cdot \sqrt{\frac{m\bar{V}_0'}{\hbar^2}}\right)^2 \cdot \left| \int_{-\infty}^{+\infty} \exp\left(-\frac{m}{\hbar^2} \cdot \right.\right.$$

probability to remain bound

$$\left.\left. \cdot (\bar{V}_0 + \bar{V}_0')|x|\right) dx \right|^2 = \frac{m^2 \bar{V}_0 \bar{V}_0'}{\hbar^4} \cdot 4 \left| \int_0^\infty e^{\alpha x} dx \right|^2 =$$

$$\alpha = -\frac{m}{\hbar^2}(\bar{V}_0 + \bar{V}_0')$$

$$= \frac{\cancel{m^2} \bar{V}_0 \bar{V}_0'}{\cancel{\hbar^4}} \cdot 4 \cdot \frac{\cancel{\hbar^4}}{\cancel{m^2}(\bar{V}_0 + \bar{V}_0')^2} = \frac{4\bar{V}_0 \bar{V}_0'}{(\bar{V}_0 + \bar{V}_0')^2}$$

(c) $E > 0 \Rightarrow$ $\Psi(x) = Ae^{ikx} + Be^{-ikx}$, $x < 0$

$\Psi(x) = Ce^{ikx}$, $x > 0$ ← transmitted wave

plane wave, incident from the left and reflected

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$\Psi(x)\Big|_{x=0} \Rightarrow$ continuous $\quad A+B=C$

$$\frac{d\Psi}{dx}\Big|_{x=0} \Rightarrow \underbrace{\frac{d\Psi}{dx}\Big|_{\varepsilon}}_{\overset{\shortparallel}{C\cdot ik}} - \underbrace{\frac{d\Psi}{dx}\Big|_{-\varepsilon}}_{\overset{\shortparallel}{ik(A-B)}} = -\frac{2m}{\hbar^2}\bar{V}_0 \underbrace{\Psi(0)}_{\overset{\shortparallel}{C}}$$

$$ik(C-A+\underbrace{B}_{\overset{\shortparallel}{C-A}}) + \frac{2m\bar{V}_0}{\hbar^2}C = 0;$$

$$2ik(C-A) + \frac{2m\bar{V}_0}{\hbar^2}C = 0 \Rightarrow 2ik - 2ik\frac{A}{C} + \frac{2m\bar{V}_0}{\hbar^2} = 0.$$

$$\frac{A}{C} = \left(\frac{2m\bar{V}_0}{\hbar^2} + 2ik\right)\cdot\frac{1}{2ik}; \quad \frac{C}{A} = \frac{\cancel{2}ik}{\cancel{2}ik + \cancel{2}\frac{m\bar{V}_0}{\hbar^2}}$$

$$T = \left|\frac{C}{A}\right|^2 = \frac{1}{1+\frac{m\bar{V}_0^2}{2\hbar^2E}}$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{1}{\frac{2\hbar^2E}{m\bar{V}_0^2}+1} \qquad R+T=1 \ \checkmark$$

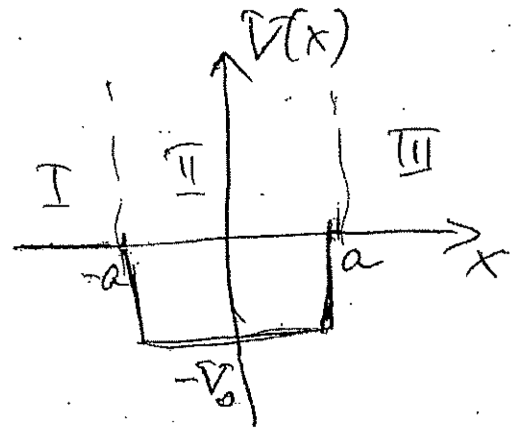
$$\frac{B}{A} = -\frac{1}{1+\frac{ik\hbar^2}{m\bar{V}_0}}$$

Problem #2 → see Lecture #19

QM I
Phys 657

Solution of HW #17 (ctd)

Problem #3



(a) Generally:

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi = E\psi$$

$$\Downarrow$$

$$\psi''(x) = \frac{2m}{\hbar^2}\left(V(x) - E\right)\psi(x)$$

1) if $E < -V_0 \Rightarrow$ no physical solution

2) if $E > 0 \Rightarrow$ unbound states $\Rightarrow \quad E = |E|$

(I) / (III): $\quad \psi''(x) = -\frac{2m}{\hbar^2}|E|\,\psi(x) \Rightarrow$

$$\psi(x) = C_1 e^{ikx} + C_2 e^{-ikx}$$

$$k = \sqrt{\frac{2m|E|}{\hbar^2}}$$

(II): $\quad \psi'' = \frac{2m}{\hbar^2}\left(-V_0 - |E|\right)\psi;$

$$\psi(x) = \tilde{C}_1 e^{ik'x} + \tilde{C}_2 e^{-ik'x}$$

$$k' = \sqrt{\frac{2m(V_0 + |E|)}{\hbar^2}}$$

3) if $-V_0 < E < 0 \Rightarrow$ bound states ; $E = -|E|$

I, III : $\psi'' = \frac{2m}{\hbar^2} \underbrace{(-E)}_{= |E|} \psi$

$$\psi_{I, III} = C_{1,3}\, e^{\kappa x} + \tilde{C}_{1,3}\, e^{-\kappa x} \quad , \quad \kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$$

II : $\psi'' = \frac{2m}{\hbar^2} \underbrace{(-V_0 - E)}_{= -(V_0 - |E|)} \psi$, $\kappa' = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}$

$$\psi_{II} = C_2 \sin \kappa' x + \tilde{C}_2 \cos \kappa' x$$

(b) Consider bound states (case $E > 0$ is physical, but not too interesting)

$$\psi_I = C_1 e^{\kappa x} \quad , \quad x < -a$$

$$\psi_{II} = C_2 \sin \kappa' x + \tilde{C}_2 \cos \kappa' x \quad , \quad -a < x < a$$

$$\psi_{III} = \tilde{C}_3 e^{-\kappa x} \quad , \quad x > a$$

Now apply boundary conditions $\Rightarrow$

$$\psi_I(-a) = \psi_{II}(-a) \quad ; \quad \psi_{II}(a) = \psi_{III}(a)$$

$$\psi_I'(-a) = \psi_{II}'(-a) \quad ; \quad \psi_{II}'(a) = \psi_{III}'(a)$$

$$C_1 e^{-\kappa a} = -C_2 \sin\kappa' a + \tilde{C}_2 \cos\kappa' a \qquad (1)$$
$$\tilde{C}_3 e^{-\kappa a} = C_2 \sin\kappa' a + \tilde{C}_2 \cos\kappa' a \qquad (2)$$

$$\kappa C_1 e^{-\kappa a} = \kappa' C_2 \cos\kappa' a + \kappa' \tilde{C}_2 \sin\kappa' a \qquad (3)$$
$$-\kappa \tilde{C}_3 e^{-\kappa a} = \kappa' C_2 \cos\kappa' a - \kappa' \tilde{C}_2 \sin\kappa' a \qquad (4)$$

$\Downarrow$

4 equations, four unknowns $(C_1, C_2, \tilde{C}_2, C_3)$

Divide (3) by (1) $\Rightarrow$ (assume none of the constants is zero)

$$\kappa = -\kappa' \cot\kappa' a + \kappa' \tan\kappa' a$$

Divide (4) by (2) $\Rightarrow$ $\quad$ same

$$-\kappa = \kappa' \cot\kappa' a - \kappa' \tan\kappa' a$$

Add (1) & (2) : $\left(C_1 + \tilde{C}_3\right) e^{-\kappa a} = 2\tilde{C}_2 \cos\kappa' a \qquad (1')$

Subtract (1) & (2) : $\left(\tilde{C}_3 - C_1\right) e^{-\kappa a} = 2 C_2 \sin\kappa' a \qquad (2')$

Add (3) & (4) ; $\kappa\left(C_1 - \tilde{C}_3\right) e^{-\kappa a} = 2\kappa' C_2 \cos\kappa' a \qquad (3')$

Subtract (3) & (4) ; $\kappa\left(C_1 + \tilde{C}_3\right) e^{-\kappa a} = 2\kappa' \tilde{C}_2 \sin\kappa' a \qquad (4')$

Divide (4') by (1') $\Rightarrow$ $\kappa = \kappa' \tan\kappa' a$

Divide (3') by (2') $\Rightarrow$ $-\kappa = \kappa' \cot\kappa' a$

can't be satisfied at the same time!

distinguish two classes of solution

1) $C_2 = 0$, $C_1 = C_3$ $\Rightarrow$ $K = K' \tan K'a$

2) $\tilde{C}_2 = 0$, $C_1 = -\tilde{C}_3$ $\Rightarrow$ $-K = K' \cot K'a$

In the <u>first</u> case:

$$\left.\begin{array}{ll} \Psi_I = C_1 e^{Kx} & x < -a \\ \Psi_{II} = \tilde{C}_2 \cos K'x & -a < x < a \\ \Psi_{III} = C_1 e^{-Kx} & x > a \end{array}\right| \Rightarrow \begin{array}{c} \text{even function} \\ \Psi(-x) = \Psi(x) \end{array}$$

In the <u>second</u> case:

$$\left.\begin{array}{ll} \Psi_I = C_1 e^{Kx} & x < -a \\ \Psi_{II} = C_2 \sin K'x & -a < x < a \\ \Psi_{III} = -C_1 e^{-Kx} & x > a \end{array}\right| \Rightarrow \begin{array}{c} \text{odd function} \\ \Psi(-x) = -\Psi(x) \end{array}$$

Generally: if your potential $V(x)$ is <u>symmetric</u> i.e. $V(-x) = V(x)$, you will have two types of solutions
- → even functions
- → odd functions

Constants $C_1, \tilde{C}_2$ in the first case and C_1, C_2 in the second case are determined from (1'),(4') and (2'),(3'), respectively, and the normalization.

(c) The corresponding energy values are given by the solutions of the transcendental equati

$$K = K' \tan K'a \quad (1\text{st case})$$

$$-K = K' \cot K'a \quad (2\text{nd case})$$

$$K = \sqrt{\frac{2m|E|}{\hbar^2}}, \quad K' = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}$$

Let's solve them graphically:

$$X = K'a, \quad Y = Ka \quad (X, Y > 0)$$

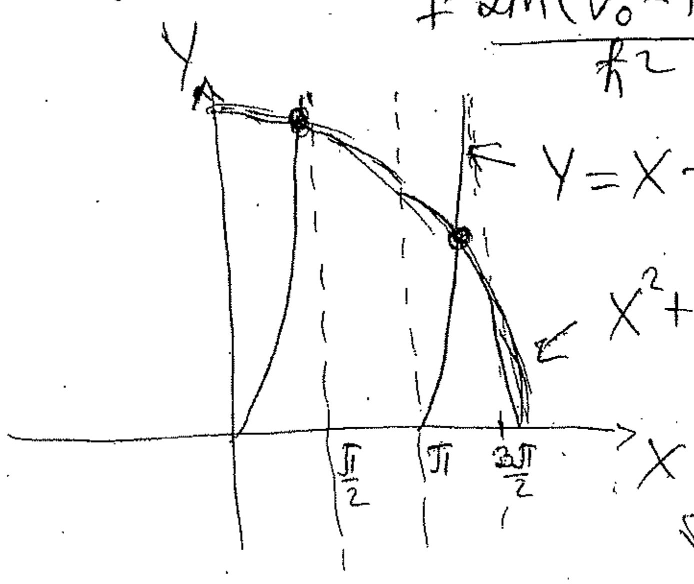
1st case : $Y = X \tan X$

For $X > 0, Y > 0$:

$$X^2 + Y^2 = (K'^2 + K^2)a^2 = \left(\frac{2m|E|}{\hbar^2} + \frac{2m(V_0 - |E|)}{\hbar^2}\right)a^2 = \frac{2mV_0}{\hbar^2}a^2$$

circle of $R = a\sqrt{\frac{2mV_0}{\hbar^2}}$

depends on V_0, a!

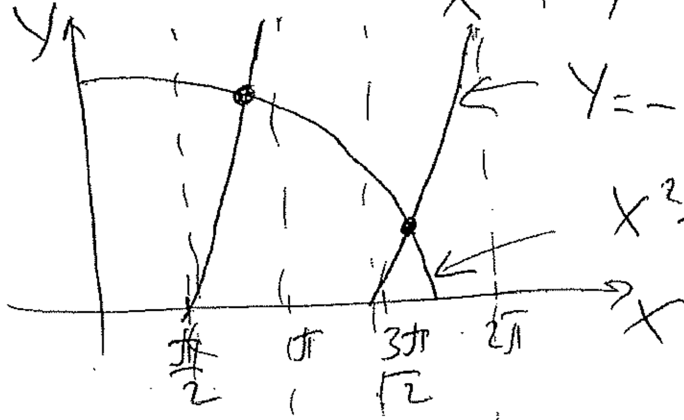


here, for example there are two solutions $\Rightarrow$ two bound states

If you increase V_0 or a $\Rightarrow$ will have more solutions

Case 2: $-Y = X \cot X$

$X^2 + Y^2 = R^2$, $\quad R = a\sqrt{\frac{2mV_0}{\hbar^2}}$

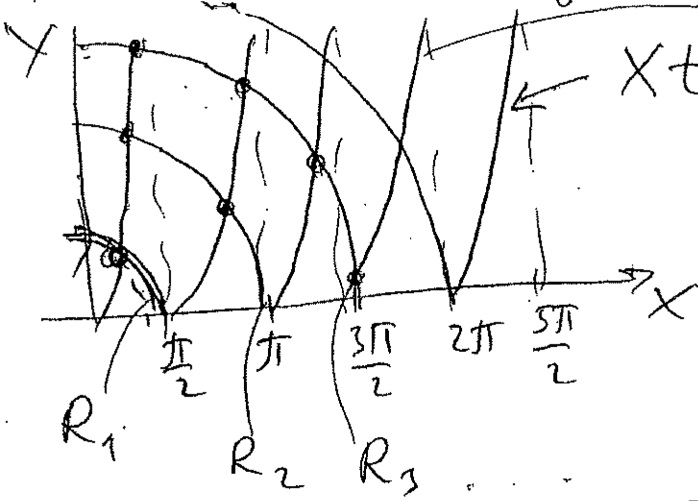


$\leftarrow Y = -X\cot X$

$X^2 + Y^2 = R^2$

$\Leftarrow$ here we have 2 solutions, but the number of solutions depends on R!

(d) Both cases together:



$-X \cot X$

$X \tan X$

$\Rightarrow \quad R = a\sqrt{\frac{2mV_0}{\hbar^2}} < \frac{\pi}{2}$

$\Downarrow$

1 bound state

If $\frac{\pi}{2} \le R \le \pi \Rightarrow$ 2 bound states

- - - -

If $\frac{N\pi}{2} \le R \le \frac{N+1}{2}\pi$

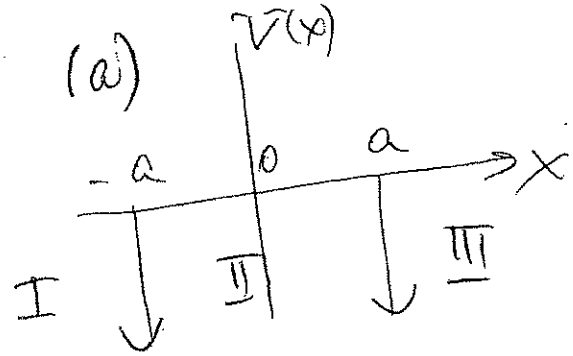
$\Rightarrow$ N+1 bound states

Problem #4

(15)

$$V(x) = -V_0\,\delta(x-a) - V_0\,\delta(x+a)$$

(a)



$$\frac{d^2\psi}{dx^2} + \frac{2mV_0}{\hbar^2}\left[\delta(x-a)+\delta(x+a)\right]\psi + \frac{2mE}{\hbar^2}\psi = 0;$$

For $x \neq \pm a \Rightarrow \psi'' + \frac{2mE}{\hbar^2}\psi = 0, \quad E<0$

$$\Downarrow$$

$$\psi = Ae^{\kappa x} + \tilde{A}e^{-\kappa x}, \quad \kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$$

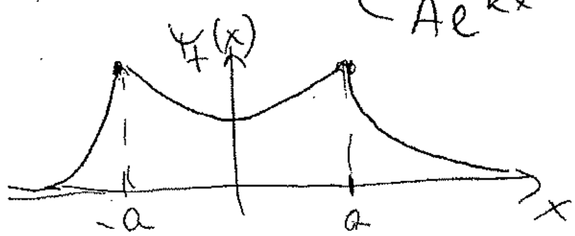
Since $V(-x) = V(x) \Rightarrow$ the wave function is either even or odd

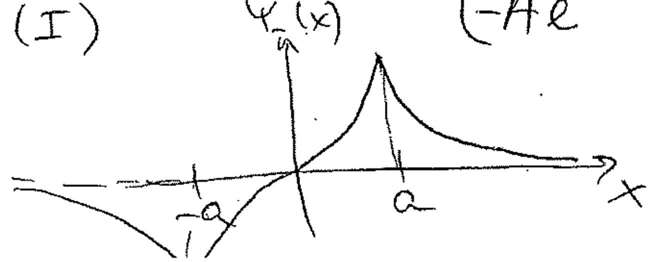
$$\Downarrow \qquad \psi_+(x) \qquad \psi_-(x)$$

$$\psi_\pm(x) = \begin{cases} Ae^{-\kappa x}, & x>a \\ \frac{B}{2}\left(e^{\kappa x} \pm e^{-\kappa x}\right), & -a<x<a; \\ \pm Ae^{\kappa x}, & x<-a \end{cases} \quad (\text{and } \psi_\pm(x\to\pm\infty)\to 0)$$

$$\Downarrow$$

$$\psi_+(x) = \begin{cases} Ae^{-\kappa x} & (\text{III}) \\ B\cosh\kappa x & (\text{II}) \\ Ae^{\kappa x} & (\text{I}) \end{cases} \quad ; \quad \psi_-(x) = \begin{cases} Ae^{-\kappa x} & (\text{III}) \\ B\sinh\kappa x & (\text{II}) \\ -Ae^{+\kappa x} & (\text{I}) \end{cases}$$





(6) Energy eigenvalues can be obtained from the boundary conditions :

$\Psi_+(x)$: $\quad \underline{A e^{-\kappa a} = B \cosh \kappa a} \quad (1)$
(continuity)

$\Psi_-(x)$: $\quad \underline{A e^{-\kappa a} = B \sinh \kappa a} \quad (2)$

We also have a condition for discontinuity of the first derivative of $\Psi_\pm(x)$ at $x = \pm a \Rightarrow$

For Ψ_+: $\lim_{\varepsilon \to 0} \left[\Psi_+'(a+\varepsilon) - \Psi_+'(a-\varepsilon) \right] + \frac{2mV_0}{\hbar^2} \Psi_+(a) = 0$

$\Downarrow$

$$\underline{A \left(\frac{2mV_0}{\kappa \hbar^2} - 1 \right) e^{-\kappa a} = B \sinh \kappa a} \quad (3)$$

For Ψ_- : $\quad \underline{A \left(\frac{2mV_0}{\kappa \hbar^2} - 1 \right) e^{-\kappa a} = B \cosh \kappa a} \quad (4)$

Divide (3) by (1) :

$$\underline{\frac{2mV_0}{\kappa \hbar^2} - 1 = \tanh \kappa a} \Rightarrow \underline{\tanh y = \frac{\gamma}{y} - 1}$$

$\uparrow$

eigenvalue equation for even solutions

$y = \kappa a \quad (y > 0)$

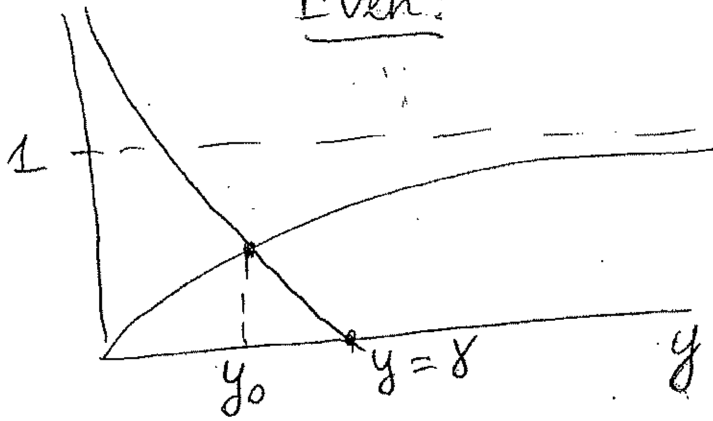
$\gamma = \frac{2maV_0}{\hbar^2}$

Divide (4) by (1):

$$\underbrace{\frac{2mV_0}{\kappa\hbar^2} - 1 = \coth \kappa a}_{\text{eigenvalue equation for the odd solutions}} \Rightarrow \tanh y = \frac{1}{\frac{\gamma}{y} - 1}$$

Solve these two equations graphically:

Even:



$$y_0 = \kappa a \Rightarrow \kappa = \sqrt{\frac{2m|E|}{\hbar^2}} = \frac{y_0}{a} \Rightarrow |\overset{\text{even}}{E}| = \frac{\hbar^2}{2m}\left(\frac{y_0}{a}\right)^2$$

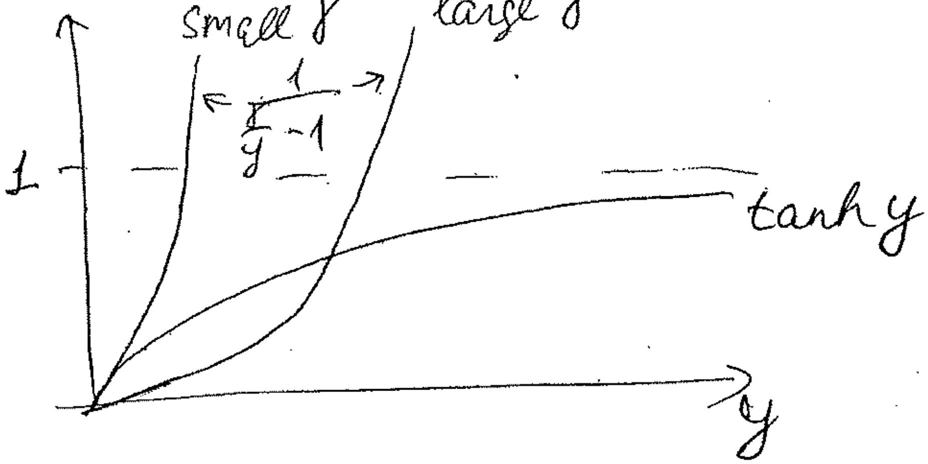
Only one bound state (even) eigenvalue

Also: $y_0 < \gamma$, but $\frac{\gamma}{y_0} - 1 < 1 \Rightarrow y_0 > \frac{\gamma}{2}$

$$\frac{\gamma}{2} < y_0 < \gamma$$

$$\boxed{-\frac{2mV_0^2}{\hbar^2} < E_{even} < -\frac{mV_0^2}{2\hbar^2}} \quad (5)$$

Odd:



Check the slopes: $\frac{d}{dy}\left(\frac{1}{\frac{\gamma}{y}-1}\right)\Big|_{y=0} = \frac{1}{\gamma}$

$$\frac{d}{dy}\tanh y\Big|_{y=0} = 1$$

If $\frac{1}{\gamma} < 1 \Rightarrow$ 1 bound state (odd)

$\Downarrow$ $\gamma > 1 \Rightarrow \frac{2maV_0}{\hbar^2} > 1 \Rightarrow V_0 > \frac{\hbar^2}{2ma}$

If $\gamma < 1 \Rightarrow$ no odd bound states

If $y = \frac{\gamma}{2} \Rightarrow \frac{1}{\frac{\gamma}{y}-1} = 1 \Rightarrow$ so, intersection of tanh y and $\frac{1}{\frac{\gamma}{y}-1}$ (if any) must occur at $y < \frac{\gamma}{2} \Rightarrow \boxed{E_{odd} > -\frac{mV_0^2}{2\hbar^2}}$ (6)

Compare (5) and (6) $\Rightarrow$

$E_{even} < E_{odd} \Rightarrow$ ground state is <u>even</u>

Altogether:

if $\gamma < 1 \Rightarrow$ there are no <u>odd</u> bound states, but there is always one <u>even</u> bound state,

$\left(V_0 < \frac{\hbar^2}{2ma}\right)$ the ground state

if $\gamma > 1 \Rightarrow$ there are <u>two</u> bound states; one is even (ground state) and one is odd (first excited state)

$\left(V_0 > \frac{\hbar^2}{2ma}\right)$

(d) $a \to 0 \Rightarrow y \to 0 \Rightarrow$

eigenvalue equation for even states

since ground state is even

$\tanh y = \frac{\gamma}{y} - 1$

turns into

$y \approx \frac{\gamma}{y} - 1 \Rightarrow y = \gamma \Rightarrow$

$y^2 = K^2 a^2 = \gamma^2 \Rightarrow \frac{2mE_{bin}}{\hbar^2} \cdot \cancel{a^2} = \frac{4m^2 \cancel{a^2} V_0^2}{\hbar^4} \Rightarrow$

$$E_{even} = -\frac{2mV_0^2}{\hbar^2}$$ ← which is what you would get with a single δ-potential

$\Rightarrow V(x) = -2V_0\,\delta(x)$

See Problem #2

$a \to \infty \Rightarrow y \to \infty \Rightarrow$

$\tanh y = \frac{\gamma}{y} - 1$ turns into $1 = \frac{\gamma}{y} - 1 \Rightarrow$

$y = \frac{\gamma}{2}$ ⇐

$$E_{even} = -\frac{mV_0^2}{2\hbar^2}$$

⇖ same as with

$V(x) = -V_0\,\delta(x)$

(see Problem #2)