QM I
Phys 651

Solution of HW #6

Problem #1

$$[\vec{R}\cdot\vec{P}, H] = [\vec{R}\cdot\vec{P}, \frac{\vec{P}^2}{2m} + V(\vec{R})] =$$

$$= [X P_x + Y P_y + Z P_z, \frac{P_x^2 + P_y^2 + P_z^2}{2m} + V(X,Y,Z)]$$

$$= [X P_x, P_x^2]\cdot\frac{1}{2m} + [Y P_y, P_y^2]\frac{1}{2m} + [Z P_z, P_z^2]\frac{1}{2m} + [X P_x + Y P_y + Z P_z, V(X,Y,Z)] =$$

$$= \frac{1}{2m}\cdot 2i\hbar\,(P_x^2 + P_y^2 + P_z^2) - i\hbar\,\vec{R}\cdot\vec{\nabla}V =$$

$$[X P_x, P_x^2] = X\underbrace{[P_x, P_x^2]}_{=0} + [X, P_x^2] P_x =$$

$$= [X, P_x] P_x + P_x [X, P_x] P_x = 2i\hbar P_x^2$$

Same for Y, Z

$$[X P_x, V(X,Y,Z)] = X[P_x, V] + \underbrace{[X, V]}_{=0} P_x =$$

$$= X(-i\hbar)\frac{\partial V}{\partial X} \quad \text{(same for Y, Z)}$$

Lecture #16

$$= 2i\hbar T - i\hbar\,\vec{R}\cdot\vec{\nabla}V$$

old edition
(2.9 in new edition)
2,10 in blue edition

Problem #2 (Sakurai 2.8)

$A|a'\rangle = a'|a'\rangle$

$H = |a'\rangle\delta\langle a''| + |a''\rangle\delta\langle a'|$

(a) Matrix representation of H in the $\{|a'\rangle\}$ basis

$\langle a'|H|a'\rangle = 0 \qquad \langle a'|H|a''\rangle = \delta$

$\langle a''|H|a''\rangle = 0 \qquad \langle a''|H|a'\rangle = \delta$

$$H \doteq \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \Rightarrow \det\begin{pmatrix} -\lambda & \delta \\ \delta & -\lambda \end{pmatrix} = 0 \Rightarrow$$

$$\lambda^2 = \delta^2,$$

$$\underline{E = \pm\delta} \Leftarrow \underline{\lambda = \pm\delta}$$

(b) Find eigenvectors:

$$|E=+\delta\rangle: \quad \begin{pmatrix} -\delta & \delta \\ \delta & -\delta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \Rightarrow c_1 = c_2 \Rightarrow$$

$$|E=+\delta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(8)

$$|E=-\delta\rangle: \begin{pmatrix} \delta & \delta \\ \delta & \delta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=0 \Rightarrow c_1=-c_2 \Rightarrow$$

$$|E=-\delta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|a'\rangle \overset{t=0}{=} \sum_{n=1}^{2} c_n |E_n\rangle$$

$$c_n = \langle E_n | a'\rangle \Rightarrow$$

$$c_1 = \langle E=+\delta | a'\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$|a'\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_2 = \langle E=-\delta | a'\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$|a'\rangle(t=0) = \frac{1}{\sqrt{2}}|E=+\delta\rangle + \frac{1}{\sqrt{2}}|E=-\delta\rangle$$

$$|a'\rangle(t) = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}\delta t}|E=+\delta\rangle + \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar}\delta t}|E=-\delta\rangle$$

(c) $\mathcal{P} = |\langle a'(t=0) | a''(t)\rangle|^2$

$$|a''\rangle(t) = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}\delta t}|E=+\delta\rangle - \frac{1}{\sqrt{2}} e^{\frac{i}{\hbar}\delta t}|E=-\delta\rangle$$

$$c_1 = \langle E=+\delta | a''\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$c_2 = \langle E=-\delta | a''\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{\sqrt{2}}$$

$$\mathcal{P} = \left| \frac{1}{\sqrt{2}} \Big(\langle E=+\delta| + \langle E=-\delta| \Big) \cdot \frac{1}{\sqrt{2}} \Big(e^{-\frac{i}{\hbar}\delta t} |E=+\delta\rangle - e^{\frac{i}{\hbar}\delta t} |E=-\delta\rangle \Big) \right|^2 = \frac{1}{4} \left| e^{-\frac{i}{\hbar}\delta t} - e^{\frac{i}{\hbar}\delta t} \right|^2 =$$

$$= \underbrace{\sin^2 \frac{\delta t}{\hbar}}$$

QM I
Phys 651

~~Solution of ...~~ Problem #3 (5)

$$\Psi(x,0) = A e^{-\frac{x^2}{2a^2} + i\frac{p_0}{\hbar}x}$$

(a) The probability to find the particle from $-\Delta$ to Δ is:

$$\frac{\int_{-\Delta}^{\Delta} |\Psi(x,0)|^2 dx}{\int_{-\infty}^{+\infty} |\Psi(x,0)|^2 dx} = \frac{A^2 \int_{-\Delta}^{\Delta} e^{-x^2/a^2} dx}{A^2 \underbrace{\int_{-\infty}^{+\infty} e^{-x^2/a^2} dx}_{\sqrt{\pi}\,a}} =$$

$$= \frac{1}{a\sqrt{\pi}} \int_{-\Delta}^{\Delta} e^{-x^2/a^2} dx \underset{\substack{\uparrow \\ \lim \\ \Delta \to 0}}{\approx} \frac{1}{a\sqrt{\pi}} \cdot \underbrace{e^{-x^2/a^2}\Big|_{x=0}}_{1} \cdot 2\Delta = \frac{2\Delta}{a\sqrt{\pi}}$$

Note: $A = \frac{1}{\sqrt{a}\,\pi^{1/4}}$

(b) $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

$$\langle x \rangle = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{+\infty} x |\Psi(x,0)|^2 dx = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{+\infty} x\, e^{-x^2/a^2} dx = 0$$

$$\langle x^2 \rangle = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{+\infty} x^2 e^{-x^2/a^2} dx = \frac{1}{a\sqrt{\pi}} \frac{-\partial}{\partial \alpha} \underbrace{\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx}_{\sqrt{\frac{\pi}{\alpha}}} =$$

$$= \frac{1}{2a}\, a^{3/2} = \frac{a^2}{2} \Rightarrow$$

$$\Delta X = \frac{a}{\sqrt{2}}$$

Note: so far everything is what we already discussed in Lecture #13, when we talked about Gaussian wave packets

(c) $$\Psi(x,0) = \int dp \underbrace{\langle x | p \rangle}_{= \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}}} \varphi(p,0)$$

For a free particle $\Rightarrow$ $H = \frac{p^2}{2m}$ $\Rightarrow$ in p-space $\Rightarrow$ $H\varphi(p,0) = E_p \varphi(p,0)$, where

$E_p = \frac{p^2}{2m}$ ← energy

Then $\varphi(p,t) = e^{-\frac{i}{\hbar}E_p t}\varphi(p,0)$

So, $$\Psi(x,t) = \int dp \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}} e^{-\frac{i}{\hbar}E_p t} \varphi(p,0)$$

Need to find $\varphi(p,0)$:

$$\varphi(p,0) = \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{px}{\hbar}} \Psi(x,0) =$$

↑ Lecture #13

$$= \frac{1}{\sqrt{a}\,\pi^{1/4}} \int_{-\infty}^{+\infty} dx\, e^{-i\frac{px}{\hbar}} e^{i\frac{p_0 x}{\hbar}} e^{-x^2/2a^2} \frac{1}{\sqrt{2\pi\hbar}} =$$

$$= \frac{1}{\sqrt{2\pi a\hbar}\,\pi^{1/4}} \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}(p-p_0)x - x^2/2a} dx \underset{\substack{\uparrow \\ \text{complete} \\ \text{the square}}}{=} \frac{1}{\sqrt{2\pi a\hbar}\cdot\pi^{1/4}} \cdot$$

$$\cdot \underbrace{\int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{2}a} + \frac{i}{\hbar}(p-p_0)\frac{\sqrt{2}}{2}a\right)^2} dx}_{= \sqrt{\pi}\cdot\sqrt{2}a} \cdot e^{-\frac{(p-p_0)^2}{2\hbar^2}a^2} =$$

$$= \frac{\sqrt{a}}{\sqrt{\hbar}\,\pi^{1/4}} e^{-\frac{(p-p_0)^2}{2\hbar^2}a^2}$$

Now, $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \frac{\sqrt{a}}{\sqrt{\hbar}\,\pi^{1/4}} \int_{-\infty}^{+\infty} dp\, e^{i\frac{px}{\hbar}} e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} \cdot$

$$\cdot e^{-\frac{(p-p_0)^2}{2\hbar^2}a^2} \underset{\uparrow}{=} \frac{\sqrt{a}}{\sqrt{2\pi}\,\hbar\,\pi^{1/4}} \cdot \frac{1}{\sqrt{c}} \cdot \sqrt{\pi} \cdot e^{b^2/4c} e^{-\frac{p_0^2 a^2}{2\hbar^2}} =$$

re-arrange the terms under exponents to complete the square $\Rightarrow \frac{i}{\hbar}px - \frac{i}{\hbar}\frac{p^2}{2m}t - \frac{p^2a^2}{2\hbar^2} +$

$$+ \frac{p p_0 a^2}{\hbar^2} - \frac{p_0^2}{2\hbar^2}a^2 = -\Big[p^2 \cdot \underbrace{\left(\frac{it}{2m\hbar} + \frac{a^2}{2\hbar^2}\right)}_{=c} - p\underbrace{\left(\frac{p_0 a^2}{\hbar^2} + \frac{i}{\hbar}x\right)}_{=b} +$$

$$+ \frac{p_0^2 a^2}{2\hbar^2}\Big] = -\left[\left(\sqrt{c}\,p - \frac{b}{2\sqrt{c}}\right)^2 - \frac{b^2}{4c} + \frac{p_0^2 a^2}{2\hbar^2}\right]$$

$$= \frac{\sqrt{a}}{\sqrt{2}\,\hbar\,\pi^{1/4}} \frac{1}{\left(\frac{it}{2m\hbar}+\frac{a^2}{2\hbar^2}\right)^{1/2}} \cdot \exp\left[\frac{\left(\frac{p_0 a^2}{\hbar^2}+\frac{i}{\hbar}x\right)^2}{4\left(\frac{it}{2m\hbar}+\frac{a^2}{2\hbar^2}\right)}\right]\cdot$$

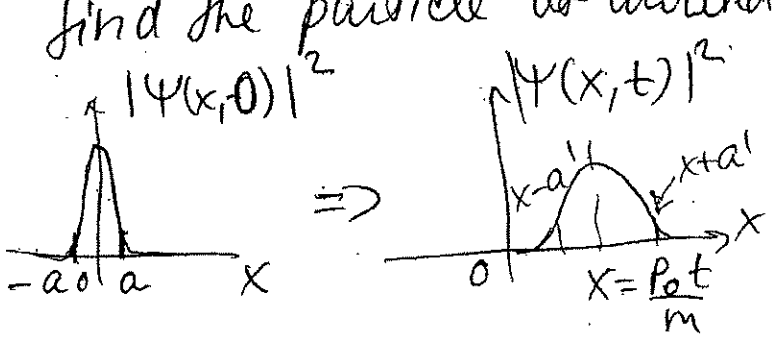
$$\cdot \exp\left[-\frac{p_0^2 a^2}{2\hbar^2}\right] = \frac{\sqrt{a}}{\sqrt{2}\,\hbar\,\pi^{1/4}} \frac{1}{\left(\frac{it}{2m\hbar}+\frac{a^2}{2\hbar^2}\right)^{1/2}}\cdot$$

$$\cdot \exp\left[\frac{-(x-p_0 t/m)^2}{2a^2(1+i\hbar t/ma^2)}\right]\cdot \exp\left[\frac{ip_0}{\hbar}\left(x-\frac{p_0 t}{2m}\right)\right]$$

Check: $\Psi(x,0) = \frac{\sqrt{a}}{\cancel{\sqrt{2}\hbar}\pi^{1/4}} \cdot \frac{1}{\frac{a}{\cancel{\sqrt{2}\hbar}}} \cdot e^{-x^2/2a^2}\, e^{i\frac{p_0 x}{\hbar}} =$

$= \frac{1}{\pi^{1/4}\sqrt{a}}\, e^{-x^2/2a^2}\, e^{i\frac{p_0}{\hbar}x}$ ✓

Now let's analyse $\Psi(x,t)$ $\Rightarrow$ It's still a Gaussian, but now it's centered around $x = \frac{p_0 t}{m}$ as opposed to $x=0$ at $t=0$. Therefore, the probability to find the particle at around $x=0$ is much smaller.



Compare $|\Psi(x,0)|^2$ and $|\Psi(x,t)|^2 \Rightarrow$ very similar $\Rightarrow$

$$\Leftarrow \begin{cases} x \to x - \frac{p_0 t}{m} \\ a \to \underbrace{a\left[1+\frac{\hbar^2 t^2}{m^2 a^4}\right]^{1/2}}_{\text{"}a'\text{"}} \end{cases}$$

Then the uncertainties:

$t=0$	$t>0$
$\frac{a}{\sqrt{2}}$	$\frac{a'}{\sqrt{2}}$ $\Rightarrow$ wave packet broadens with time!